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POLYNOMIAL REPRESENTATIONS OF KNOTS*

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Abstract. In this paper, we show that every 1-dimensional knot-type in the three-dimensional Euclidean space has polynomial representations. We also write down specific polynomial expressions for the trefoil knot and the figure eight knot. This result strengthens a conjecture of Abhyankar, that there exist non-rectifiable polynomial embeddings of the complex line in the three-dimensional complex affine space.

1. Introduction. In 1977, at a Kyoto conference, Abhyankar [1] conjectured that there exist polynomial embeddings of the affine line A^1 in A^3, which are inequivalent under the polynomial automorphisms of A^3. If our field k is algebraically closed, then this is equivalent to saying that there exist ring-theoretic epimorphisms $\alpha, \beta: k[x, y, z] \to k[t]$ such that for no automorphism φ of $k[x, y, z]$, we have $\alpha \circ \varphi = \beta$. In support of this conjecture, Abhyankar further conjectured that the embeddings $\theta(m, n, l)$ given by

$$t \mapsto (t^m, t^n, t^l + t)$$

where the natural number m, n, l are such that none of them belongs to the additive semigroup generated by the other two are not equivalent to the standard embedding:

$$t \mapsto (t, 0, 0),$$

i.e., $\theta(m, n, l)$ are *non-rectifiable*.

Subsequently, several authors have shown that, this latter conjecture is somewhat far-fetched! (For relevant literature, see [2], [3], [4], and [6].) To be precise, for instance, Creighero [3] showed that the embedding $\theta(m, n, l)$ is indeed equivalent to the standard embedding. Perhaps, after presenting a proof of the now famous EPIMORPHISM THEOREM, it was quite natural for Abhyankar to make the above conjecture, since the main step in the proof of this theorem is that if $x \mapsto f(t)$, $y \mapsto g(t)$ defines an epimorphism of $k[x, y]$ onto $k[t]$, then $\deg f$ divides $\deg g$ or $\deg g$ divides $\deg f$. (Here, $\mathrm{ch}(k) = 0$). However, a more important aspect of the results of these authors is that they obtain plausible candidates for non-tame automorphisms of A^3.

To bring in the topological point of view, let us from now on assume that the field

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is either $\boldsymbol{R}$ or $\boldsymbol{C}$. It is not very difficult to see that $\theta(m, n, l)$ over $\boldsymbol{R}$ defines a trivial knot. Thus one is not very much surprised that they are actually equivalent to the standard embedding, even in this strong (algebraic) sense. If we are looking for embeddings which are not rectifiable, then why not take an embedding θ over $\boldsymbol{C}$ which is defined over $\boldsymbol{R}$, and which defines a nontrivial knot? To be precise, we are led to the following questions:

Q.1 Are there any real polynomial functions

$$t \mapsto (f(t), g(t), h(t))$$

representing a nontrivial knot and defining an embedding of $\boldsymbol{C}$ in $\boldsymbol{C}^3$?

Q.2 Does every knot have a real polynomial representation

$$t \mapsto (f(t), g(t), h(t))$$

which defines an embedding of $\boldsymbol{C}$ in $\boldsymbol{C}^3$?

Here, we are going to answer Q.2 in the affirmative. So, Q.1 becomes redundant anyway. However, writing down some specific polynomial representations for at least a few specific knots has its own importance, particularly from the algebraists' point of view. We shall do this for the trefoil knot and the figure eight knot, these being the simplest knots.

Finally, we note that Madhav Nori has proved that any two embeddings of $\boldsymbol{A}^n$ in $\boldsymbol{A}^{2n+1}$ are isotopic (see, for instance, [6]). Here is a simple version of his argument for $n=1$.

Given an embedding $\varphi : t \mapsto (\alpha(t), \beta(t), \gamma(t))$, by performing a linear change of coordinates one can assume that $t \mapsto (\alpha(t), \beta(t))$ is a generic immersion. Then consider the set X of points λ in $\boldsymbol{C}$ such that

$$t \mapsto (\alpha(t), \beta(t), \lambda\gamma(t) + (1-\lambda)t)$$

is not an embedding. This set X is finite, and hence, there exists a path $\omega : [0, 1] \to \boldsymbol{C} \setminus X$, joining $\lambda = 0$ and $\lambda = 1$. Define the isotopy

$$\varphi : \boldsymbol{C} \times [0, 1] \to \boldsymbol{C}^3$$

by the formula

$$\varphi(t, s) = (\alpha(t), \beta(t), \omega(s)\gamma(t) + (1-\omega(s))t)$$

from the embedding φ to the embedding $t \mapsto (\alpha(t), \beta(t), t)$. Repeating this procedure or otherwise, one can produce an isotopy from the given embedding and the embedding $t \mapsto (t, t, t)$.

