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ABEL-ERGODIC PROPERTIES OF PSEUDO-RESOLVENTS AND APPLICATIONS TO SEMIGROUPS

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Abstract. This paper is devoted to the study of ergodic properties of strongly and weakly continuous semigroups of operators on Banach spaces. Some new equivalent conditions are given for strong and weak ergodic properties in the locally integrable case. Such conditions are applied to the study of the quasi-weakly Y-integrable semigroups.

1. Introduction. In this paper, we examine the strong and weak Abel-ergodic properties of a general pseudo-resolvent and apply the results to two kinds of semigroups. In a forthcoming paper, we shall apply some new criteria obtained in this paper to the solvability problem of the equation $AC-CB=Q$, where A and B are generators of certain kinds of semigroups, Q is a bounded linear operator and C is unknown.

For a complex Banach space X, let $L(X)$ denote the Banach algebra of bounded linear operators from X into X. For $T\in L(X)$, we denote by $\mathcal{N}(T)$, $\mathcal{D}(T)$ and $\mathcal{R}(T)$ the null space, the domain and the range, respectively. $R(\cdot)$ is pseudo-resolvent on X, i.e. an $L(X)$-valued function, defined on an open subset Ω of the complex plane $\boldsymbol{C}$, satisfying the first resolvent equation

$$R(\lambda)-R(\mu)=(\mu-\lambda)R(\lambda)R(\mu)\,. \tag{1}$$

$R(\cdot)$ has a unique maximal extension satisfying (1) (e.g. [3, pp. 188–9]), which we invariantly denote by $R(\cdot)$ with its domain Ω. The subspaces $\mathcal{N}(\lambda R(\lambda)-I)$ and $\mathcal{R}(\lambda R(\lambda)-I)$ are known to be independent of the choice of $\lambda\in\Omega$ (e.g. [9, p. 215]).

Let $R(\cdot)$ be a pseudo-resolvent defined on a domain Ω. If $0\in\bar{\Omega}$, define

$$P_s x=s\text{-}\lim_{\mu\to 0}\mu R(\mu)x\,; \tag{2}$$

$$P_w x=w\text{-}\lim_{\mu\to 0}\mu R(\mu)x\,, \tag{3}$$

whenever the limits on the right hand sides exist in X. Clearly, both P_s and P_w are projections.

Moreover, the following equalities hold:

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$$\mathcal{N}(\lambda R(\lambda)-I)=\mathcal{R}(P_s)=\mathcal{R}(P_w)\,. \tag{4}$$

Indeed, if $x\in\mathcal{N}(\lambda R(\lambda)-I)$, the latter being independent of the choice of $\lambda\in\Omega$, one has $\mu R(\mu)x=x$ for all $\mu\in\Omega$, hence $P_s x=x$ and

$$\mathcal{N}(\lambda R(\lambda)-I)\subset\mathcal{R}(P_s)\,.$$

Next, suppose $x\in\mathcal{R}(P_w)$. Then

$$\lambda R(\lambda)x=w\text{-}\lim_{\mu\to 0}\lambda\mu R(\lambda)R(\mu)x=w\text{-}\lim_{\mu\to 0}\frac{\lambda\mu}{\lambda-\mu}[R(\mu)-R(\lambda)]x=P_w x=x\,.$$

Hence $\mathcal{R}(P_w)\subset\mathcal{N}(\lambda R(\lambda)-I)$. This, combined with the opposite inclusion, proved above, and with the evident fact $\mathcal{R}(P_s)\subset\mathcal{R}(P_w)$, gives (4).

Moreover, if $x\in\mathcal{N}(P_w)$, then

$$x=w\text{-}\lim_{\mu\to 0}[x-\mu R(\mu)x]\in(\mathcal{R}(\lambda R(\lambda)-I))^-$$

and hence

$$\mathcal{N}(P_s)\subset\mathcal{N}(P_w)\subset(\mathcal{R}(\lambda R(\lambda)-I))^-\,. \tag{5}$$

2. Abel-ergodic properties of pseudo-resolvents. To simplify writing, we shall use z for s or w, in the sense that P_z will express P_s or P_w, z-lim will stand for s-lim or w-lim, and z-dense will mean s-dense or w-dense. When we have to distinguish between s and w, we shall do so.

