Tôhoku Math. J.
45 (1993), 393–422

HOLOMORPHIC MAPS FROM COMPACT MANIFOLDS INTO LOOP GROUPS AS BLASCHKE PRODUCTS

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(Received March 2, 1992, revised November 19, 1992)

Abstract. We describe a factorization theorem for holomorphic maps from a compact manifold M into the loop group of $U(N)$. We prove that any such map is a finite Blaschke product of maps into Grassmann manifolds (unitons), satisfying recursive holomorphicity conditions; each map being attached to a point in the open unit disc. This factorization is essentially unique. Using a theorem of Atiyah and Donaldson, we construct a stratification of the moduli space of framed $SU(2)$ Yang-Mills instanton over the 4-sphere, in which the strata are iterated fibrations of spaces of polynomials, indexed by plane partitions; and the unique open stratum of "generic" instantons of charge d, is the configuration space of d distinct points in the disc, labelled with d biholomorphisms of the 2-sphere.

Introduction. Let $\Omega U(N)=\{\gamma: S^1 \to U(N) \mid \gamma \text{ real analytic}, \gamma(1)=I\}$ be the real analytic loop group of the unitary group $U(N)$. By using Fourier series expansions, $\Omega U(N)$ may be given a Kähler manifold structure (cf. [A]).

In this paper we study holomorphic maps (or, more generally, rational maps (cf. the definition in §2), from a compact complex manifold M into $\Omega U(N)$.

The motivation for this study comes from two different results, both in the realm of gauge theory and twistor geometry.

(1) By a theorem of Atiyah and Donaldson (cf. [A]), for any classical group G, the parameter space of based holomorphic maps $S^2 \to \Omega G$ is diffeomorphic to the space of Yang-Mills instantons over S^4, modulo based gauge transformations. The instanton number corresponds to the degree of the map, defined via $H^2(\Omega G, \boldsymbol{Z}) \cong \boldsymbol{Z}$.

(2) Uhlenbeck [U] associated a holomorphic map $F: S^2 \to \Omega U(N)$ to any harmonic map $f: S^2 \to U(N)$, using methods from the theory of completely integrable systems. She gave a recursive procedure, similar to a Bäcklund transformation, to generate new F's from given ones by the choice of appropriate holomorphic vector bundles over S^2, called unitons. Then she proved a unique factorization theorem of any such F as a product of unitons.

Moreover, generalizing the paper of Uhlenbeck, Segal [Seg] has showed that any holomorphic map from a compact manifold into $\Omega U(N)$ has values in the space of rational loops. But it is relatively well known that any based rational matrix valued function, unitary on the circle, has a finite factorization as a "Blaschke product" (cf. [G]).

Key words. Loop group, Blaschke product, instanton, uniton.
1991 *Mathematics Subject Classification.* Primary 81T13; Secondary 14F05, 32H02, 32L25, 47A56.

In this paper we describe the factorization of holomorphic maps $F: M \to \Omega U(N)$, M being a compact complex manifold, induced by the Blaschke product decompositions. This includes Uhlenbeck's work as a special case.

The appropriate notion of uniton is a torsion-free coherent sheaf over M, "based" at a point α of the open unit disc D. After giving a new proof of Segal's result, we show that any holomorphic map $M \to \Omega U(N)$ is a Blaschke product of such unitons; moreover, this product is essentially unique (by the constructions in [V3]).

In §1 we give the basic facts about the loop group $\Omega U(N)$; we also define Blaschke products. In §2 we give the definition of rational maps, explaining some basic material; and we recall the definition of the *-product (a commutative meromorphic product on $\Omega U(N)$, generalizing the ordinary product for rational functions) from [V3]. Chapter 3 is more specific: we give the definition of uniton, and we show that "adding a uniton" increases the degree of the map by the degree of the uniton (this generalizes the energy formula in [V1]). In §4 we show that any holomorphic map $M \to \Omega U(N)$ is a (Blaschke) product of unitons, if M is compact; and in §5 we give a unique factorization theorem. Finally, in §6 we apply our results to based holomorphic maps from S^2 into $\Omega SU(2)$, thus describing a holomorphic stratification of the moduli space of $SU(2)$ instantons, and computing the dimensions of the strata; we also give some open problems.

This is a revised version of a preprint of November 1990. A short version of this paper has already appeared in an informal lecture notes (in Japanese) of Tokyo Metropolitan University, August 1990. While finishing this paper, we learnt of the paper of Lerner [L] about factorization of instantons. Presumably, our results are very much related to Lerner's, via the Atiyah-Donaldson theorem quoted above (cf. also [Mu]). Moreover, Boyer, Mann, Hurtubise, and Milgram [BHMM] have recently proved the Atiyah-Jones conjecture, about the topology of the moduli space of framed $SU(2)$ instantons: in this context they defined a stratification of this moduli space which has many striking similarities with the one we discuss in §6 of this paper. It would be interesting to compare the two approaches.

ACKNOWLEDGEMENTS. We are indebted to Professor Masaki Maruyama, for the statement and proof of Lemma 4.5, and with Yoshihiro Ohnita, for many explanations; we thank them deeply. We also benefited from discussions with a very large number of mathematicians; we thank them all.

We wish to thank the University of Warwick and Tokyo Metropolitan University for their hospitality at various stages of this research and for financial support. We also thank the University of Rome 2, and Cassa Rurale ed Artigiana di Faenza for financial support.

1. The loop group $\Omega U(N)$. The material in this section is quite standard; it may be found in (for example) [A], [EL], [PS].

Let $U(N)$ be the unitary group of degree N. We define the loop group by

$$\Omega U(N)=\{\gamma : S^1 \to U(N) \,|\, \gamma(1)=I\} . \tag{1}$$

It has a group structure defined by pointwise multiplication. Different choices of regularity for the loops γ's are possible. For example, one can consider:

(i) The real analytic loop group, which we simply denote $\Omega U(N)$.

(ii) The smooth loop group $\Omega_{\mathrm{sm}} U(N)$: it is a Fréchet manifold.

(iii) The "Hilbert loop group", of loops of Sobolev class $L^2_{1/2}$. It is a Hilbert manifold, and it may be seen as a Hilbert space Grassmannian (cf. [PS]).

(iv) The rational loop group $\Omega_{\mathrm{rat}} U(N)=\{\gamma \in \Omega U(N) \,|\, \gamma$ extends to a rational matrix-valued function on $S^2\}$. By the result of Segal quoted in the introduction, it is of fundamental importance.

In the following we will always consider the real analytic loop group (Case (i)), unless we say otherwise. This choice does not seem to be particularly restrictive; in particular, Uhlenbeck's "extended solution" of the harmonic map equation (cf. [U]) falls in this class; and also Atiyah's paper [A] actually treats this case. We have to remark that Segal's result and proofs in [Seg], although stated for the smooth case, hold for the Hilbert case.

Let $\lambda=e^{it}$ be the complexified loop variable. The Lie algebra of $\Omega U(N)$ is

$$\Omega \mathrm{u}(N)=\{\eta : S^1 \to \mathrm{u}(N) \,|\, \eta(1)=0\} \tag{2}$$

with the appropriate choice of regularity. We identify $\Omega \mathrm{u}(N)$ with the tangent space to $\Omega U(N)$ at each point, via left translation. It is possible to define a complex structure J on $\Omega U(N)$ using Fourier series expansion; if $\eta \in \Omega \mathrm{u}(N)$, then we have $\eta=\sum_{\alpha \neq 0}(1-\lambda^\alpha)\eta^\alpha$, and we define $J^\sim : \Omega \mathrm{u}(N) \to \Omega \mathrm{u}(N)$ by

$$J^\sim \eta = -i\Big(-\sum_{\alpha<0}(1-\lambda^\alpha)\eta^\alpha + \sum_{\alpha>0}(1-\lambda^\alpha)\eta^\alpha\Big) . \tag{3}$$

We then define J as the left translation of $J^\sim$.

The standard Riemannian metric on $\Omega U(N)$ is defined by the left translation of the norm

$$|\eta|^2 = \sum_{\alpha \neq 0} |\alpha|\,|\eta^\alpha|^2 . \tag{4}$$

This metric is Kähler with respect to the complex structure J. The real analytic and the smooth loop groups are not complete with respect to this metric. Their topological completion is given by the Hilbert loop group mentioned above.

The associated symplectic 2-form S, normalized so as to be the positive generator of $H^2(\Omega U(N), \boldsymbol{Z}) \cong \boldsymbol{Z}$, is given by the left translation of the alternating form

$$S^\sim(\eta, \xi) = \frac{1}{\pi^2}\int_{S^1} \mathrm{Tr}(\eta \xi') . \tag{5}$$

S is a closed integral 2-form of type (1, 1): since $\Omega U(N)$ is simply connected, $-2i\pi S$ is the curvature form of a holomorphic line bundle over $\Omega U(N)$, called the determinant line bundle (cf. [PS]).

The complex structure on $\Omega U(N)$ may also be described as follows (cf. [PS], [A]): let $D=\{\lambda \in \boldsymbol{C} \mid |\lambda|<1\}$. Define the two complex groups

$$LGL(N, \boldsymbol{C})=\{\gamma : S^1 \to GL(N, \boldsymbol{C})\} \quad \text{and}$$
$$L^+GL(N, \boldsymbol{C})=\{\gamma \in LGL(N, \boldsymbol{C}) \mid \gamma \text{ extends continuously to a holomorphic map } D \to GL(N, \boldsymbol{C})\}.$$

Then $\Omega U(N)$ is a homogeneous space for $LGL(N, \boldsymbol{C})$, with left isotropy group $L^+GL(N, \boldsymbol{C})$; and this gives a complex structure to $\Omega U(N)$, which coincides with the one previously defined.

Let M be a differentiable manifold. Let $f: M \to \Omega_{\mathrm{sm}} U(N)$ be a map into the smooth loop group. Suppose that f is smooth with respect to the Fréchet manifold structure on $\Omega_{\mathrm{sm}} U(N)$. Then the map

$$f^{\$}: M \times S^1 \to U(N), \quad f^{\$}(z, e^{it})=f(z)(e^{it})$$

is smooth (cf. [Mi] for the general theory of manifolds of smooth maps).

Let now $f: M \to \Omega U(N)$ be any smooth map; suppose M is a compact complex manifold of complex dimension m; and let Ω be any real closed 2-form on M. Then we can define the *degree of f with respect to Ω*, by

$$\deg_\Omega(f)=\int_M f^*(S) \wedge \Omega^{m-1}=\frac{1}{2\pi^2}\int_M \mathrm{Tr}(f^{-1}df \wedge (f^{-1}df)') \wedge \Omega^{m-1}. \tag{6}$$

If Ω is an integral form, then $\deg_\Omega(f)$ is also integral, and we have

$$\deg_\Omega(f)=\langle f^*[S] \cup [\Omega]^{m-1}, [M]\rangle \tag{7}$$

where $[S]$ is the (integral) cohomology class of S, $\cup$ is the cup product in cohomology, and $\langle\ , [M]\rangle$ is the evaluation on the fundamental cycle of M.

Let $G_k(\boldsymbol{C}^N)$ be the Grassmann manifold of complex k-planes in $\boldsymbol{C}^N$. Let $e_{-1}: \Omega U(N) \to U(N)$ be loop evaluation at $\lambda=-1$.

LEMMA 1.1. *There is a totally geodesic embedding ψ, and a family (parametrized by D) of holomorphic embedding ϕ_α, such that the following diagram commutes for any $\alpha \in D$:*

$$\begin{tikzcd} & \Omega U(N) \arrow[d, "e_{-1}"] \\ \psi: G_k(\boldsymbol{C}^N) \arrow[ur, "\phi_\alpha"] \arrow[r] & U(N). \end{tikzcd} \tag{8}$$

Moreover, ϕ_α induces an isomorphism of 2-dimensional integral homology groups.

PROOF. If $V \in G_k(\mathbf{C}^N)$, let p be the Hermitian projection operator $p: \mathbf{C}^N \to V$; we have $p^* = p$, $p^2 = p$; moreover $p^\perp = I - p$ is the Hermitian projection onto $(V)^\perp$. We also denote V by p, as image space of p. The map defined by $\psi(V) = p - p^\perp$ is well-known to be a totally geodesic embedding, called "Cartan embedding".