This may lead one to thinking that, perhaps, there are no non-rectifiable embeddings of $\boldsymbol{A}$ in $\boldsymbol{A}^3$. However, we feel that, unlike in the topological case, there is a large gap between *isotopy* and *equivalence through affine automorphisms*.

2. The main theorem. Let us first recall a few basic ways of describing a knot-type. We consider piecewise linear (PL-) embeddings $k: S^1 \to S^3$ which map the base point $(0, 1) \in S^1$ to the base point $(0, 0, 0, 1) \in S^3$. Two such embeddings k_1, k_2 are said to be equivalent if there is an orientation preserving PL-homeomorphism $f: S^3 \to S^3$ such that $k_1 \circ f = k_2$. Such an equivalence class is called a *knot-type*. Using standard techniques in topology, one sees that a knot-type is the same as the base-point preserving PL-isotopy classes of base-point preserving PL-embeddings of S^1 in S^3. By taking one-point compactifications one checks that a knot-type is the same as a proper PL-isotopy class of a proper PL-embedding of $\boldsymbol{R}$ in $\boldsymbol{R}^3$ which are linear outside a closed interval. This is the model of a knot-type that we are going to use in our input.

On the other hand, there is also the piecewise smooth model in which one considers piecewise smooth embeddings of S^1 in S^3 and their isotopy classes to represent a knot-type. Under one-point compactification, this corresponds to taking piecewise smooth proper embeddings of $\boldsymbol{R}$ in $\boldsymbol{R}^3$ given by coordinate functions which are monotone outside a closed interval and their proper isotopy classes. This is the model of a knot-type that we are going to use for our output.

Recall further (see [5]) that a knot-type is determined by a *knot-data*, which consists of

(1) a *knot-projection*, viz., a generic proper PL-immersion φ:

$$t \mapsto (\alpha(t), \beta(t))$$

of $\boldsymbol{R}$ in $\boldsymbol{R}^2$ with finitely many crossings (each such crossing being an ordinary double point), and

(2) a set of under(over)-crossing data at each of these crossings.

THEOREM 1. *Every knot-type has a polynomial representation.*

PROOF. Given a knot-type, our task is to find real polynomials $f(t), g(t), h(t)$ such that the map $t \mapsto (f(t), g(t), h(t))$ defines an embedding of $\boldsymbol{C}$ in $\boldsymbol{C}^3$, and as an embedding of $\boldsymbol{R}$ in $\boldsymbol{R}^3$, it is in the given knot-type. So, given a knot-type, let $[a, b]$ be the interval in which all the crossings of the knot-projection φ occur. Though strictly not necessary, we may (and are going to) assume that, $\varphi((-\infty, a])$ lies in the third quadrant, and $\varphi([b, \infty))$ lies in the first quadrant. Choose points $a < t_1 < \cdots < t_{2n+1} < b$ in $\boldsymbol{R}$ such that

(i) $\varphi(t_i)$ are smooth points of φ,

(ii) in the interval $[a, b]$ all the crossings are either over-crossings or all are under-crossings and

(iii) if in $[t_{i-1}, t_i]$ we have under-crossings (resp. over-crossings), then in $[t_i, t_{i+1}]$ we have over-crossings (resp. under-crossings). Now define

$$h_1(t) = \pm \prod_{i=1}^{2n+1} (t - t_i)\,.$$

Here, the sign of h_1 should be determined depending on the first crossing from the left.

Now the given knot-type is represented by the map

$$t \mapsto (\alpha(t), \beta(t), h_1(t)).$$

It is fairly easy to see that there exists $c<a<b<d$ such that $|h_1(c)|=|h_1(d)|>|h_1(t)|$ for all $t \in (c, d)$ and h_1 is monotonic in $(-\infty, c]$ as well as in $[d, \infty)$. This observation is crucial to our argument. In particular, note that h_1 is injective outside the interval $[c, d]$.