THEOREM 1. *Given a pseudo-resolvent $R(\cdot)$, if $0\in\bar{\Omega}$, then the following are equivalent*:

(i) $\mathcal{N}(P_z)=(\mathcal{R}(\lambda_0 R(\lambda_0)-I))^-$, *where $\lambda_0\in\Omega$, $\lambda_0\neq 0$ is fixed*;

(ii) *for each $x\in X$*,

$$z\text{-}\lim_{\lambda\to 0}\lambda^2 R(\lambda)x=0 \tag{6}$$

and there exist a neighborhood δ of 0 and a constant $M>0$, such that

$$\|\lambda R(\lambda)y\|\le M\|y\| \tag{7}$$

for all $y\in\mathcal{R}(\lambda_0 R(\lambda_0)-I)$ and all $\lambda\in\delta\cap\Omega$;

(iii) (7) *holds and $\mathcal{R}([\lambda_0 R(\lambda_0)-I]^2)$ is s-dense in $\mathcal{R}(\lambda_0 R(\lambda_0)-I)$*.

PROOF. (i) $\Rightarrow$ (ii): Let $x\in X$ and let $y=[\lambda_0 R(\lambda_0)-I]x$. In view of the definition of y and the condition (i), one has

$$z\text{-}\lim_{\lambda\to 0}\lambda R(\lambda)y=0\,. \tag{8}$$

The computation

$$\lambda R(\lambda)y=\frac{\lambda^2}{\lambda_0-\lambda}R(\lambda)x-\frac{\lambda\lambda_0}{\lambda_0-\lambda}R(\lambda_0)x \tag{9}$$

and the relation (8), combined with the evident fact

$$s\text{-}\lim_{\lambda\to 0}\frac{\lambda\lambda_0}{\lambda_0-\lambda}R(\lambda_0)x=0 \tag{10}$$

imply

$$z\text{-}\lim_{\lambda\to 0}\frac{\lambda^2}{\lambda_0-\lambda}R(\lambda)x=0\,.$$

Thus, (6) follows.

Further, (i) implies (8) for all $y\in(\mathscr{R}(\lambda_0R(\lambda_0)-I))^-$. The latter being closed, (7) follows from the uniform boundedness principle.

(ii)$\Rightarrow$(iii): It suffices to verify the second statement of (iii). Since the strong convergence implies the weak one, we may assume that the limit in (6) holds in the weak topology i.e., $z=w$. Let $x\in X$ and $y=[\lambda_0R(\lambda_0)-I]x$. Relations (9), (10) together with (6) for $z=w$, assert that $w\text{-}\lim_{\lambda\to 0}\lambda R(\lambda)y=0$ and hence

$$w\text{-}\lim_{\lambda\to 0}[\lambda R(\lambda)-I]y=-y\,. \tag{11}$$

Since $[\lambda R(\lambda)-I]y\in\mathscr{R}([\lambda_0R(\lambda_0)-I]^2)$ by the definition of y, (11) implies the w-density and hence the s-density of $\mathscr{R}([\lambda_0R(\lambda_0)-I]^2)$ in $\mathscr{R}(\lambda_0R(\lambda_0)-I)$.

(iii)$\Rightarrow$(i): Let $y\in\mathscr{R}([\lambda_0R(\lambda_0)-I]^2)$, so $y=\lambda_0R(\lambda_0)x-x$, for some $x\in\mathscr{R}(\lambda_0R(\lambda_0)-I)$. Clearly, (9) holds for such a pair of x and y. Now (7) applied to x, under consideration, asserts the boundedness of $\{\lambda R(\lambda)x\}$ as $\lambda\to 0$. Hence the right hand side of (9) strongly converges to zero and so does the left hand side as $\lambda\to 0$. Consequently, the conditions (iii) infer that (8) holds for $z=s$ and $y\in(\mathscr{R}(\lambda_0R(\lambda_0)-I))^-$. Therefore, $(\mathscr{R}(\lambda_0R(\lambda_0)-I))^-\subset\mathscr{N}(P_s)$. This, together with (5) gives (i). □