Similarly, if $V \in G_k(\mathbf{C}^N)$, and $\alpha \in D$, let $\phi_\alpha: V \mapsto p + \xi_\alpha p^\perp$, where p is the Hermitian projection onto V, and $\xi_\alpha \in \Omega U(1)$ is the rational function

$$\xi_\alpha(\lambda) = (\lambda - \alpha)(\bar{\alpha} - 1)(\bar{\alpha}\lambda - 1)^{-1}(1 - \alpha)^{-1} . \tag{9}$$

It is easy to prove that, for $\lambda \in S^1$, $p + \xi_\alpha p^\perp$ is unitary. It is also easy to see that ϕ_α is holomorphic, because, by composing it with holomorphic maps into the Grassmannian, we get holomorphic maps into $\Omega U(N)$ (cf. §3).

The commutativity of the diagram is obvious. □

LEMMA 1.2. *Any rational loop $\gamma \in \Omega U(N)$ admits a factorization*

$$\gamma = \xi_{\alpha_1}\xi_{\alpha_2} \dots \xi_{\alpha_h}(p_1 + \xi_{\beta_1}p_1^\perp)(p_2 + \xi_{\beta_2}p_2^\perp) \dots (p_K + \xi_{\beta_K}p_K^\perp) \tag{10}$$

which is called a (finite) Blaschke product.

PROOF. Lemma 1.2 is a special case of a factorization theorem for rational matrix-valued functions of Hardy class, and unitary on the disc, due to Potapov and Masani (see [Ma], [Po], [G]). A simple proof is in [U]; see also [Be], and the proof of Theorem 4.1 in the following. The idea is first to multiply γ with a scalar, so as to obtain a function of Hardy class. Then one chooses as p's the Hermitian projections onto the image (or kernel) spaces of γ, at those points $\alpha \in D$, where γ is singular; proceeding by induction on the order of zero of $\det(\gamma)$ at these points, one proves the result. □

Of course the orders of the α's and β's are quite arbitrary. To overcome this difficulty, we present the approach in [V3]. For any finite subset $a \subset D$, we define $a^* = \{1/\bar{\alpha} \mid \alpha \in a\} \subset D$, and a subgroup of $\Omega_{\mathrm{rat}}U(N)$ by

$$\Omega^a = \{\gamma \in \Omega_{\mathrm{rat}}U(N) \mid \gamma \text{ is smooth and invertible outside } a \cup a^*\} .$$

In particular, $\Omega^{\{0\}}$ is the space of *algebraic loops* $\Omega_{\mathrm{alg}}U(N)$ (those which admit a finite Laurent decomposition into powers of λ). For any $\alpha \in D$, $\Omega^{\{\alpha\}} \cong \Omega^{\{0\}}$, by the change of variable $\lambda \mapsto \xi_\alpha(\lambda)$.

The subgroups Ω^a satisfy the lattice conditions

$$\Omega^a \cap \Omega^b = \Omega^{a \cap b} ; \tag{11A}$$

$$\Omega^a \Omega^b = \Omega^b \Omega^a = \Omega^{a \cup b} . \tag{11B}$$

For $\gamma \in \Omega^a$, $\eta \in \Omega^b$ with $a \cap b = \varnothing$ we define the $*$-product (a sort of least common multiple) $\gamma * \eta \in \Omega^{a \cup b}$ as follows: there exist $\gamma' \in \Omega^a$, $\eta' \in \Omega^b$, satisfying: $\gamma\eta' = \eta\gamma' = \gamma * \eta$. It is proved in [V3] that this recipe gives a well-defined, meromorphic, associative and commutative product on $\Omega_{\mathrm{rat}}U(N)$. Lemma 1.2 may now be rephrased.

LEMMA 1.3. *Let $\gamma \in \Omega_{\mathrm{rat}}U(N)$. Then there exist $\alpha_1 \ldots \alpha_k \in D$, uniquely determined by γ, and loops $\gamma_i \in \Omega^{\{\alpha_i\}}$, algebraic in $\xi_{\alpha_i}(\lambda)$, so that*

$$\gamma = \gamma_{\alpha_1} * \cdots * \gamma_{\alpha_k}. \tag{12}$$

Moreover, any γ_{α_i} is a product of a power of ξ_{α_i} and of factors $p_i + \xi_{\alpha_i} p_i^{\perp}$.

In the following, we will apply Lemma 1.2 (and its proof) in order to decompose holomorphic maps from compact manifolds M into $\Omega U(N)$, into the product of factors $(p_i + \xi_{\alpha_i} p_i^{\perp})(z)$. This factorization will depend on the order of the α_i's; but Lemma 1.3 will give, in a very precise sense, the meaning by which our factorizations are intrinsic, and unique.

2. Holomorphic and rational maps into $\Omega U(N)$. Let M be a complex manifold of complex dimension m. In the following we will always suppose M is connected. Let $d = \partial' + \partial''$ be the decomposition of the exterior derivative into components of types (1, 0) and (0, 1).

By our definitions, a *holomorphic map* $f: M \to \Omega U(N)$ is a smooth map f: $M \to \Omega_{\mathrm{sm}} U(N)$, with image in the real analytic loops, such that $A_{e^{it}} = (f^{-1}\partial'' f)_{e^{it}}$ extends continuously, and holomorphically in λ, from $S^1 = \{\lambda = e^{it}\}$ to the disc $D = \{\lambda \in \boldsymbol{C} \mid |\lambda| < 1\}$. We call A_λ ($\lambda \in D$) the holomorphic extension of $A_{e^{it}}$. We remark that, (cf. §1), any holomorphic map induces a smooth map $f^{\$}: M \times S^1 \to U(N)$, which is real analytic in the loop variable.

The main purpose of this paper is to give a quite explicit description of every holomorphic map from a compact manifold M into $\Omega U(N)$, as a Blaschke product of maps into Grassmannians. When M has complex dimension greater than 1, one has to consider anyway a wider class of maps: rational maps. We give a definition which is the most useful for our purposes (cf. [OV]).

Let M be a compact complex manifold. A *rational* map $f: M \to \Omega U(N)$ is given by

(i) a complex analytic subset $\mathscr{S}$ of M, of complex codimension at least 2; and a smooth holomorphic map $f: M - \mathscr{S} \to \Omega U(N)$;

(ii) a compact complex manifold M^*, and a holomorphic map $\tau: M^* \to M$, which is a biholomorphism outside $\mathscr{S}$;

(iii) a holomorphic map $f^*: M^* \to \Omega U(N)$ such that $f^* = f\tau$ outside $\tau^{-1}(\mathscr{S})$.

We say that (M^*, τ, f^*) is a *resolution* of $f: M \to \Omega U(N)$. Moreover, we identify two rational maps, if they have the same f. Of course, when M is a Riemann surface, every rational map is holomorphic, in an obvious sense.

REMARK. It is proved in [SV] that, if M is compact, any smooth holomorphic map $f: M \setminus \mathscr{S} \to \Omega U(N)$, with $\mathscr{S}$ complex analytic subset of M, of complex codimension at least 2, is rational. This result will not be used in this paper, anyway.

In the following we will always argue supposing M is a Riemann surface, and f is a holomorphic map, unless we say otherwise. Anyway, we will say explicitly how to modify the statements and proofs, in order to deal with the general case. For example, a natural assumption in higher dimensions, is to require M Moishezon, i.e. birational to a compact projective manifold. As such, M inherits a Hermitian holomorphic line bundle, with curvature $-2i\pi\Omega$, where Ω is a closed integral semipositive 2-form of type $(1, 1)$, which is strictly positive outside some 0-measure subset. Conversely, this condition characterizes Moishezon manifolds, by a well known result of Siu (cf. [Si]). This allows one to consider degrees of maps, and the first Chern classes of sheaves, as integers.

We remark that any compact Riemann surface is obviously Moishezon; and that, if $\tau: M^* \to M$ is a resolution of M, and M is Moishezon with semipositive 2-form Ω, then M^* is also Moishezon, with 2-form $\tau^*\Omega$. This is not true for the category of Kähler manifolds. The following remark shows that this assumption is not too heavy.

PROPOSITION 2.1. *Let M be a complex manifold, and suppose there exists a nonconstant holomorphic map $f: M \to \Omega U(N)$. Then either there exists a foliation of M into closed analytic subsets, such that f is constant along the leaves, or the differential of f has maximal rank somewhere, and M is Moishezon.*

PROOF. Suppose df has never maximal rank. Then the sets $\{z \in M \mid f(z) = \gamma\}$ are analytic subsets of M, for any loop $\gamma \in \Omega U(N)$.

Suppose df has maximal rank somewhere. Then, by holomorphicity, the set where df has not maximal rank has measure 0, with respect to any metric on M. Let L be the pullback of the dual of the determinant line bundle over $\Omega U(N)$ (cf. [PS]); it has as curvature the pullback $2i\pi f^*S$, where S is the symplectic 2-form on $\Omega U(N)$. We can use Siu's result concluding that M is Moishezon, because it admits a semipositive holomorphic line bundle, with strictly positive curvature outside a 0-measure subset.

We give now some applications of the definitions, which we will need in the following. Let M be a compact complex manifold. Let $f: M \to \Omega U(N)$ be a rational map.

PROPOSITION 2.2. *Let Ω be a real closed 2-form of type $(1, 1)$ form.*

(i) $\deg_\Omega(f)$ *is well-defined, independently of the resolution* (M^*, τ, f^*) *of* f;

(ii) *if Ω represents an integral cohomology class,* $\deg_\Omega(f)$ *is an integer*;

(iii) *if M is Moishezon, with semipositive 2-form Ω, and* $\deg_\Omega(f) = 0$, *then f is constant.*

PROOF. Let (M^*, τ, f^*) be a resolution of f. We define $\deg_\Omega(f) = \deg_{\tau^*\Omega}(f^*)$. By functoriality of pull-backs, (ii) is obvious. Moreover, using the usual notation, we have:

$$\deg_{\tau^*\Omega}(f^*) = \int_{M^*} (f^*)^*(S) \wedge (\tau^*\Omega)^{m-1} = \int_{M^* - \tau^{-1}(\mathcal{S})} (f^*)^*(S) \wedge (\tau^*\Omega)^{m-1}$$
$$= \int_{M - \mathcal{S}} f^*(S) \wedge \Omega^{m-1} .$$

Therefore $\deg_\Omega(f)$ does not depend on the chosen resolution, but only of f. Let $\mathscr{T} \subset M$ be a 0-measure subset, such that Ω is positive outside $\mathscr{T}$. Then we have

$$\deg_\Omega(f) = \int_{M-(\mathscr{S} \cup \mathscr{T})} f^*(S) \wedge \Omega^{m-1} = 0 .$$

But $f^*S \wedge \Omega^{m-1}$ is a positive (m, m) form on $M-(\mathscr{S} \cup \mathscr{T} \cup \{df=0\})$. Therefore f must be constant. □

Let M be a complex manifold, and $E \to M$ a smooth complex vector bundle. Let ∇ be a ∂''-operator on $E \to M$, i.e. a first order differential operator

$$\nabla \colon C^\infty(E) \to C^\infty(E \otimes T^*_{(0,1)}(M))$$

which has the Cauchy-Riemann-Dolbeault operator ∂'' as principal part. In local coordinates we have $\nabla = \partial'' + \sum A_i(z) dz_i = \partial''_A$, where $A = \sum A_i(z) d\bar{z}_i$.

Suppose the integrability condition $(\nabla)^2 = 0$ is satisfied. This means $\partial''(A) + 1/2[A, A] = 0$ in local coordinates. Then we have the following.

THEOREM 2.3 (Koszul-Malgrange). *There exists a unique holomorphic structure on E, such that a local section v of E is holomorphic if and only if $\nabla v = 0$.*

We denote $M \times \boldsymbol{C}^N$, with the holomorphic structure induced by a ∂''-operator ∇, by $(\underline{\boldsymbol{C}}^N, \nabla)$.