By Weierstrass's approximation theorem, we can now choose polynomials f, g, *sufficiently* close to α and β such that their derivatives are also *sufficiently* close to the derivatives of α and β, respectively, in the interval $[c, d]$ and such that h_1 separates the singularities of the curve $t \mapsto (f(t), g(t))$, in the interval $[c, d]$. Since outside this interval h_1 is injective, the mapping $t \mapsto (f(t), g(t), h_1(t))$ is an embedding of $\boldsymbol{R}$ in $\boldsymbol{R}^3$. This embedding has the 'same' projection data within the interval $[c, d]$ as that given by $t \mapsto (\alpha(t), \beta(t))$. However, this is enough for our purpose: the knot-type represented by (f, g, h_1) is the same as that represented by (α, β, h_1). One has only to look at the piece of the knot lying between the two horizontal planes $Z = \pm h_1(c)$. Neither the portions of the knot for $t<c$ and $t>d$ get entangled with the rest of the curve nor they are knotted by themselves.

Finally, we shall modify h_1 to h so that (f, g, h) defines an embedding of $\boldsymbol{C}$ as follows. Let $s_1, \ldots, s_n$ be the set of points in $\boldsymbol{C}$ at which $f'(s_j)=g'(s_j)=0$, and let (z_k, w_k) be the set of all pairs of points in $\boldsymbol{C}$ such that $(f(z_k), g(z_k))=(f(w_k), g(w_k))$. These sets being finite, we can choose the coefficients of h very close to that of h_1 such that $h_1'(s_j)$ do not vanish and such that $h(z_k) \neq h(w_k)$. This assures that (f, g, h) is an embedding of $\boldsymbol{C}$. If the coefficients of h are chosen sufficiently close to that of h_1, then we can assure that h provides the same under(over)-crossing data as provided by h_1. Hence, the knot-type does not change. This completes the proof.

3. The two examples.

1. The trefoil knot:

Set

$$f(t)=t^3-3t, \quad g(t)=t^4-4t^2, \quad h(t)=t^5-10t.$$

First, check that the corresponding ring-homomorphism $\varphi: k[X, Y, Z] \to k[t]$ defined by

$$X \mapsto f(t), \quad Y \mapsto g(t), \quad Z \mapsto h(t)$$

is surjective: infact, it is easily verified that

$$\varphi(YZ - X^3 - 5XY + 2Z - 7X) = t.$$

This porves that the mapping defines an embedding of $\boldsymbol{C}$ in $\boldsymbol{C}^3$. Let us now consider the knot $\boldsymbol{R} \subset \boldsymbol{R}^3$ defined by this mapping. Note that the derivatives of f and g do not have any common zeros. So the XY-projection of φ is an immersion. To find the multiple points on the curve, we proceed as follows: we seek points $t_1 \neq t_2$ in $\boldsymbol{R}$ such that $f(t_1)=f(t_2)$ and $g(t_1)=g(t_2)$. So, we must solve the following equations simultaneously:

$$t_1^2+t_1t_2+t_2^2-3=0$$
$$t_1^3+t_1^2t_2+t_1t_2^2+t_2^3-4(t_1+t_2)=0\,.$$

Putting $t_1+t_2=0$ (from the second equation) into the first equation yields, $(t_1, t_2)=(\sqrt{3}, -\sqrt{3})$. The other factor of the second equation, viz., $t_1^2-t_2^2=0$, yields

$$(t_1, t_2)=(-(\sqrt{6}+\sqrt{2})/2, (\sqrt{6}-\sqrt{2})/2)$$

or

$$(t_1, t_2)=((\sqrt{2}-\sqrt{6})/2, (\sqrt{2}+\sqrt{6})/2)\,.$$

This shows that, there are precisely three multiple points. One can easily verify that these are actually ordinary double points.

Finally note that

$$-(\sqrt{6}+\sqrt{2})/2<-\sqrt{3}<(\sqrt{2}-\sqrt{6})/2<(\sqrt{6}-\sqrt{2})/2<\sqrt{3}<(\sqrt{6}+\sqrt{2})/2$$

and

$$h(-(\sqrt{6}+\sqrt{2})/2)<h((\sqrt{6}-\sqrt{2})/2)$$
$$h(-\sqrt{3})>h(\sqrt{3})$$
$$h((\sqrt{2}-\sqrt{6})/2)<h((\sqrt{2}+\sqrt{6})/2)\,.$$

Thus the under(over)-crossings are as required in the trefoil-knot (see Figure 1 and Figure 2).