COROLLARY 2. *If one of* (i), (ii) *and* (iii) *in Theorem* 1 *holds, then* $P_s=P_w$ *and*

$$\mathscr{D}(P_s)=\mathscr{D}(P_w)=\mathscr{N}(\lambda R(\lambda)-I)\ \oplus\ (\mathscr{R}(\lambda R(\lambda)-I))^-\,. \tag{12}$$

PROOF. We may assume that (i) holds. Then

$$\mathscr{N}(P_s)=\mathscr{N}(P_w)=(\mathscr{R}(\lambda_0R(\lambda_0)-I))^-$$

and this combined with (4) gives $P_s=P_w$ and (12). Here we use the fact that P_s and P_w are projections. □

Since the direct sum in (12) may not be closed, P_s and hence P_w may not be bounded. The following corollary gives a sufficient condition for P_s (and P_w) to be bounded. With this we provide an affirmative answer to a question raised by Freeman [1] in a similar context.

COROLLARY 3. *Let $R(\cdot)$ be as in Theorem* 1. *If there exist a neighborhood δ of* 0 *and a constant $M>0$ such that*

$$\|\lambda R(\lambda)\| \leq M \tag{13}$$

for all $\lambda \in \delta \cap \Omega$, then $P_s = P_w$ are bounded with $\mathscr{D}(P_s)$ closed and the direct sum (12) *holds.*

PROOF. Condition (13) implies (ii) of Theorem 1, hence $P_s = P_w$ and (12) holds. Further, (13) implies that P_s $(=P_w)$ is bounded and $\mathscr{D}(P_s)$ $(=\mathscr{D}(P_w))$ is closed. $\square$

We say that the pseudo-resolvent $R(\cdot)$ as in Theorem 1 has the strong (weak) Abel-ergodic property, if for each $x \in X$, the limit

$$P_s x = s\text{-}\lim_{\lambda \to 0} \lambda R(\lambda) x \qquad \left(P_w x = w\text{-}\lim_{\lambda \to 0} \lambda R(\lambda) x\right)$$

exists.

The following corollary gives an equivalent condition for the strong (weak) Abel-ergodic property of $R(\cdot)$, different from those given in [3, XVIII, §2].

COROLLARY 4. *Let $R(\cdot)$ be as in Theorem* 1. *The following are equivalent*:

(i) *$R(\cdot)$ has the strong Abel-ergodic property*;

(ii) *$R(\cdot)$ has the weak Abel-ergodic property*;

(iii) *The conditions* (6), (7) *are satisfied and, for each $x \in X$, there exists a sequence $\{\lambda_n\} \subset \Omega$ converging to zero as $n \to \infty$ such that*

$$w\text{-}\lim_{n \to \infty} \lambda_n R(\lambda_n) x \tag{14}$$

exists.

PROOF. Implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (i): Let $x \in X$ and set

$$x_1 = w\text{-}\lim_{n \to \infty} \lambda_n R(\lambda_n) x\,, \qquad x_2 = x - x_1\,.$$

Let $\lambda \in \Omega$. Since, by the first resolvent equation,

$$\lambda R(\lambda) x_1 = w\text{-}\lim_{n \to \infty} \lambda \lambda_n R(\lambda) R(\lambda_n) x = x_1\,,$$

one has

$$w\text{-}\lim_{n \to \infty} \lambda_n R(\lambda_n) x_2 = w\text{-}\lim_{n \to \infty} \lambda_n R(\lambda_n) x - x_1 = 0\,.$$

Consequently, $x_1 \in \mathscr{N}(\lambda_0 R(\lambda_0) - I)$ and

$$x_2 = w\text{-}\lim_{n\to\infty} [x_2 - \lambda_n R(\lambda_n) x_2] \in (\mathscr{R}(\lambda_0 R(\lambda_0) - I))^- .$$

Thus x has a representation

$$x = x_1 + x_2 , \quad \text{with} \quad x_1 \in \mathscr{N}(\lambda_0 R(\lambda_0) - I) , \quad x_2 \in (\mathscr{R}(\lambda_0 R(\lambda_0) - I))^- .$$

This together with (6), (7) and Corollary 2, gives rise to

$$X = \mathscr{D}(P_s) = \mathscr{D}(P_w) = \mathscr{N}(\lambda_0 R(\lambda_0) - I) \oplus (\mathscr{R}(\lambda_0 R(\lambda_0) - I))^- .$$