Let $f \colon M \to \Omega U(N)$ be a holomorphic map. Let A_λ, or $A(\lambda)$, be the holomorphic (in λ) extension of $f^{-1}\partial'' f$ to the disc D. Let $\partial''_{A(\lambda)} = \partial'' + A(\lambda)$ be the associated ∂''-operator on the trivial bundle $M \times \boldsymbol{C}^N$. For each $\lambda \in S^1$, we have $\Gamma = (\partial''_{A(\lambda)})^2 = 0$. Since Γ is holomorphic in D, by the unicity of the solution to the Dirichlet problem, it must be 0 on all the disc. Therefore Koszul-Malgrange theorem defines a vector bundle $(\underline{\boldsymbol{C}}^N, \partial''_{A(\lambda)})$, for any $\lambda \in D$. This vector bundle will play a major role in the following.

In the rational case, the following holds.

LEMMA 2.4. *Let M be a compact complex manifold, and let $f \colon M \to \Omega U(N)$ be a rational map, singular on $\mathscr{S}$. Then, for each $\lambda \in D$, there exists a torsion-free coherent sheaf $\mathscr{F}$ over M, which coincides with $(\underline{\boldsymbol{C}}^N, \partial''_{A(\lambda)})$, over $M - \mathscr{S}$, and which has the first Chern class 0.*

PROOF. The proof is as in [OV, §7]. We just remark that the sheaf $\mathscr{F}$ coincides with a topologically trivial vector bundle outside a set of complex codimension ≥ 2. Therefore it has the first Chern class 0.

3. Unitons. In [U] Uhlenbeck associated to any harmonic map $g \colon S^2 \to U(N)$ a holomorphic map (called extended solution) $f \colon S^2 \to \Omega U(N)$, satisfying $f_{-1} = g$. In our language, extended solutions are holomorphic maps $f \colon S^2 \to \Omega U(N)$, which have an $A(\lambda)$ of the form $A(\lambda) = (1-\lambda)B$, with B independent of the loop variable.

Uhlenbeck gave a one-to-one correspondence between based extended solutions and based harmonic maps $S^2 \to U(N)$. She proved that any such extended solution is a Blaschke product of at most $N-1$ factors $(p+\lambda p^{\perp})(z)$, which she called unitons, each one satisfying holomorphicity equations; and she also gave a unique factorization theorem. This immediately produced a unique factorization theorem for harmonic maps $S^2 \to U(N)$.

We want to generalize this work to the case of general rational maps $M \to \Omega U(N)$, with M a compact complex manifold. As a first step, we generalize the notion of unitons. In §§4, 5 we will give a unique factorization theorem (Theorem 5.1). In the case M has complex dimension greater than 1, one has to use the theory of coherent sheaves (cf. [OV], where the work of Uhlenbeck is generalised to pluriharmonic maps $M \to U(N)$, with M compact, complex, simply-connected).

Let M be a complex manifold, and let $\alpha \in D$. Let $f\colon M \to G_k(C^N)$ be a smooth map; for any $\alpha \in D$, f defines a map $g=\phi_\alpha f=(p+\xi_\alpha p^{\perp})\colon M \to G_k(C^N) \subset \Omega U(N)$, with $\boldsymbol{p}=\operatorname{Im} p \subset M \times C^N$ complex subbundle; $\boldsymbol{p}$ may be identified with the pullback, via f, of the tautological bundle. Moreover, f is holomorphic if and only if $\boldsymbol{p}$ is holomorphic, since both conditions are equivalent to $p^{\perp}\partial'' p=0$. We then call g a 1-*uniton*, *based at* α. More generally, if f is rational in the standard sense, i.e. smooth and holomprhic outside an analytic subset of complex codimension ≥ 2, then f defines a rational map into $\Omega U(N)$, and a coherent subsheaf p of the trivial bundle $M \times C^N$ (cf., for example, [UY], [OV]).

One way to iterate this procedure, would be to take *-products of 1-unitons; but this does not cover the general case (cf. Theorem 5.1 and Lemma 1.3). We therefore study the holomorphicity properties of the Blaschke product factorization of Lemma 1.2. Since the pointwise product of $\Omega U(N)$ is not holomorphic, we are going to get twisted Cauchy-Riemann equations. We proceed as follows.

Let $f\colon M \to \Omega U(N)$ be a given holomorphic map; we do not assume M is compact. As usual, let $A(\lambda)$ be the holomorphic extension of $f^{-1}\partial'' f$ to the disc. Let $p(z)$ be a Hermitian projection operator of constant rank, which is a function of the point z in M. It defines a subbundle $\boldsymbol{p} \subset M \times C^N$. Let $p^{\perp}=I-p$, as in Lemma 1.1; and we fix $\alpha \in D$. We want to know when the map $f^{\sim}=f(p+\xi_\alpha p^{\perp})\colon M \to \Omega U(N)$ is holomorphic.

Let $A^{\sim}=(f^{\sim})^{-1}\partial'' f^{\sim}$. Then $f^{\sim}$ is holomorphic if and only if $A^{\sim}$ extends holomorphically to the disc D. But we have

$$A^{\sim}=(p+\xi^{-1}p^{\perp})A(p+\xi p^{\perp})+(1-\xi)p\partial'' p+(\xi^{-1}-1)p^{\perp}\partial'' p\,. \tag{1}$$

$A(\lambda)$ is extendable to the disc, and so is ξ_α; therefore the unique obstruction to extendability lies in the coefficient of $(\xi_\alpha)^{-1}$ which is $p^{\perp}A(\lambda)p+p^{\perp}\partial'' p$. We may write

$$p^{\perp}A(\lambda)p+p^{\perp}\partial'' p=p^{\perp}A(\alpha)p+p^{\perp}\partial'' p+p^{\perp}(A(\lambda)-A(\alpha))p\,.$$

But we have

$$\xi^{-1}p^{\perp}(A(\lambda)-A(\alpha))p = p^{\perp}B(\lambda)p(\bar{\alpha}\lambda-1)(1-\alpha)(\bar{\alpha}-1)^{-1}$$

where $A(\lambda)-A(\alpha)=B(\lambda)(\lambda-\alpha)$ is extendable holomorphically to the disc. Therefore we have that $A^{\sim}$ is extendable holomorphically to the disc D if and only if

$$p^{\perp}\partial''p + p^{\perp}A(\alpha)p = 0\,. \tag{2}$$

This is a Cauchy-Riemann equation for the subbundle $\boldsymbol{p}$ (cf. the discussion on Koszul-Malgrange theorem in §2).

PROPOSITION 3.1. *Let $f\colon M \to \Omega U(N)$ be a holomorphic map. Let $f^{\sim} = f(p+\xi_\alpha p^{\perp})$, with $|\alpha|<1$; and p, ξ as above. Then $f^{\sim}$ is holomorphic if and only if $\boldsymbol{p}$ is a holomorphic subbundle of $M \times \boldsymbol{C}^N$, with respect to the holomorphic structure induced by the ∂''-operator $\partial''_{A(\alpha)}$.*

Slightly extending the terminology in [U], we say that $f^{\sim}$ *has been obtained by addition of the uniton $\boldsymbol{p}$, based at α, to the map f.* We remark that the case considered by Uhlenbeck is when $\alpha=0$. Moreover she gets an extra condition, motivated by the requirements to obtain extended solutions $f^{\sim}$'s, starting from extended solutions f's.

Actually, Uhlenbeck also considers unitons based at $\alpha \neq 0$; but the requirement to get extended solutions forces the bundles $\boldsymbol{p}$'s to be not only holomorphic, but also co-variant constant, (with respect to an appropriate flat connection), and holomorphically trivial: their use in [U] is to study a "dressing action" of $\Omega U(N)$ on the space of extended solutions.

Let us consider now the case of rational maps. Let $f\colon M \to \Omega U(N)$ be a rational map; we suppose M is compact. Let $\mathscr{S}(f)$ be the singularity set of f: it is an analytic subset of M of complex codimension at least 2. We want to add "rational unitons" to f. We can use the arguments in [OV §7]. Let (M^*, τ, f^*) be a resolution of the map f. By Lemma 2.4, for each $\alpha \in D$ there exists a torsion-free coherent sheaf $v = \tau_*(\underline{\boldsymbol{C}}^N, \partial''_{A^*(\alpha)})$ over M, which coincides with $(\underline{\boldsymbol{C}}^N, \partial''_{A(\alpha)})$ outside the singularity set $\mathscr{S}(f)$ of f.

We call any holomorphic vector subbundle η of $(\underline{\boldsymbol{C}}^N{}_{|M-\mathscr{S}(f)-\mathscr{S}(\eta)}, \partial''_{A(\alpha)})$, (where $\mathscr{S}(\eta)$ is also an analytic subset of M of complex codimension at least 2), a *rational uniton for f, based at α.* Equivalently (cf. [OV]), we can call a rational uniton for f

(1) a coherent subsheaf $\mathscr{F}$ of $\tau_*(\underline{\boldsymbol{C}}^N, \partial''_{A^*(\alpha)})$, with torsion-free quotient; or

(2) a coherent subsheaf $\mathscr{F}^*$ of $(\underline{\boldsymbol{C}}^N, \partial''_{A^*(\alpha)})$, with torsion-free quotient, which coincides with the pull-back of η, on $M^* - \tau^{-1}(\mathscr{S}(f) \cup \mathscr{S}(\eta))$.

We remark that, if f is a rational map, and η is a uniton for f, with associated projection operator p, then there exists a resolution M^* of M, such that both f and p are smooth, when "read" on M^*.

PROPOSITION 3.2. *Let M be a compact complex manifold, and let $f\colon M \to \Omega U(N)$ be a rational map. Let $\boldsymbol{p}$ be a rational uniton for f based at $\alpha \in D$. Let $f^{\sim} = f(p+\xi_\alpha p^{\perp})$. Then, for any real closed 2-form Ω on M we have*

(3) $$\deg_\Omega f^\sim - \deg_\Omega f = -\deg_\Omega \boldsymbol{p}$$

where $\deg_\Omega f$ *and* $\deg_\Omega f^\sim$ *are defined as in* §2, *and* $\deg_\Omega \boldsymbol{p}$ *is defined by contraction of the first Chern class of the coherent sheaf defined by* $\boldsymbol{p}$, *with* Ω^{m-1}.

PROOF. The proof generalizes the proofs in [V1] and [OV], with some minor variations. We divide the proof into four steps.

(i) We prove a topological lemma on the additivity of degrees, under pointwise multiplication. It is not the most general statement; formulas are probably related to cyclic cohomology.

(ii) We prove the formula in the case $\alpha=0$.

(iii) We use invariance of the problem under a group of biholomorphism of the disc to prove the case of general α.

(iv) We use the technique of resolution of singularities to pass from the smooth case to the general case.

(i) The following is a special case of a property of 1-dimensional cycles in gauge groups (cf. [V2]).