2. The figure eight knot:

Here we define $\psi: \boldsymbol{C} \to \boldsymbol{C}^3$, by taking

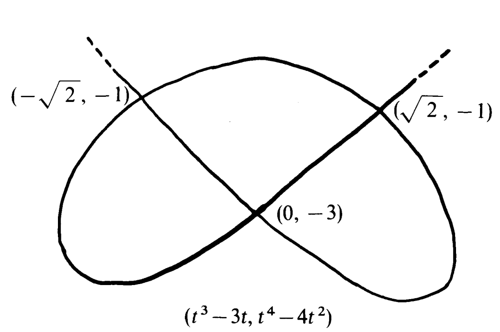


FIGURE 1.

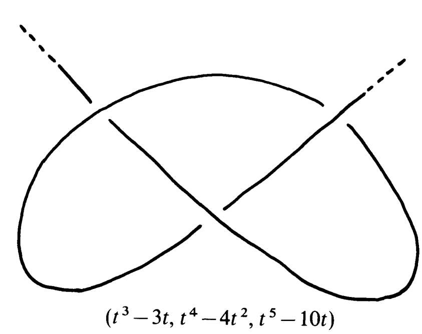


FIGURE 2.

$$f(t) = t^3 - 3t$$
$$g(t) = t(t^2-1)(t^2-4)$$
$$h(t) = t^7 - 42t\,.$$

One verifies that the corresponding ring homomorphism

$$k[X, Y, Z] \to k[t]$$

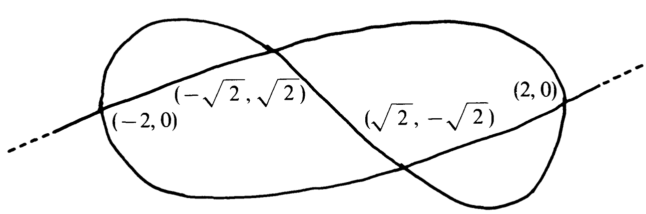
maps $X^2Z - XY^2 - 7X^2Y - 23X^3 - 3Z + 22Y + 71X$ to t and hence is surjective. As before, we proceed to show that the XY-projection is a generic immersion. The computations are necessarily messier here, though not very difficult. We find that there are four ordinary double points corresponding to the parametric values

$$(t_1, t_2) = \begin{cases} (-2, 1)\,, \\ (-(\sqrt{6}+\sqrt{2})/2, (\sqrt{6}-\sqrt{2})/2)\,, \\ (-1, 2)\,, \\ ((\sqrt{2}-\sqrt{6})/2, (\sqrt{2}+\sqrt{6})/2)\,. \end{cases}$$

Again we note that

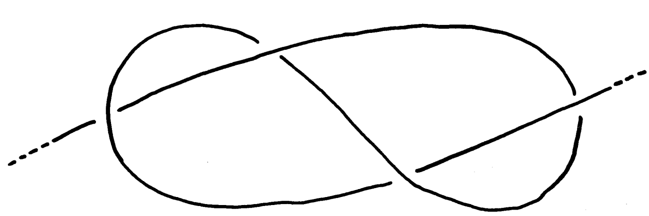
$$-2 < -(\sqrt{2}+\sqrt{6})/2 < (\sqrt{2}-\sqrt{6})/2 < 1 < (\sqrt{2}+\sqrt{6})/2 < 2$$

and finally,



(t^3-3t, t^5-5t^3+4t)

FIGURE 3.



$(t^3-3t, t^5-5t^3+4t, t^7-42t)$

FIGURE 4.

$$h(-2) < h(1)$$
$$h(-(\sqrt{2}+\sqrt{6})/2) > h((\sqrt{6}-\sqrt{2})/2)$$
$$h(-1) < h(2)$$
$$h((\sqrt{2}-\sqrt{6})/2) > h((\sqrt{2}+\sqrt{6})/2)\,.$$

This shows that ψ represents the figure eight knot (see Figure 3 and Figure 4).

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