□

COROLLARY 5. *Let $R(\cdot)$ be as in Theorem 1. If X is reflexive, then the following are equivalent:*

(i) *$R(\cdot)$ has the strong (or equivalently the weak) Abel-ergodic property;*

(ii) *There exist a neighborhood δ of 0 and a constant $M > 0$ such that*

$$\|\mu R(\mu)\| \leq M \qquad \text{for all} \quad \mu \in \delta \cap \Omega .$$

PROOF. X being reflexive, (14) of Corollary 4 is automatically satisfied. □

3. Strong and weak ergodic properties for semigroups. As applications of Theorem 1 and its corollaries, we shall study ergodic properties of strongly and weakly continuous semigroups of operators on Banach spaces. In [6], uniform ergodic properties were considered for locally integrable semigroups. So far, the corresponding strong and weak ergodic properties for the same kind of semigroups seem to be unknown, except for the classical works in [3, XVIII, §2] for more restricted semigroups. In the first part of this section, we shall provide some equivalent conditions for strong and weak ergodic properties other than those given in [3, XVIII, §2] for the locally integrable semigroups. In the second part, we shall focus our attention on the *quasi-weakly Y-integrable semigroups*, whose definition will be given later after Corollary 8.

Let $\{T(t) : t > 0\}$ be a locally integrable semigroup, i.e. $T(\cdot)$ is a semigroup and for each $x \in X$, the function $T(\cdot)x$ is Bochner integrable with respect to the Lebesgue measure over every finite subinterval of $(0, \infty)$. $T(\cdot)$ is known to be strongly continuous on $(0, \infty)$. The type of $T(\cdot)$ is the number

$$\omega_0 = \inf\{t^{-1} \|T(t)\| : t > 0\} < \infty .$$

Now, we assume that $T(\cdot)$ is a locally integrable semigroup. For every complex λ with $\operatorname{Re}\lambda > \omega_0$ and every $x \in X$, the Bochner integral

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t) x dt \tag{15}$$

exists and defines a bounded linear operator $R(\lambda)$ on X for $\operatorname{Re}\lambda > \omega_0$. $R(\lambda)$ satisfies the first resolvent equation (1) and hence it is a pseudo-resolvent on $\{\lambda \in \boldsymbol{C} : \operatorname{Re}\lambda > \omega_0\}$, referred to as the pseudo-resolvent of $T(\cdot)$.

One may define the Laplace transform of $T(\cdot)$ in a weaker sense:

$$R_s(\lambda)x=\lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)x\,ds\,,\qquad x\in X\,.$$

It has been proved in [2, Proposition 1.1] that if $R_s(\lambda_0)$ exists for some λ_0, then $R_s(\lambda)$ exists for all λ with $\operatorname{Re}\lambda>\operatorname{Re}\lambda_0$. This leads us to define

$$\begin{aligned}\sigma&=\inf\{\mu\in(-\infty,\infty):R_s(\lambda)\text{ exists for all }\lambda\text{ with }\operatorname{Re}\lambda>\mu\}\\&=\inf\{\mu\in(-\infty,\infty):R_s(\mu)\text{ exists}\}\,;\end{aligned}$$

$$\sigma_a=\inf\{\mu\in(-\infty,\infty):R_s(\lambda)\text{ is analytic for all }\lambda\text{ with }\operatorname{Re}\lambda>\mu\}\,.$$

Clearly, $\sigma\le\sigma_a\le\omega_0$ and $R_s(\lambda)=R(\lambda)$ for all λ with $\operatorname{Re}\lambda>\omega_0$. Therefore, we may write $R(\lambda)$ for $R_s(\lambda)$, whenever $\operatorname{Re}\lambda>\sigma_a$. An example in [6] shows that it may happen that $-\infty=\sigma_a<0<\omega_0$. Consequently, the well-known results given in [3] and elsewhere do not apply universally.

In view of the above definitions and comments, it follows easily from (15) and the uniqueness of the Laplace transform that

$$\bigcap_{t>0}\mathcal{N}(T(t)-I)=\mathcal{N}(\lambda R(\lambda)-I)\,,\qquad\lambda>\sigma_a\,.\tag{16}$$

For the locally integrable semigroup $T(\cdot)$, the operator $S(t)$ defined by

$$S(t)x=\int_0^t T(s)x\,ds$$

is bounded in X.