LEMMA 3.3. *Let M be a compact manifold, and let $g, h\colon M \to \Omega U(N)$ be smooth maps. Let Ω be any real closed 2-form on M. Then we have*

(4) $$\deg_\Omega(gh) = \deg_\Omega g + \deg_\Omega h\,.$$

PROOF. Let $f=gh$. Then we have

$$2\pi^2 \deg_\Omega g = \int_{M\times S^1} \mathrm{Tr}(g^{-1}dg \wedge (g^{-1}dg)' \wedge \Omega^{m-1}$$

$$2\pi^2 \deg_\Omega h = \int_{M\times S^1} \mathrm{Tr}(h^{-1}dh \wedge (h^{-1}dh)') \wedge \Omega^{m-1}$$

$$f^{-1}df = h^{-1}g^{-1}dgh + h^{-1}dh$$

$$(f^{-1}df)' = -h^{-1}[h'h^{-1}, g^{-1}dg]h + h^{-1}(g^{-1}dg)' + (h^{-1}dh)' \quad (h^{-1}dh)' = h^{-1}d(h'h^{-1})h\,.$$

Therefore

(5) $$\begin{aligned} &2\pi^2(\deg_\Omega f - \deg_\Omega g - \deg_\Omega h) \\ &\quad = \int_{M\times S^1} \mathrm{Tr}\{g^{-1}dg \wedge [-h'h^{-1}, g^{-1}dg] + g^{-1}dg \wedge d(h'h^{-1}) \\ &\qquad + dhh^{-1} \wedge [-h'h^{-1}, g^{-1}dg] + dhh^{-1} \wedge (g^{-1}dg)'\} \wedge \Omega^{m-1}\,. \end{aligned}$$

Integrating by parts on S^1 and on M, we get

$$\int_{M\times S^1} dhh^{-1} \wedge (g^{-1}dg)' \wedge \Omega^{m-1} = \int_{M\times S^1} g^{-1}dg \wedge (dhh^{-1})' \wedge \Omega^{m-1}$$

$$=\int_{M\times S^1}\mathrm{Tr}(g^{-1}dg\wedge(d(h'h^{-1})+[h'h^{-1},dhh^{-1}]))\wedge\Omega^{m-1}$$

$$=\int_{M\times S^1}\mathrm{Tr}(-g^{-1}dg\wedge g^{-1}dgh'h^{-1}+g^{-1}dg\wedge[h'h^{-1},dhh^{-1}])\wedge\Omega^{m-1}\,.$$

Integrating by parts on M also the second term in (5) we easily get, by elementary algebraic manipulations, the result:

$$\deg_\Omega f-\deg_\Omega g-\deg_\Omega h=0\,.$$

□

(ii) We observe the following.

LEMMA 3.4. *Let $\boldsymbol{p}\subset M\times\boldsymbol{C}^N$ be a vector subbundle of constant rank, and let p be the associated projection. Then, for any real closed $2m-2$ form Ω on M, we have*

$$\deg_\Omega(p+\lambda p^\perp)=\deg_\Omega\boldsymbol{p}\,. \tag{6}$$

PROOF. The proof is a quite simple and purely algebraic computation. We endowe $\boldsymbol{p}$ with the connection induced by the trivial connection on $M\times\boldsymbol{C}^N$, by Hermitian projection, and we compute its curvature, as in [OV §5]. Then we use Chern-Weil formulas for characteristic classes to evaluate the right hand side. □

Lemmas 3.3 and 3.4 prove Proposition 3.2 in the case when f and p are smooth on M, and the uniton is based at $\alpha=0$. We must now allow any α, and singularities in p, f.

(iii) We could prove the result for any α, by a direct argument, but we would get involved in messy computations. We prefer to use another argument, using a group action. We define the group

$\mathcal{G}=\{$biholomorphisms of the disc D which extend smoothly, together with their inverses, to orientation preserving diffeomorphisms of the circle, sending 1 to 1$\}$.

By a standard "reflection across the boundary" argument $\mathcal{G}$ is a subgroup of the group of automorphisms of the Riemann sphere $SL(2,\boldsymbol{C})/\{+I,-I\}$. More precisely, $\mathcal{G}$ is made precisely of all rational maps of type $\lambda\mapsto\xi_\alpha(\lambda)$ with $\alpha\in D$, considered in §1. The group $\mathcal{G}$ acts transitively on D, because $\xi_\alpha(\alpha)=0$, for each $\alpha\in D$.

We remark that, by identifying the disc with the Poincare' half-plane, $\mathcal{G}$ gets identified with the group of affinities of the real line.

The group $\mathcal{G}$ acts on $\Omega U(N)$ by change of the loop parameter. This action is symplectic, because it preserves the 2-form $S^\sim$ on the Lie algebra $\Omega\mathrm{u}(N)$. Moreover $\mathcal{G}$ acts by holomorphic transformations on $\Omega U(N)$; therefore $\mathcal{G}$ also acts on the space of rational maps $M\to\Omega U(N)$, for any M, preserving the 2-dimensional degree.

This proves Proposition 3.2 for any α. Indeed, acting by a change of parameter in $\mathscr{G}$, we can suppose the uniton $\boldsymbol{p}$ is based at $\alpha=0$; and so we are in the case previously considered.

(iv) Suppose now f, and p are smooth only outside analytic subsets of codimension ≥ 2. By the arguments of some pages ago, we may suppose there exists a resolution (M^*, τ) of both f and p. The map $f^{\sim}=f(p+\xi_\alpha p^\perp)$ will also be smooth, when defined on M^*. But the degrees of f (and $f^{\sim}$) agree when computed on M or on M^*, by Proposition 2.2. Moreover the degree of $\boldsymbol{p}$, as a vector bundle η^* on M^*, agrees with its degree, when considered as a coherent sheaf η over M. Indeed, let $\mathscr{S}$ be the subset of M where p and f are not smooth. Let $c_1(\eta, \nabla_{A(\lambda)})$ be the 2-form on $M-\mathscr{S}$ obtained by applying the Chern-Weil formulas to the connection induced by $A(\lambda)$ on η. Similarly, we define $c_1(\eta^*, \nabla_{A^*(\lambda)})$. We have

$$\deg_{\tau^*\Omega}(\eta^*)=\int_{M^*} c_1(\eta^*, \nabla_{A^*(\lambda)}) \wedge (\tau^*\Omega)^{m-1}=\int_{M-\mathscr{S}} c_1(\eta, \nabla_{A(\lambda)}) \wedge \Omega^{m-1}=\deg_\Omega(\eta)\,;$$

(the first equality is by definition, the second is obvious, and the third may be proved by considering determinant line bundles, and using the fact that $\mathscr{S}$ has complex codimension ≥ 2: cf. [OV], [K]). $\square$

REMARKS. A different proof, in the case M Riemann surface, and $\alpha=0$, is in [V2]. It uses Fourier series expansion.

The use of unbased maps into Grassmannians to parametrize (based, cf. §6) maps into $\Omega U(N)$ is just a generalization of the well-known J-homomorphism in topology (see [BM]).

4. The factorization theorem. As in the case of the extended solutions studied by Uhlenbeck, adding unitons produces all holomorphic maps $M \to \Omega U(N)$.

THEOREM 4.1. *Let M be a compact complex manifold. Let $f\colon M \to \Omega U(N)$ be a rational map into the real analytic loop group. Then f is obtained from a constant loop $Q(\lambda) \in \Omega U(N)$ by a finite number of additions of rational unitons. More precisely, f is a Blaschke product of a constant loop Q and of a finite number of rational unitons*

$$f=Q(p_1+\xi_{\beta_1}p_1^\perp)(p_2+\xi_{\beta_2}p_2^\perp)\dots(p_K+\xi_{\beta_K}p_K^\perp)\,. \tag{1}$$

Moreover, there exists a resolution M^* of M such that each p_i is smooth on M^*; and, if M is Moishezon, then $K \leq \deg f$.

REMARK. By the Atiyah-Donaldson theorem quoted in the introduction, holomorphic maps $S^2 \to \Omega SU(N)$ correspond to Yang-Mills $SU(N)$-instantons; since maps into flag manifolds (in particular, Grassmannians) correspond to monopoles, we can heuristically rephrase Theorem 4.1 as: *instantons are product of monopoles.*

NOTE. In most of the following, we will assume M to be a compact Riemann surface. If M has greater dimension, one has to consider resolutions, as usual, in order to prove similar statements.

Let M be a compact Riemann surface; and let $f: M \to \Omega U(N)$ be a holomorphic map. Let $A(\lambda)$, for λ in D be the holomorphic extension of $f^{-1}\partial'' f$ from S^1 to D. An *expansion* of f is a smooth map $G: D \times M \to \mathrm{gl}(N, \boldsymbol{C})$, holomorphic in the loop variable λ, such that

(i) $\partial'' G + A(\lambda) G = 0$ for $(\lambda, z) \in D \times M$;

(ii) There exist $\alpha_1 \dots \alpha_n \in D$ so that $G_{|M \times (D - \{\alpha_1 \dots \alpha_n\})}$ is invertible;

(iii) G extends smoothly to the closure of the disc, and it satisfies $G(1, z) = I$;

(iv) $G = f^{-1} k$ on the circle, where $k \in LGL(N, \boldsymbol{C})$.

A *unitary expansion* is an expansion which is unitary for $\lambda \in S^1$.

We may introduce a partial ordering among expansions of f as follows: $G_1 \le G_2$ if $G_1 = G_2 q(\lambda)$, with $q: D \to \mathrm{gl}(N, \boldsymbol{C})$ holomorphic, smooth up to the boundary, (and invertible except at a finite number of points, since G_1, G_2 are). An expansion is said to be *maximal*, if it is maximal with respect to $\le$. We remark that maximal expansions are defined modulo right multiplication by elements q in $L^+GL(N, \boldsymbol{C})$ satisfying $q(1) = I$. In particular, the points $\alpha_1 \dots \alpha_n$ are uniquely determined by f.

The following is a key lemma. It is an analogue of Theorem 13.2 in [U].

LEMMA 4.2. *Let M be a compact Riemann surface; and let $f: M \to \Omega U(N)$ be a holomorphic map. The there exists a unique unitary maximal expansion of f.*

PROOF. Uniqueness: let G_1, G_2 be maximal unitary expansions of f. Then $G_1 = G_2 q$, with $q \in \Omega U(N)$. Since both G_1, G_2 are maximal, q and q^{-1} are both holomorphic and invertible in D. Therefore $q \in L^+GL(N, \boldsymbol{C}) \cap \Omega U(N) = (I)$.

Existence. Let η be the holomorphic vector bundle on $D \times M$ induced by the ∂''-operator

$$(\partial''_\lambda + \partial''_{A(\lambda)}) = \nabla \quad \text{on} \quad D \times M \times \boldsymbol{C}^N .$$

We have indeed: $(\nabla)^2 = 0$, because $A(\lambda)$ is holomorphic in λ. Therefore we may apply the Koszul-Malgrange theorem 2.3.

The bundle η extends to a holomorphic vector bundle on $D_\varepsilon \times M$, (where $D_\varepsilon = \{\lambda \in \boldsymbol{C} \mid |\lambda| < 1 + \varepsilon\}$, $\varepsilon > 0$), because A is real analytic on the circle.

The projection $\pi: D_\varepsilon \times M \to D_\varepsilon$ is holomorphic and proper. By Grauert's direct image theorem, $\nu = \pi_* \eta$ is a (torsion-free) coherent sheaf over D_ε; therefore it is locally free, because the disc has complex dimension 1. But every holomorphic vector bundle over the disc is holomorphically trivial, i.e. it is globally free. Moreover ν must have rank N, because it has rank N in a neighbourhood of the circle, where a local holomorphic extension of f gives a trivialization.

Therefore ν has N spanning holomorphic sections $v_1 \dots v_N$; they satisfy $\partial'' v_i / \partial'' \lambda = 0$, $\partial''_{A(\lambda)} v_i = 0$. Let K be the matrix-valued function which has $v_1 \dots v_N$ as column vectors. Of course K satisfies equation (i) above.

Det K is a holomorphic section of Det(η), the determinant bundle of η; therefore its zero locus $T=\{(\lambda, z)\in D_\varepsilon\times M \mid \det(K)(\lambda, z)=0\}$ is an analytic subset of $D_\varepsilon\times M$.

Let us fix $\mu\in D$. We claim that $T\cap(\{\lambda=\mu\})\times M$ is either empty or all of $\{\mu\}\times M$. Indeed, the restriction of $\det K$ is either identically 0 or nowhere vanishing, because it is a holomorphic section of the topologically trivial line bundle $\mathrm{Det}(\eta)_{|\{\mu\}\times M}$. Therefore, if $(\mu, z)\in T$, then $(\mu, w)\in T$ for any $w\in M$. The set T is a closed analytic subset of $D_\varepsilon\times M$, therefore its intersection with the closure of $D\times M$ is of the form $\{\alpha_1\dots\alpha_n\}\times M$. On S^1 we have, by a trivial computation, $\partial''(fK)=0$, which in turn implies (iv). If we set $\lambda=1$, $\partial''K(1, z)=0$, therefore $K(1, z)$ is independent of z.

Our construction does not guarantee that G is unitary on the unit circle. We have anyway, for $\lambda\in S^1$

$$\begin{aligned}\partial''K+A(\lambda)K=0 &\Leftrightarrow \partial''(fK)=0\Leftrightarrow \partial'(fK)=0\Leftrightarrow \partial'K+f^{-1}\partial'fK=0\\ &\Leftrightarrow \partial''K^*-K^*f^{-1}\partial''f=0\,.\end{aligned}$$

Therefore we have

$$\partial''(K^*K)=\partial''(K^*)K+K^*\partial''K=K^*A(\lambda)K-K^*A(\lambda)K=0$$

which implies that K^*K is constant on M, for any fixed $\lambda\in S^1$.

We fix a point $p\in M$: then $K(\lambda, p)\in LGL(N, \boldsymbol{C})$ has a (unique) factorization $K(\lambda, p)=\rho\sigma$, with $\rho\in\Omega U(N)$ and $\sigma\in L^+GL(N, \boldsymbol{C})$ (cf. Theorem 8.1.1 in [PS], or §2 of this paper). Let $G(\lambda, z)=K(\lambda, z)\sigma^{-1}$. $G(\lambda, z)$ is also a maximal expansion of f, because $\sigma\in L^+GL(N, \boldsymbol{C})$. Moreover, we have $G^*G=(\sigma^{-1})^*K(\lambda, z)^*K(\lambda, z)\sigma^{-1}$, and G^*G is a constant function of z, for $|\lambda|=1$. Evaluating at $z=p$ we get

$$(G^*G)(\lambda, z)=(\sigma^{-1})^*K(\lambda, p)^*K(\lambda, p)\sigma^{-1}=\rho^*\rho=I\,.$$

Therefore G is a maximal expansion of f, unitary for $\lambda\in S^1$. □

REMARKS. The map G is a sort of holomorphic "extension" of the map f^{-1} from S^1 to the disc D. Since the word "extension" is overused in mathematics, we prefer to call the map G expansion. Lemma 4.2 is an analogue of similar normalization conditions in [U] and [Seg]. The inverse of the unique maximal expansion gives a canonical choice between all left translates of the given rational map.

PROOF OF THEOREM 4.1. The proof of Theorem 4.1 is quite similar to the proof of the analogous factorization theorem in [U], and to the proof of Lemma 1.2. As usual, we suppose that M is a compact Riemann surface.

Let G be a maximal unitary expansion of the holomorphic map f. Fix an order for the α_i's. We denote, for a fixed $\alpha\in D$, the function $G(\alpha, z)$ on M by $G(\alpha)$.

From the definition of G, we have

$$\partial''G(\alpha_i)+A(\alpha_i)G(\alpha_i)=0 \qquad \text{for each} \quad i\,.$$

Therefore $G(\alpha_i)$ defines a (non-zero, because of maximality) holomorphic map between holomorphic vector bundles over M

$$G(\alpha_i)\colon (\underline{C}^N, \partial'') \to (\underline{C}^N, \partial''_{A(\alpha_i)})\,. \tag{2}$$

Outside a finite number of points in M, $G(\alpha_i)$ has constant rank $r<N$. Let us choose $i=1$. Let $\boldsymbol{p}$ be the rank r subbundle of $(\underline{C}^N, \partial''_{A(\alpha_1)})$ obtained by taking $\operatorname{Im} G(\alpha_1)$ outside this finite set of points, and "filling out zeroes".

By (2), $\boldsymbol{p}$ is a uniton for f, based at α_1. Let us add $\boldsymbol{p}$ to f. We get a new holomorphic map $f^\sim = f(p+\xi_{\alpha_1}p^\perp)\colon M \to \Omega U(N)$. Now $f^\sim$ has an expansion $G^\sim = (p+(\xi_{\alpha_1})^{-1}p^\perp)G$; we will see in §5 that $G^\sim$ is also maximal. We have

$$\det G^\sim = (\xi_{\alpha_1})^{-N+\mathrm{rk}(p)} \det G\,. \tag{3}$$

Therefore $\det G^\sim$ has an order of zero at α_1 which is strictly less than the order of zero of $\det G$.

We repeat the procedure at α_1, adding another uniton. By the above argument, after a finite number of times, we finish up with a G which is invertible at α_1. Moreover, the rank of the new G has not changed, for $\lambda \neq \alpha_1$.

Repeating the procedure at α_2, then at $\alpha_3, \ldots$ then at α_n, we finish up with a G' which is invertible everywhere, and unitary for $|\lambda|=1$. But $\Omega U(N) \cap L^+GL(N, C) = (I)$ (the constant loop). Therefore $G'(\lambda, z) \equiv I$.

The last statement may be proved by elementary algebraic geometry: arguing as in [OV, Lemma 6.6], we see that each image bundle of the maximal unitary expansion has positive integral first Chern class, by using Lemma 5.2. Therefore, at each step of the factorization above, the degree of the holomorphic map decreases by a positive integer. Since the degree is not negative, it follows that the number of steps, i.e. of unitons, is bounded above by the degree of the map □

REMARK. A possible alternative proof, avoiding the use of theorem in [PS], is the following. Suppose that M is a Moishezon manifold. We can take a maximal expansion G of f, and factor out unitons, as above. After a finite number of steps, we reach a holomorphic map g and an expansion G' of g, which is invertible everywhere inside the disc; and we must prove that g is constant. For λ on the unit circle we have: $(G')^{-1} = q(\lambda)g$, with a based loop q in $GL(N, C)$. Since the inclusion $U(N) \to GL(N, C)$ is a deformation retract, we can compute the degrees of loops in $GL(N, C)$ using Maurer-Cartan forms. We get

$$\deg g = \deg(G')^{-1} + \deg q = -\deg G' + \deg q = 0 + 0 = 0\,.$$

Indeed, we have $\deg G' = 0$, because G is extendable to a map defined on the disc (cf. the proof of Proposition 4.3 (ii)); and $\deg q = 0$ because q is independent of the variable in M. Therefore the map g is holomorphic of degree 0; it is therefore constant, by Proposition 2.2 (iii).

REMARK. We can extend Theorem 4.1 to the case of holomorphic maps f into the smooth loop group $\Omega_{\mathrm{sm}}U(N)$. The key point is to take as η, in the proof of Lemma 4.2, the restriction of the bundle on $S^2 \times M$ defined by f (see [PS], Theorem 8.10.2) to a neighbourhood of $\bar{D} \times M$. We do not know if our proof can be extended to the case of holomorphic maps into the Hilbert loop group, quoted in §1: maybe the key point would be a Koszul-Malgrange theorem for Sobolev connections, and the analysis of families of ∂''-operators, and of their determinants (cf. [Q], [GMS]).

We have therefore given a canonical decomposition of any given holomorphic map $f\colon M \to \Omega U(N)$ as a product of unitons. The only thing that was not canonical in our construction was the order of the points α's. A different order produces a different factorization.

The α's are canonically and uniquely defined by f. Indeed the α's are the zeroes of the determinant of a maximal expansion of f; moreover, it is easy to see that they do not depend on the choice of the maximal expansion. We call the α's the *poles* of f. Moreover, they have a natural notion of multiplicity, defined as follows.

PROPOSITION 4.3. *Let $f\colon M \to \Omega U(N)$ be a holomorphic map, with M compact. Then we have the following.*

(i) *The poles of f are uniquely defined by f.*

(ii) *Let $f = Qf_{\alpha_1} \dots f_{\alpha_k}$ be a factorization with $Q \in \Omega U(N)$, $f_{\alpha_i}\colon M \to \Omega^{\{f_{\alpha_i}\}}$. Then $d_i = \deg f_{\alpha_i}$ is uniquely defined by f, independently of a choice of an order of the poles α's. When have*:

$$\sum_i d_i = \deg f\,. \tag{4}$$

NOTE. We call d_i the *degree of f at α_i*. These are not topological invariants, since they are not stable under small deformations of f.

PROOF. As usual, we suppose that M is a compact Riemann surface, for simplicity of exposition. (i) has just been observed. We prove (ii).

The definition of degree for a map $f\colon M \to \Omega U(N)$ given in §2 was essentially the following. Let $\theta = g^{-1}dg$ be the Maurer-Cartan form on $U(N)$. Then the 3-form $\phi = \mathrm{Tr}(\theta \wedge \theta \wedge \theta)$ represents, properly normalized, the positive generator of $H^3(U(N), \boldsymbol{Z}) = \boldsymbol{Z}$. Let ρ be the normalization of ϕ. If $f\colon M \times S^1 \to U(N)$ is a smooth map, then $f^*\rho = \omega^{2,1} + \omega^{3,0}$, where the indexes refer to the cartesian product $M \times S^1$. We have

$$d_{M \times S^1} f^*\rho = 0\,.$$

Therefore

$$d_M \omega^{2,1} + d_{S^1}\omega^{3,0} = 0\,, \quad d_{S^1}\omega^{2,1} = 0\,,$$

with the obvious notation. Integrating $\omega^{2,1}$ over S^1, we get a closed integral 2-form on M, which is, by definition, the degree of f (as a cohomology class: if we want a number, we must integrate over M). Since the inclusion $U(N) \to GL(N, \boldsymbol{C})$ is a deformation retract, we could take f with values in $GL(N, \boldsymbol{C})$, and nothing would change.

If f is smoothly extendible to a map $g: M \times D \to GL(N, \boldsymbol{C})$, then we have $g^*\rho = \psi^{2,1} + \psi^{1,2} + \psi^{3,0}$ on D; and $d_M\psi^{2,1} + d_D\psi^{3,0} = 0$, $d_D\psi^{2,1} + d_M\psi^{1,2} = 0$. Therefore

$$\begin{aligned}\langle [f^*\rho], [S^1]\rangle &= \langle [\psi^{2,1}], [S^1]\rangle = \langle [\psi^{2,1}], \partial[D]\rangle = \langle [d_D\psi^{2,1}], [D]\rangle \\ &= -\langle [d_M\psi^{1,2}], [D]\rangle = -d_M(\langle [\psi^{1,2}], [D]\rangle) = 0 \quad \text{in } H^2(M, \boldsymbol{Z}).\end{aligned}$$

Let $f: M \to \Omega U(N)$ be a holomorphic map. Then, using a maximal expansion $G(\lambda, z)$, we see that the degree of f is "concentrated" at the points were G is not invertible, i.e. at the "poles" of f. More precisely, if $f = Qf_{\alpha_1} \ldots f_{\alpha_k}$, then we have $\deg f = \sum_i \deg f_{\alpha_i}$. Now we deform S^1 as a sum of a finite number of loops $\gamma_1 \ldots \gamma_k$, each one winding once around the respective α_i; we have

$$\deg f_{\alpha_i \mid \gamma_j \times M} = 0 \quad \text{if } i \neq j, \quad \text{and} \quad \deg f_{\mid \gamma_i \times M} = \deg f_{\alpha_i \mid \gamma_i \times M} = \deg f_{\alpha_i}.$$

If we take another maximal expansion G', we have $G' = Gq(\lambda)$ with q a constant loop, invertible on the disc. By the above discussion, this does not change the degrees. $\square$

The poles of a based holomorphic function $f: M \to \Omega U(N)$ have also an algebro-geometric interpretation in terms of generalized "jumping lines". Let $p \in M$, and suppose $f(p) = I$. Let G be the maximal unitary expansion of f. We have $f = G(\lambda, p)G(\lambda, z)^{-1}$. Let η be the holomorphic vector bundle on $D \times M$ constructed in the proof of Lemma 4.2. We can attach η to the trivial bundle on $(S^2 - D) \times M$, using f as the transition function. In this way we get a $\boldsymbol{C}^N$-bundle $E \to S^2 \times M$, with a canonical trivialization on $(S^2 - D) \times M$ and $S^2 \times \{p\}$. We call *E the bundle associated to f*; it coincides with the bundle associated to f considered in [A]. A canonical set of N meromorphic sections of E is given by G itself (on $D \times M$) and by the identity matrix (on $(S^2 - D) \times M$). Generalizing the algebro-geometric notion of jumping lines, we define a "jumping manifold" of E, to be a submanifold $N \subset S^2 \times M$ such that $E_{\mid N}$ is not trivial.