Let

$$Q_sx=s\text{-}\lim_{t\to\infty}t^{-1}S(t)x\,;$$

$$Q_wx=w\text{-}\lim_{t\to\infty}t^{-1}S(t)x\,,$$

whenever the limits on the right hand sides exist in X.

As in §2, we shall use z for s or w, in the sense that Q_z will stand for Q_s or Q_w, etc.

For $x\in\mathcal{D}(Q_s)$ and $t>0$, we have

$$T(t)Q_zx=T(t)\left[z\text{-}\lim_{u\to\infty}u^{-1}S(u)x\right]=z\text{-}\lim_{u\to\infty}u^{-1}[S(t+u)-S(t)]x=Q_zx\,.$$

This shows that Q_z is a projection and

$$\mathcal{R}(Q_z)\subset\bigcap_{t>0}\mathcal{N}(T(t)-I)\,.\tag{17}$$

Since each x in $\bigcap_{t>0} \mathcal{N}(T(t)-I)$ is fixed by $T(t)$ with $t>0$, it is also fixed by $S(t)$ and so is by Q_z. Therefore,

$$\bigcap_{t>0} \mathcal{N}(T(t)-I) \subset \mathcal{R}(Q_z) .$$

This, together with (16), (17) and the evident inclusion $\mathcal{R}(Q_s) \subset \mathcal{R}(Q_w)$, gives rise to

$$\mathcal{R}(Q_s) = \mathcal{R}(Q_w) = \mathcal{N}(\lambda R(\lambda) - I) , \qquad \lambda > \sigma_a . \tag{18}$$

As for $\mathcal{N}(Q_w)$, we note that for $x \in X$ and $y \in X^*$, we have

$$\begin{aligned} |\langle \mu R(\mu)x, y\rangle| &= \left| \int_0^\infty \mu e^{-\mu t} \langle T(t)x, y\rangle dt \right| \le \int_0^\infty \mu^2 e^{-\mu t} |\langle S(t)x, y\rangle| dt \\ &\le M \int_0^N \mu^2 e^{-\mu t} dt \|x\| \, \|y\| + \int_N^\infty \mu^2 e^{-\mu t} t \, |t^{-1}\langle S(t)x, y\rangle| dt , \end{aligned} \tag{19}$$

where $M>0$ is a constant satisfying the inequality $\|S(t)\| \le M$ for $t \in [0, N]$. If $x \in \mathcal{N}(Q_w)$, i.e., $\lim_{t\to\infty} t^{-1}\langle S(t)x, y\rangle = 0$ for all y in X^*, then (19) implies that $\lim_{\mu\to 0} \langle \mu R(\mu)x, y\rangle = 0$ and hence

$$x = w\text{-}\lim_{\mu\to 0} [x - \mu R(\mu)x] \in (\mathcal{R}(\lambda R(\lambda) - I))^- .$$

Here we assume that $\sigma_a \le 0$. Consequently,

$$\mathcal{N}(Q_s) \subset \mathcal{N}(Q_w) \subset (\mathcal{R}(\lambda R(\lambda) - I))^- . \tag{20}$$

The following lemma, which will be used for the proof of the forthcoming Theorem 7, is a generalized version of [7, Theorem 3.3]. Since the generator of a locally integrable semigroup may not always exist, the lemma takes this fact in consideration. We continue to assume that $\sigma_a \le 0$.

LEMMA 6. *Let $T(\cdot)$ be a locally integrable semigroup. Assume that there exists a constant $M>0$ such that for sufficiently large $t>0$,*

$$t^{-1}\|S(t)\| \le M . \tag{21}$$

Then the following are equivalent:

(i) *for each $x \in X$ and $u>0$*

$$z\text{-}\lim_{t\to\infty} t^{-1} T(t) S(u) x = 0 ; \tag{22}$$

(ii)
$$\mathcal{N}(Q_z) = (\mathcal{R}(\lambda R(\lambda) - I))^- ;$$

(iii) *for each $x \in X$,*

$$z\text{-}\lim_{t\to\infty} t^{-1} T(t) R(1) x = 0 .$$

PROOF. We confine the proof to the strong case, that of the weak one being similar.

(i) $\Rightarrow$ (ii): In view of (20), it suffices to show that $\mathcal{N}(Q_s)\supset(\mathcal{R}(\lambda R(\lambda)-I))^-$. For $x\in X$, $t>0$ and fixed $\lambda>\max\{\omega_0, 0\}$,

$$\begin{aligned} t^{-1}S(t)[\lambda R(\lambda)-I]x &= \int_0^\infty \lambda e^{-\lambda u}t^{-1}S(t)[T(u)-I]xdu \\ &= \left(\int_0^N + \int_N^\infty\right)\lambda e^{-\lambda u}t^{-1}S(t)[T(u)-I]xdu\,. \end{aligned}$$

It follows from (21) that for given $\varepsilon>0$, there exists a constant $N>0$ such that

$$\int_N^\infty \lambda e^{-\lambda u}\|t^{-1}S(t)\|\,\|T(u)-I\|\,\|x\|du<\varepsilon\|x\|\,. \tag{23}$$

To evaluate the integral over $[0, N]$, we use the following equality

$$S(t)[T(u)-I]=[T(t)-I]S(u)$$

from [5, Lemma 2.3]. Then, for $u\in[0, N]$,

$$s\text{-}\lim_{t\to\infty} t^{-1}S(t)[T(u)-I]x = s\text{-}\lim_{t\to\infty} t^{-1}[T(t)-I]S(u)x=0$$

and

$$\|t^{-1}T(t)S(u)\|=t^{-1}\|S(t+u)-S(t)\|\le t^{-1}(t+u)M+M\le 3M\,, \qquad \text{for} \quad t\ge N\,.$$

Consequently, the Lebesgue's dominated convergence theorem is applicable and it gives

$$\lim_{t\to\infty}\int_0^N \lambda e^{-\lambda u}\|t^{-1}S(t)[T(u)-I]x\|du=0\,. \tag{24}$$

Relations (23) and (24) give rise to

$$s\text{-}\lim_{t\to\infty} t^{-1}S(t)[\lambda R(\lambda)-I]x=0$$

and hence

$$\mathcal{N}(Q_s)\supset(\mathcal{R}(\lambda R(\lambda)-I))^-\,.$$

This, together with (20), gives (ii).

(ii) $\Rightarrow$ (iii): Since, by [6, Lemma 3],

$$(T(t)-I)R(1)=S(t)(R(1)-I)\,, \tag{25}$$

for each $x\in X$, one has

$$s\text{-}\lim_{t\to\infty} t^{-1}T(t)R(1)x = s\text{-}\lim_{t\to\infty} t^{-1}[T(t)-I]R(1)x = s\text{-}\lim_{t\to\infty} t^{-1}S(t)[R(1)-I]x=0\,.$$

(iii) $\Rightarrow$ (i) is a straightforward consequence of (25) and the inclusion

$$\mathscr{R}(S(u))=\mathscr{R}(R(1)S(u)-R(1)[T(u)-I])\subset\mathscr{R}(R(1)) .$$

$\square$

We say that $T(\cdot)$ has the strong (weak) Cesàro-ergodic property if

$$s\text{-}\lim_{t\to\infty} t^{-1}S(t)x \qquad \left(\text{resp. } w\text{-}\lim_{t\to\infty} t^{-1}S(t)x\right)$$

exists for every $x\in X$, i.e. if $\mathscr{D}(Q_s)=X$ (resp. $\mathscr{D}(Q_w)=X$).

Moreover, we say that $T(\cdot)$ has the strong (weak) Abel-ergodic property if its pseudo-resolvent $R(\cdot)$ is strongly (weakly) Abel-ergodic.

THEOREM 7. *Let $T(\cdot)$ be a locally integrable semigroup. If $\sigma_a\le 0$, then the following are equivalent*:

(i) *$T(\cdot)$ is strongly Cesàro-ergodic*;

(ii) *$T(\cdot)$ is strongly Abel-ergodic, for each $x\in X$,*

$$s\text{-}\lim_{t\to\infty} t^{-1}T(t)R(1)x=0 \tag{26}$$

and there exists a constant $M>0$ such that

$$\|t^{-1}S(t)x\|\le M\|x\| \tag{27}$$

for all $x\in\mathscr{R}(R(1)-I)$ and $t>0$ sufficiently large;

(iii) *For each $x\in X$, there exists a sequence $\{\lambda_n\}$ converging to zero as $n\to\infty$ such that $w\text{-}\lim_{n\to\infty}\lambda_n R(\lambda_n)x$ exists*, (26) *and* (27) *hold.*

PROOF. (i) $\Rightarrow$ (ii): The fact that (i) implies (27) and (26) follows from the uniform boundedness principle and Lemma 6, respectively. Now, we claim that (i) implies the strong Abel-ergodicity of $T(\cdot)$.