PROPOSITION 4.4. *The poles α_i of the based holomorphic map $f: M \to \Omega U(N)$ are the "vertical" jumping manifolds $\{\alpha_i\} \times M$ of the bundle E associated to f.*

PROOF. The only possible vertical jumping manifolds are the α_i's, since we have just remarked that E is trivialized by N meromorphic sections outside $\{\alpha_i\} \times M$. Moreover, semistability is an open condition for families of holomorphic vector bundles; therefore, by using Lemma 4.5 below, we can easily see that, if $\{\alpha\} \times M$ is not a jumping manifolds, a maximal expansion of f should be invertible on $\{\alpha\} \times M$, by construction. $\square$

REMARK. Our proof of Theorem 4.1 is a generalization of Uhlenbeck's proof of the factorization theorem for extended solutions, together with the proof of the Blaschke product factorization. This will perhaps become clearer in §5, when we will give a unicity result for the factorization. It is possible to prove analogous statements, using Quillen's "Grassmannian model" of $\Omega U(N)$, in the style of [Seg]. This was shown to me by F.

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It is also possible to generalize instead the proof in [V1] of Uhlenbeck's factorization, arguing as follows. By Proposition 2.1, it is not really a restrictive assumption to take M a complex projective manifold. Let $f\colon M \to \Omega U(N)$ be a holomorphic map. Then, for λ near S^1, the holomorphic vector bundle $(\underline{C}^N, \partial''_{A(\lambda)})$ is holomorphically trivial. Suppose that there exists λ_0, such that the bundle $(\underline{C}^N, \partial''_{A(\lambda_0)})$ is not semistable, in the sense of algebraic geometry (cf. [K]). Then, using Proposition 3.2, we see that it is possible to decrease the degree of f by an integer, by adding a uniton based at λ_0. Since the degree is finite, and everything is integral, after repeating the procedure a finite number of times, we finally get a new holomorphic map $f^{\sim}$, such that $(\underline{C}^N, \partial''_{A^{\sim}(\lambda_0)}$ is semistable. We repeat the procedure at the other points of the disc. Moreover, we do not pass back through the previous λ's, because adding a uniton at λ_0 acts by a gauge transformation on $\partial''_{A(\lambda)}$, for $\lambda \neq \lambda_0$, hence preserving semistability and triviality.

We finally get be a holomorphic family $\{E_\lambda\}$ of semistable holomorphic vector bundles over M, parametrized by the disc. Moreover, $\{E_\lambda\}$ is a holomorphically trivial family of holomorphically trivial vector bundles in a neighbourhood of S^1. Then, if we know that this implies that $\{E_\lambda\}$ is a trivial family of trivial vector bundles, we have finished, since this would imply the existence of an invertible expansion of f. But this is a consequence of the following lemma, which was showed to us by Professor Masaki Maruyama.

LEMMA 4.5. *Let $E \to M$ be a holomorphic vector bundle of rank N and the first Chern class* 0 *over a compact projective manifold M. Suppose E is semistable, and* $\dim H^0(E) \geq N$. *Then E is trivial.*

PROOF. Let $\mathcal{O}$ be the sheaf of holomorphic functions on M. Let us consider the natural map, given by multiplication, $g\colon H^0(E) \otimes \mathcal{O} \to E$; and the quotient map $\pi\colon E \to E/\mathrm{Im}(g)$. Let $T \subset E/\mathrm{Im}(g)$ be the torsion part; and let $\pi^{-1}(T) = F$. The map g induces a (generically surjective) map

$$\alpha\colon (\mathcal{O})^s \to F \qquad s = \mathrm{rank}(F)\,.$$

If the complex codimension of the supporting set $\mathrm{Supp}(F/\mathrm{Im}(\alpha))$ is 1, then $c_1(F) > 0$; but this is not possible, because E is semistable. Therefore we have: $\mathrm{codim}\,\mathrm{Supp}(F/\mathrm{Im}(\alpha)) \geq 2$. The quotient sheaf E/F is torsion-free, and therefore F is reflexive. There is an injective map $(\mathcal{O})^s \to F$, and F is locally free on an open set $U \subset M$, with $\mathrm{codim}(M-U) \geq 2$. Let $j\colon U \to M$. We have $(\mathcal{O})^s \cong F_{|U}$; but F is reflexive, and therefore normal. Therefore

$$(\mathcal{O})^s = j_*(\mathcal{O}^s{}_{|U}) \cong j_*(F_{|U}) = F\,.$$

The map g factors through F: $g\colon H^0(E) \otimes \mathcal{O} \twoheadrightarrow F = (\mathcal{O})^s \to E$. At the level of zeroth cohomology, we have

$$\begin{tikzcd} H^0(H^0(E)\otimes\mathcal{O})\cong H^0(E) \arrow[rr, "i"] \arrow[dr] & & H^0(E)\cong \boldsymbol{C}^N . \\ & H^0(F)\cong \boldsymbol{C}^s \arrow[ur] & \end{tikzcd}$$

Therefore $s=N$, since i is an isomorphism; and $F=E$, since they are free sheaves of the same rank.

REMARK. In the case $M=S^2$, the proof is much simpler, because the unique semistable vector bundle with zero first Chern class over the Riemann sphere is trivial.

COROLLARY 4.7. *Let M be a compact complex manifold, and let $f: M\to\Omega U(N)$ be a rational map, smooth and holomorphic on $M-\mathscr{S}$. Then f is real analytic on $M-\mathscr{S}$. Moreover, let $x\in M$, and suppose $f(x)=I$. Then f is rational in the loop variable λ, for each $z\in M$.*

PROOF OF 4.7. We decompose f as a product of unitons $\boldsymbol{p}_i$'s. Let (M^*, τ) be a resolution of M such that each uniton $\boldsymbol{p}_i$ is smooth on M^*. Using induction, we easily see that each $\boldsymbol{p}_i$ is real analytic, because it is the solution of an elliptic equation with real analytic coefficients. The second statement is a direct consequence of Theorem 4.1. Its first proof appeared in [Seg]. □

REMARK. It is proved in [V3] that if f is holomorphic, and M is a compact projective manifold, then f is real algebraic.

5. Uniqueness theorems for the factorization. Theorem 4.1 gives a canonical decomposition of any rational map f from a compact manifold M into the loop group $\Omega U(N)$. The main idea of the proof, in its different versions, was, starting from f, to "factor out" canonically chosen unitons, so as to decrease the "complexity" of f.

Of course the factorization one finds in this way depends on the order of the "poles" α_j; moreover it is not unique, since no recipe for constructing uniquely maps from simpler ones was given. The first problem may be overtaken by the use of the * product of [V3], briefly described in §2; while the second problem leads us to a generalization of Uhlenbeck's proof of the unique factorization theorem for extended solutions in [U]. We shall arrive at a unique factorization theorem (Theorem 5.1).

REMARKS. Let M be a compact connected complex manifold, and let $p\in M$. By the arguments in §4, the space of based (i.e. $f(p)=I$) rational maps f: $M\to\Omega SU(N)$ is in natural one-to-one correspondence with the space of corresponding maximal unitary expansions; or, equivalently, with the space of unbased rational maps $M\to\Omega U(N)$, modulo left multiplication by constant loops. This is because any rational map $M\to\Omega U(N)$ must have constant determinant, because any rational map $M\to\Omega U(1)$ is constant. This will be implicitly used in the following, when we will switch to the definition occasionally more convenient (see also §6, with the description of the moduli space of $SU(2)$-instantons).

In the rest of this section, we will suppose M to be a compact Riemann surface. In the general case one would have to consider resolutions of M and f, but the statements and the proofs would be essentially unchanged.

Let $f\colon M \to \Omega U(N)$ be a holomorphic map; and let G be a maximal expansion of f. Let $\alpha_1 \dots \alpha_n \in D$ be the poles of f. We define the *uniton number k_i of f at α_i* as the order of the pole of G, at $1/\bar{\alpha}_i$. (By order of the pole at one point, of a meromorphic matrix-valued function, we mean the maximum pole order of its entries, at the point). In other words, k_i is the smallest positive integer such that $(\xi_{\alpha_i})^{-k_i}G$ is holomorphic at $1/\bar{\alpha}$. This notion generalizes Uhlenbeck's notion of uniton number; but Uhlenbeck's extended solutions have non-zero uniton number only at $\alpha = 0$.

We call the *total uniton number K* of f the sum of the uniton numbers k_i of f at its poles. We also define the *McMillan degree r_i of f at α_i* (cf. [G], [MH]) as the order of zero of the determinant of G at α_i; and the total McMillan degree R of f as the sum of the McMillan degrees r_i of f at its poles. We will see that, if $N=2$, we have $r_i = k_i$.

The following is the main result of this section; an equivalent version is Theorem 5.6.

THEOREM 5.1. *Let $f\colon M \to \Omega U(N)$ be a holomorphic map, with M compact. Then there exists a unique $Q \in \Omega U(N)$, unique points $\alpha_1 \dots \alpha_n \in D$, and unique holomorphic maps $f_i\colon M \to \Omega^{\{\alpha_i\}}$, satisfying the following.*

(1) (i) $f = Q(f_1 * \cdots * f_n)$.

(ii) *The maps f_i are products of k_i unitons based at α_i:*

$$f_i = (p_{1,i} + \xi_{\alpha_i} p_{1,i}^{\perp})(p_{2,i} + \xi_{\alpha_i} p_{2,i}^{\perp}) \dots (p_{K_i,i} + \xi_{\alpha_i} p_{K_i,i}^{\perp}) \,. \tag{2}$$

(iii) $p_{h+1}^{\perp}(p_{h,i}^{\perp}) = p_{h+1,i}^{\perp}$ *as holomorphic vector bundles, for any h, i.*

(iv) *The holomorphic subbundle* $\mathrm{Ker}(p_{h,i}^{\perp} p_{h-1,i}^{\perp} \cdots p_{2,i}^{\perp} p_{1,i}^{\perp}) \subset M \times \boldsymbol{C}^N$ *has no holomorphic sections, for any h, i.*

Moreover we have:

1) *the above factorization coincides with the one produced in the proof of Theorem 4.1; and k_i coincides with the uniton number of f at α_i;*

2) $\mathrm{rk}(p_{h,i}) \leq \mathrm{rk}(p_{h+1,i})$ *for any h, i.*

3) $\deg(p_{h,i}) < 0$ *for any h, i.* □

REMARK. Arguing by induction on h, it is easy to show that (iii) is equivalent to:

(iii)′ The image bundle $\mathrm{Im}(p_{h,i}^{\perp} p_{h-1,i}^{\perp} \cdots p_{2,i}^{\perp} p_{1,i}^{\perp})$ coincides with $p_{h,i}^{\perp}$, for any h, i.

The idea of the proof of Theorem 5.1 is simple; given a factorization of a maximal expansion of f into unitons, we look for conditions on the unitons to ensure that the factorization coincides with the one constructed in the proof of Theorem 4.1.

As the first step, we need to study a bit more the notion of "maximal expansion" of a given holomorphic map f. Remember that G was essentially a holomorphic extension of f^{-1} from S^1 to the disc. The following lemma relates our notion of maximal expansion with Uhlenbeck's normalization condition for extended solutions.

LEMMA 5.2. *Let f: $M \to \Omega U(N)$ be a holomorphic map. Let H be an expansion of f. Then H is a maximal expansion if and only if for any vector subspace $V \subset C^N$, and for any $\lambda_0 \in D$, $H(\lambda_0, z)V$ is not identically* 0.

PROOF. Suppose there exists a V and a λ_0 satisfying $H(\lambda_0, z)V=0$ for each z. Let p be the Hermitian projection onto V, and $p^\perp = I-p$. Then $K(\lambda, z)=H(\lambda, z)(p+(\xi_{\lambda_0})^{-1}p^\perp)$ has no pole at $\lambda=\lambda_0$, and we have $K>H$. Therefore H is not maximal.

Conversely, suppose H is not maximal; let G be a maximal expansion of f. Then there exists a loop $q(\lambda)$, holomorphic in λ, matrix-valued, satisfying

$$Gq(\lambda)=H\,. \tag{3}$$

If q is everywhere invertible, the H is maximal; otherwise there exists $\lambda_0 \in D$ so that $\det(q(\lambda_0))=0$. Then q^{-1} is a meromorphic matrix-valued function on D, with poles of finite order, less than or equal to the order of zero of $\det q$ at the same points. Around $\lambda=\lambda_0$ we have

$$\begin{aligned} q^{-1}(\lambda)&=(\lambda-\lambda_0)^{-j}(q_0+(\lambda-\lambda_0)q_1(\lambda, z)) \qquad j>0\\ G(\lambda, z)&=(\lambda-\lambda_0)^k(G_0+(\lambda-\lambda_0)G_1(\lambda, z))\\ H(\lambda, z)&=(\lambda-\lambda_0)^h(H_0+(\lambda-\lambda_0)H_1(\lambda, z)) \end{aligned}$$

with $q_0(z)$, $G_0(z)$, $H_0(z)$ not identically 0.

If $h>0$ the thesis is obvious. Suppose then $h=0$. We must also have $k=0$, for otherwise G is not maximal. Therefore we get from (3):

$$(\lambda-\lambda_0)^{-j}H_0(z)q_0(z)=0\,.$$

Let V be the image bundle of q_0. Then $H(\lambda_0, z)V$ is identically 0. □

The condition of Lemma 5.2 may be rephrased as follows. Remember that by (4.2), for any $\lambda_0 \in D$, $\operatorname{Ker} G(\lambda_0, z) \subset M \times C^N$ is a holomorphic subbundle, with respect to the standard complex structure.

LEMMA 5.3. *Let f: $M^2 \to \Omega U(N)$ be a holomorphic map. Let H be an expansion of f. Then H is a maximal expansion if and only if, for any $\lambda_0 \in D$, the holomorphic subbundle* $\operatorname{Ker} H(\lambda_0, z) \subset M \times C^N$ *does not have holomorphic sections.*

If $N=2$ this is equivalent to $\operatorname{Ker} H(\lambda_0, z)$ *not being a holomorphically trivial subbundle of $M \times C^2$.*

PROOF. The holomorphic vector bundle $M \times C^N$ does not have non-constant holomorphic sections. □

Let f: $M \to \Omega U(N)$ be a rational map, and let G be its maximal unitary expansion.

Then G^{-1} has poles for $\lambda=\alpha_i$, $i=1\dots n$ (where α_i are the poles of f). The order of these poles is equal to the uniton number of f at the respective α_i, because G is unitary on S^1. Therefore the function $\prod(\xi_{\alpha_i})^{k_i}G^{-1}=H$ is defined and holomorphic inside the disc; the k_i's are the uniton numbers of f at α_i's. Then, for each fixed α in the disc, $H(\alpha, z)$ defines a holomorphic map of holomorphic vector bundles over M

$$H\colon(\underline{C}^N, \partial''_{A(\alpha)})\to(\underline{C}^N, \partial'') . \tag{4}$$

In particular, the kernel bundle of H is a uniton for f, based at α. It is easy to prove that we have: $\operatorname{Im} G(\alpha, z)\subset\operatorname{Ker} H(\alpha, z)$. (We could have used $\operatorname{Ker} H(\alpha, z)$ all along in our proofs; this would have been more similar to the approach in [U], but the factorization would have been different). By a slight abuse of language, by $\operatorname{Ker} G^{-1}(\alpha)$ we denote $\operatorname{Ker} H(\alpha, z)$; and let $G(\alpha)$ be the function $G(\alpha, z)$ on M.

LEMMA 5.4. *Let f be a holomorphic map $M\to\Omega U(N)$, with M compact. Suppose f has the uniton number n at a point $\alpha\in D$. Let G be the maximal unitary expansion of f. Let $\boldsymbol{p}$ be a uniton for f, based at α. Let $f^{\sim}=f(p+\xi_\alpha p^\perp)$ be the rational map obtained by addition of the uniton $\boldsymbol{p}$ to f. Let $G^{\sim}$ be the maximal unitary expansion of $f^{\sim}$. Then we have the following.*

(i) $\operatorname{Im} G(\alpha)\subset\boldsymbol{p}$ *if and only if* $G^{\sim}(\lambda)=(\xi_\alpha^{-1}p^\perp+p)G(\lambda)$.

(ii) $f^{\sim}$ *has uniton number* $n-1$ *at* α, *and* $G^{\sim}(\lambda)=(\xi_\alpha^{-1}p^\perp+p)G(\lambda)$ *if and only if* $\operatorname{Im} G(\alpha)\subset\boldsymbol{p}\subset\operatorname{Ker} G^{-1}(\alpha)$.

(iii) $f^{\sim}$ *has uniton number* $n+1$ *at* α, *and* $G^{\sim}=(p^\perp+\xi_\alpha p)G$ *if and only if the following* (A) *and* (B) *are satisfied*:

(A) *For each constant vector* $v\in\boldsymbol{C}^N$, $p^\perp G(\alpha)v\not\equiv 0$.

(B) $\operatorname{Im} G(\alpha)\not\subset\boldsymbol{p}\not\subset\operatorname{Ker} G^{-1}(\alpha)$.

(iv) *Suppose* $\boldsymbol{p}$ *satisfies*:

(A) $p^\perp(\operatorname{Im} G(\alpha))=\boldsymbol{p}^\perp$.

(B) *For each* $v\in\boldsymbol{C}^N$, $p^\perp G(\alpha)v\not\equiv 0$.

Then $G^{\sim}=(p^\perp+\xi_\alpha p)G$, *and* f *is obtained from* $f^{\sim}$ *by addition of the uniton* $\boldsymbol{p}^\perp=\operatorname{Im} G^{\sim}(\alpha)$; *in particular*, $f^{\sim}$ *has uniton number* $n+1$ *at* α.

PROOF. The proof of this lemma is tedious. The first computations of this type appeared in [U], but we cannot rely on them, since our construction is dual to Uhlenbeck's. We have

$$G^{\sim}q(\lambda)=(p^\perp+\xi_\alpha p)G . \tag{5}$$

The loop q exists because $G^{\sim}$ is a maximal expansion. Moreover, q is invertible at every point in the disc $\lambda\neq\alpha$; otherwise, evaluating (5) at $\lambda=\alpha$ would show $G(\alpha)$ has some constant kernel, and this is against the maximality of G.

(i) and (ii) are more or less obvious. One has to expand $G^{\sim}$ (or $(G^{\sim})^{-1}$) in power series of ξ_α, and check directly the uniton number. Similar computations are also in [U].

(iii) We first remark that the condition (iii) (A) is precisely the condition of

maximality for $(p^\perp+\xi_\alpha p)G$, according to Lemma 5.1.
To prove the sufficiency, we write

$$(\xi_\alpha)^n(G^\sim)^{-1}=(\xi_\alpha)^{-1}((\xi_\alpha)^n G^{-1})p+((\xi_\alpha)^n G^{-1})p^\perp . \tag{6}$$

If the left hand side is holomorphic inside the disc, so that the uniton number of $f^\sim$ at α is $\leq n$, we must have: $((\xi_\alpha)^n G^{-1})p=0$, i.e. $\boldsymbol{p}\subset \operatorname{Ker} G^{-1}(\alpha)$. Therefore the uniton number of $f^\sim$ at α is $>n$. It is easy to prove that it must be $\leq n+1$.
The converse is also easy to prove, examining (6).

In order to prove (iv) observe that condition (iv) (B) implies that $(p^\perp+\xi_\alpha p)G$ is a maximal expansion. Moreover, we have $G^\sim(\alpha)=p^\perp G(\alpha)$, therefore (iv) (A) implies that $\boldsymbol{p}^\perp=\operatorname{Im} G^\sim(\alpha)$. Applying (ii), one gets the equality for the uniton numbers. □

The following lemma reduces the study of maximal expansions to the study of maximal expansions with only one pole; or, equivalently, to the study of holomorphic maps into the space of algebraic loops.

LEMMA 5.5. *Let $f\colon M\to\Omega U(N)$ be a holomorphic map; let $\alpha_1\ldots\alpha_n$ be its poles; and let G be its maximal unitary expansion. Let $G^{-1}=(G_1)^{-1}*\cdots*(G_n)^{-1}$, with $G_i^{-1}\colon M\to\Omega^{\{\alpha_i\}}$ holomorphic, be the decomposition given by Lemma* 1.3. *Then we have the following.*

(i) *G is maximal if and only if each G_i is maximal.*

(ii) *The uniton number and the McMillan degree of f at each α_i are equal to the uniton number of G_i at α_i.*

PROOF. (i) Let $G^{-1}=H_1\ldots H_n$, with $H_i\colon M\to\Omega^{\{\alpha_i\}}$; and $H_1=G_1^{-1}$.
Then, $G=H_n^{-1}\ldots H_2^{-1}G_1$; by using Lemma 5.2, we see that G is a maximal expansion around α_1 if and only if G_1 is, because the other factors are all invertible around α_1. By permuting the poles α_i's, and repeating this argument, we get the conclusion.

(ii) This is a pointwise statement on the matrix-valued rational functions G_i's, and it may be easily deduced from the alternative definition of the * product presented in [V3], in terms of union of "zero pairs" (generalized divisors) of Gohberg's theory (cf. [G]). □

PROOF OF THEOREM 5.1. The statements are a direct consequence of Lemmas 1.2, 5.5, and 5.4 (iv). Conditions (iii) and (iv) come from 5.4 (iv), (A), (B), by induction. Part (1) is by definition, while part (2) is a trivial consequence of (iii). □

The following is a version of Theorem 5.1, if one does not want to use the * product. The proof is analogous, but we do not use Lemma 5.5.

THEOREM 5.6. *Let $f\colon M\to\Omega U(N)$ be a holomorphic map, with M compact. Suppose we choose an order for the poles $\alpha_1\ldots\alpha_n$ of f. Let k_i, $i=1\ldots n$ be the uniton numbers of f at the poles α_i. Let $K=\sum_{1\leq i\leq n}k_i$ be the total uniton number. Then there exists a unique factorization of f into unitons*

(7) $$f = Q(p_1 + \xi_{\beta_2} p_1^\perp)(p_2 + \xi_{\beta_2} p_2^\perp) \dots (p_K + \xi_{\beta_K} p_K^\perp)$$

such that:

(i) *the first $k_1 \beta$'s are equal to α_1, the second k_2's are equal to α_2, and so on.*

(ii) *For each h with $1 \le h \le K-1$, if $\beta_h = \beta_{h+1}$, we have*

(8) $$p_{h+1}^\perp(\boldsymbol{p}_h^\perp) = \boldsymbol{p}_{h+1}^\perp .$$

(iii) *For each constant vector $v \in \boldsymbol{C}^N$, and for each h with $1 \le h \le K$,*

$$p_h^\perp(\xi_{\beta_{h-1}} p_{h-1} + p_{h-1}^\perp) \dots (\xi_{\beta_2} p_2 + p_2^\perp)(\xi_{\beta_2} p_1 + p_1^\perp)(\beta_h) v$$

is not identically 0.

In the case $N=2$ the statements are simpler.

THEOREM 5.7. *Let $f: M \to \Omega U(2)$ be a rational map, with M compact. Then there exists a unique $Q \in \Omega U(2)$, unique points $\alpha_1 \dots \alpha_n \in D$, and unique holomorphic maps $f_i: M \to \Omega_\bullet^{\{\alpha_i\}}$ satisfying the following.*

(i) $f = Q(f_1 * \cdots * f_n)$.

(ii) *The maps f_i are products of k_i unitons based at α_i*

$$f_i = (p_{1,i} + \xi_{\alpha_i} p_{1,i}^\perp)(p_{2,i} + \xi_{\alpha_i} p_{2,i}^\perp) \dots (p_{K_i,i} + \xi_{\alpha_i} p_{K_i,i}^\perp) .$$

(9) (iii) $\boldsymbol{p}_{h,i}^\perp \ne \boldsymbol{p}_{h+1,i}$ *as holomorphic line bundles.*

(iv) $\boldsymbol{p}_{1,i}$ *is not trivial for any i.*

Moreover we have

(10) $$\deg(\boldsymbol{p}_{h+1,i}) \le \deg(\boldsymbol{p}_{h,i}) < 0 \qquad \textit{for any} \quad i.$$

PROOF. Because every uniton is a holomorphic line bundle, (9) is equivalent to

(11) $$\boldsymbol{p}_{h,i}^\perp \cap \boldsymbol{p}_{h+1,i} = 0 \text{ almost everywhere} .$$

This is in turn equivalent to the condition (iii) in Theorem 5.1. The existence of a factorization satisfying the condition (iii) comes from Theorem 5.1. We remark that, for each i, h, $\boldsymbol{p}_{h,i}^\perp$ and $\boldsymbol{p}_{h+1,i}$ are holomorphic with respect to the same complex structure on $M \times \boldsymbol{C}^N$.

Condition (iv) implies that the holomorphic line bundle $\mathrm{Ker}(p_{i,h}^\perp p_{i,h-1}^\perp \dots p_{i,1}^\perp) = \boldsymbol{p}_{1,i}$ is not trivial. Therefore 5.7 (iv) is equivalent to 5.1 (iv), by Lemma 5.3.

It is quite straightforward to check that the Hermitian projection onto $\boldsymbol{p}_{h+1,i}^\perp$ defines a holomorphic map between holomorphic line bundles $\boldsymbol{p}_{h,i}^\perp \to \boldsymbol{p}_{h+1,i}^\perp$. This implies (10). ((10) is also a consequence of (11)).

6. Holomorphic maps from S^2 into $\Omega SU(2)$. We want to examine in more detail the case of holomorphic maps from the Riemann sphere into the loop group of $SU(2)$. By the theorem of Atiyah and Donaldson quoted in the introduction (cf. [A]) the space $\mathrm{Hol}(d)$ of based holomorphic maps from S^2 into $\Omega SU(2)$ of topological degree d, is naturally diffeomorphic to the space M_d of Yang-Mills $SU(2)$-instantons over the 4-sphere, with instanton number d, modulo framed gauge equivalence. A similar statement holds for any classical group G. For $G=SU(2)$ this moduli space M_d is an hyper-Kähler manifold with components of real dimension $8d$ (cf. [Sal]). From the point of view of complex structures the two spaces are different; the space Hol(d) has a complex structure coming from the complex structure on the space of associated holomorphic vector bundles (cf. [A] and §4); and the argument in [A] identifies Hol(d), with an open subset of M_d (with one of the complex structures).

We quote some well-known results, which we will need in the following.

(1) Because $M=S^2$ has complex dimension 1, it is sufficient to treat the case of holomorphic maps, and vector bundles. Moreover the first Chern class of a holomorphic vector bundle may be identified with a number, the degree, by integration on M.

(3) For any integer h, there exists a unique isomorphism class of holomorphic line bundles of degree h on S^2, which we denote by $\mathcal{O}(h)$.

(4) By a theorem of Birkhoff and Grothendieck, every holomorphic vector bundle E over S^2 decomposes as direct sum of holomorphic subbundles of rank one. The degrees of these line subbundles are uniquely determined by E.

(5) In particular, if E is a topologically trivial holomorphic vector bundle over S^2, of rank 2, then $E\cong\mathcal{O}(h)\oplus\mathcal{O}(-h)$, where $h>0$ (with the subbundle $\mathcal{O}(h)$ uniquely determined). We denote such a bundle by E_h.

(6) Let $E_h\to S^2$ be the holomorphic vector bundle in (5). Then we have the following:

(A) There exists a unique holomorphic line subbundle of E_h of degree h.

(B) Let $L\subset E_h$ be a holomorphic line subbundle of degree k. Then either $k=h$, and L is the $\mathcal{O}(h)$ factor in E_h; or $k\leq -h$.

We want now to discuss the moduli space $F_{h,k}$ for the space of holomorphic line subbundles of E_h, of degree $-k\leq -h$. We have to study holomorphic inclusions $\mathcal{O}(-k)\to\mathcal{O}(h)\oplus\mathcal{O}(-h)$, modulo the action of $\boldsymbol{C}^*$. This is equivalent to studying holomorphic inclusions $\mathcal{O}(0)\to\mathcal{O}(k+h)\oplus\mathcal{O}(k-h)$, modulo the action of $\boldsymbol{C}^*$. But holomorphic sections of $\mathcal{O}(r)$, for $r\geq 0$ are in natural one-to-one correspondence with complex polynomials of degree $\leq r$ (where, if the degree is less than r, the polynomial is considered to have roots at infinity).

Therefore $F_{h,k}$ is biholomorphic to the space of pairs of polynomials of degree $\leq h+k$, $\leq -h+k$, with no roots in common (including those at infinity), modulo the action of $\boldsymbol{C}^*$. In particular, $F_{h,k}$ is a smooth manifold of complex dimension $2k+1$. (This type of results have a very old history; see, for example, [GO]). As a special case, we observe that $F_{h,h}$ is a contractible space. Some computations in [V3] identify $F_{h,h}$ with a fibre of the tangent space to the space of unbased rational maps $S^2\to S^2$ of degree h.

Finally, we can now construct a parametrization of the space of holomorphic maps from S^2 into $\Omega SU(2)$.

For d positive integer, we define a *plane partition* of d, of size $n\leq d$, to be an $n\times d$

matrix $c=(c_{ij})$ of non-negative integers, satisfying $c_{ij}\leq c_{i,j+1}$, and $\sum_{i,j}c_{ij}=d$; modulo permutation of the rows. To any plane partition, we may associate the numbers k_i of non-zero elements on the i-th row; $K=\sum_{1\leq i\leq d}k_i$ total uniton number; and $d_i=\sum_{1\leq j\leq d}c_{i,j}$. We have $n\leq K\leq d$.

PROPOSITION 6.1. *Any based holomorphic map $S^2\to\Omega SU(2)$ of degree d is obtained by the following data.*

(1) *a plane partition $c=(c_{ij})$ of size n.*

(2) *a choice of n points $\alpha_1\ldots\alpha_n\in D$;*

(3) *a choice of holomorphic line subbundles $p_{i,j+1}$ of $E\cong\mathcal{O}(c_{ij})\oplus\mathcal{O}(-c_{ij})$ of degree $-c_{i,j+1}$, for each i, j such that $c_{i,j+1}>0$.*

Then by the above procedures we obtain a holomorphic map with poles $\alpha_1\ldots\alpha_n\in D$; with degree $\sum_{1\leq j\leq d}c_{ij}=d_i$ and with uniton number k_i at α_i.

Denote the space of maps $f\in\mathrm{Hol}(d)$, with a given plane partition c, by $X(c)$. The spaces $X(c)$ define a stratification of $\mathrm{Hol}(d)$ by complex manifolds. We remark that the complex submanifold structure on each $X(c)$, is different from the complex structure coming from the Blaschke product factorization. This is a consequence of the fact that the complex structures on the space of based holomorphic maps, is different from the complex structure on the space of maximal expansions.

COROLLARY 6.2. *For any plane partition c of d, of size n, the space $X(c)$ is a complex manifold of complex dimension $2d+n+K$ (where K is the total uniton number). $X(c)$ has maximal complex dimension $4d$ if and only if $d=n=K$, and $X(c)$ is made up of those maps which are products of d unitons of degree -1, based at d different α's.*

PROOFS. The proof of 6.1 is almost finished. One has only to observe that, if $f^{\sim}$ is obtained from f by addition of the uniton p, of degree $-h$, $h>0$, based at α, then, if we want to add other unitons, also based at α, we have to find a line subbundle of $E=\mathcal{O}(h)\oplus\mathcal{O}(-h)$. This is because $p^{\perp}$ may be added to $f^{\sim}$, giving back f. Therefore, by Remark (6) above, we get that $p^{\perp}$ must be the factor $\mathcal{O}(h)$. The other parts are corollaries of our previous results. Conversely, given the above data, one can construct a holomorphic map, by repeated addition of the unitons p_{ij}.

Concerning 6.2, the fact that $X(c)$ is smooth follows from 6.1, since it is a "configuration space" of points α's in the disc, each one "labelled" by iterated fibrations of spaces of polynomials $F_{i,j}$'s. The total number of complex parameters is, by the discussion above:

$$\sum_{1\leq i\leq n\ 1\leq j\leq k_i}(2c_{ij}+1)=2\sum_{1\leq i\leq n\ 1\leq j\leq k_i}c_{ij}+\sum_{1\leq i\leq n\ 1\leq j\leq k_i}(+1)$$

$$=2\sum_{1\leq i\leq n}d_i+\sum_{1\leq i\leq n}k_i=2d+K$$

for the choice of the unitons. This must be added to n, for the choice of the poles α_i's.

Therefore we get $2d+K+n$. This number is equal to $4d$ if and only if $d=n=K$. $\square$

For any space Y, and positive integer d, we denote by $C_d(D, Y)$ the "configuration space" of d ordered points in the disc D, each one associated ("labelled") with a point in Y, modulo the permutation group in d letters. This is a well-known notion in algebraic topology.

The following is a rephrasing of part of Proposition 6.1 and Corollary 6.2, because the space of holomorphic subbundles of $S^2 \times \boldsymbol{C}^2$ of degree -1 is isomorphic to the Möbius group $PSL(2, \boldsymbol{C})$.

COROLLARY 6.3. *In the stratification of* Hol(d) *by the* $X(\boldsymbol{c})$*'s, there is a unique open stratum; it is diffeomorphic to* $C_d(D, PSL(2, \boldsymbol{C}))$.

EXAMPLE. We describe the lower-dimension moduli space ($d=1, 2$).

A holomorphic map $S^2 \to \Omega SU(2)$ of degree 1 is of the form

$$f=Q(p+\xi_\alpha p^\perp)$$

where $Q \in \Omega U(2)$, and p is associated to a holomorphic map $S^2 \to S^2$ of degree 1. The moduli space is diffeomorphic to $D \times PSL(2, \boldsymbol{C})$.

A holomorphic map $S^2 \to \Omega SU(2)$ of degree 2 is one of the following:

(A) $f=Q(p+\xi_\alpha p^\perp)$

where $Q \in \Omega SU(2)$, and p is associated to a holomorphic map $S^2 \to S^2$ of degree 2. The number of complex parameters is $1+5=6$.

(B) f is a product of two unitons, based at the same α. The number of complex parameters is $1+3+3=7$.

(C) f is a product of two unitons based at different α's. The number of complex parameters is $1+1+3+3=8$.

REMARK. Let $\Omega_{\text{alg}}U(N)$ be the space of algebraic loops. As a special case of the arguments above, we see that the space of holomorphic maps $S^2 \to \Omega_{\text{alg}}SU(2)$, of uniton number K and degree d, is described by $2d+K$ complex parameters (this agrees with previous results in [GP]). Therefore the generic holomorphic map $S^2 \to \Omega_{\text{alg}}SU(2)$ of degree d is a product of d unitons $\boldsymbol{p}$'s of degree -1, based at $\lambda=0$; and the moduli space of these generic maps is a complex vector bundle over $PSL(2, \boldsymbol{C})$.

We finish up with some questions.

(i) Is it possible to use the stratification above in order to give a proof of the Atiyah-Jones conjecture (about the topology of the moduli space of instantons)? The stratification used in [BHMM] for this purpose, is very similar to ours; and it has the natural complex structure.

(ii) Which is the analogue of our stratification, for the moduli space of $SU(N)$-instantons? The combinatorics looks more complicated.

(iii) Is it possible to construct a similar stratification for the moduli space of based harmonic maps from a compact Riemann surface M into $\Omega U(2)$? This is likely to produce a proof of the Atiyah-Jones conjecture for certain moduli spaces of framed instantons over $M \times S^2$.

(iv) A conjecture by Atiyah states that any harmonic map $S^2 \to \Omega SU(2)$ is either holomorphic or anti-holomorphic. We wonder if it is possible to construct a counterexample to Atiyah's conjecture by adding a uniton and an antiuniton (in some possible sense: a map into S^2 which is antiholomorphic with respect to a certain connection). Preliminary computations seem to deny this possibility.

(v) Is there an analogue of Theorem 4.1 for non-compact M's?

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