For each $x\in X$ and $\lambda>0$, we have

$$\lambda R(\lambda)x=\int_0^\infty \lambda^2 e^{-\lambda u}S(u)x\,du . \tag{28}$$

Thus, we obtain

$$\begin{aligned}\|\lambda R(\lambda)x-Q_sx\| &\le \lambda^2\int_0^N e^{-\lambda u}(\|S(u)\|+u\|Q_s\|)du\|x\| \\ &\quad+\lambda^2\int_N^\infty e^{-\lambda u}u\|u^{-1}S(u)x-Q_sx\|du \\ &\le\left(\sup_{0\le u\le N}\|S(u)\|+N\|Q_s\|\right)\lambda^2N\|x\|+\sup_{u\ge N}\|u^{-1}S(u)x-Q_sx\| .\end{aligned} \tag{29}$$

Since $\lim_{u\to\infty} \|u^{-1}S(u)x - Q_s x\| = 0$, it follows from (29) that

$$\lim_{\lambda\to 0} \|\lambda R(\lambda)x - Q_s x\| = 0 .$$

Consequently, $T(\cdot)$ is strongly Abel-ergodic.

(ii) ⇒ (iii): clear.

(iii) ⇒ (i): The assumption on $R(\cdot)$ implies the following decomposition

$$X = \mathcal{N}(R(1)-I) + (\mathcal{R}(R(1)-I))^- \tag{30}$$

by the argument (iii) ⇒ (i) in Corollary 4. It follows from (25) and (26) that

$$s\text{-}\lim_{t\to\infty} t^{-1}S(t)[R(1)-I]x = s\text{-}\lim_{t\to\infty} t^{-1}[T(t)-I]R(1)x = 0 .$$

Thus, the inequality (27) asserts that

$$s\text{-}\lim_{t\to\infty} t^{-1}S(t)x = 0 \qquad \text{for each} \quad x \in (\mathcal{R}(R(1)-I))^- .$$

Consequently,

$$\mathcal{N}(Q_s) \supset (\mathcal{R}(R(1)-I))^- . \tag{31}$$

Thus, it follows that

$$\mathcal{N}(Q_s) = (\mathcal{R}(R(1)-I))^- .$$

This, together with (18) and (30) asserts that $\mathcal{D}(Q_s) = X$, thus proving that $T(\cdot)$ is strongly Cesàro-ergodic. □

The weak versions of Abel and Cesàro ergodic properties have a similar proof.

COROLLARY 8. *Let $T(\cdot)$ be a locally integrable semigroup. If $\sigma_a \le 0$, then the following are equivalent*:

(i) *$T(\cdot)$ is weakly Cesàro-ergodic;*

(ii) *$T(\cdot)$ is weakly Abel-ergodic, for each $x \in X$;*

$$w\text{-}\lim_{t\to\infty} t^{-1}T(t)R(1)x = 0 \tag{26'}$$

and there exists a constant $M > 0$ such that

$$\|t^{-1}S(t)x\| \le M\|x\| \tag{27'}$$

for all $x \in \mathcal{R}(R(1)-I)$ and $t > 0$ sufficiently large;

(iii) *for each $x \in X$, there exists a sequence $\{\lambda_n\}$ converging to zero as $n \to \infty$ such that $w\text{-}\lim_{n\to\infty} \lambda_n R(\lambda_n)x$ exists,* (26′) *and* (27′) *hold.*

Assume that Y is a closed subspace of the dual space X^* such that X and Y are reciprocal, i.e., $\|x\|=\sup\{\langle x, y\rangle/\|y\|: y\in Y, y\neq 0\}$. Then $\{T(t):t>0\}$ is a weakly Y-integrable semigroup of operators on X (cf. [8]), if $T(\cdot)$ satisfies the following conditions:

(W1) Y is invariant under $T(t)^*$ for each $t>0$;

(W2) $T(\cdot)x$ is $\sigma(X, Y)$-continuous on $(0, \infty)$ for each $x\in X$;

(W3) (a) for each $x\in X$ and $y\in Y$, the function $\langle T(t)x, y\rangle$ of t is L-integrable on $[0, 1]$,

(b) the integral $\int_0^1\langle T(t)x, y\rangle dt$ is $\sigma(Y, X)$-continuous with respect to $y\in Y$, for each $x\in X$;

(W4) $X_0=\bigcup\{\mathscr{R}(T(\eta)):\eta>0\}$ is $\sigma(X, Y)$-dense in X, and $\bigcap\{\mathscr{N}(T(\eta)):\eta>0\}=\{0\}$.

If $T(\cdot)$ satisfies the conditions (W1), (W2) and (W3) only, we say that $T(\cdot)$ is quasi-weakly Y-integrable.

A special case of a weakly Y-integrable semigroup is a locally Y-integrable semigroup with an additional condition. These concepts were introduced by S. Y. Shaw in [5], in terms of the following definition.

A semigroup $\{T(t):t>0\}$ is called locally Y-integrable if it satisfies conditions (A1)–(A4) below.

(A1) X and Y are reciprocal, i.e.

$$\|x\|=\sup\{|\langle x, y\rangle|/\|y\|:y\in Y, y\neq 0\}:$$

(A2) Y is invariant under $T(t)^*$ for all $t>0$;

(A3) For each $x\in X$, $T(\cdot)x$ is $\sigma(X, Y)$-continuous on $(0, \infty)$.

It follows from (A1) and (A3) that $\|T(\cdot)\|$ is bounded in every closed interval $[u, t]\subset(0, \infty)$. Therefore, for each $x\in X$, the Riemann integral $\int_u^t\langle T(s)x, y\rangle ds$ $(y\in Y)$ defines an element $x_{u,t}\in X$, by the argument of [8, Lemma 3.1]. In [5], the existence of $x_{u,t}$ is an assumption and $T(\cdot)x$ is called Y-Riemann integrable under this assumption.

(A4) For each $x\in X$, $T(\cdot)x$ is Y-Riemann integrable on every $[u, t]\subset(0, \infty)$, and for each $t>0$, the Y-improper Riemann integral

$$Y-\int_0^t T(s)xds=Y-\lim_{u\to 0^+}\left\{Y-\int_u^t T(s)xds\right\}$$

exists in X. This means that there exists $x_t\in X$ such that

$$x_t=Y-\int_0^t T(s)xds\,.$$

The additional condition is

$$\text{(A5)}\qquad Y-\lim_{t\to 0^+}T(t)x=x\qquad\text{for all}\quad x\in X\,.$$

If the locally Y-integrable semigroup $T(\cdot)$ satisfies (A5), then the existence of x_t is a consequence of [8, Proposition 3.4].

It has been shown in [8] that the semigroup $T(\cdot)$, $[T(t)x](s)=x(s+t)$, defined on $X=L^\infty(-\infty, \infty)$ is not strongly continuous but weakly Y-integrable with $Y=L(-\infty, \infty)$. For completeness, we give the proof of this fact.

For each $x\in X$, $y\in Y$, it is known that

$$\langle T(t)x, y\rangle = \int_{-\infty}^{\infty} x(t+s)\overline{y(s)}ds \to \int_{0}^{\infty} x(s)\overline{y(s)}ds = \langle x, y\rangle$$

and hence (W2) follows. (W1), (W3, a) and (W4) are evident. To verify (W3, b), it suffices to note that $\langle T(t)x, y\rangle$, as a function of t, is continuous on $[0, 1]$. The application of Lebesgue's dominated convergence theorem asserts that $\int_0^1 \langle T(t)x, y\rangle dt$ is $\sigma(Y, X)$-continuous with respect to $y\in Y$, for each $x\in X$. To show that $T(\cdot)$ is not strongly continuous, let

$$x_0(s)=\begin{cases} 1, & \text{if } s\le 0; \\ 0, & \text{if } s>0. \end{cases}$$

Then $\|T(t)x_0 - x_0\| = 1$ for $t>0$. Therefore, $T(t)x_0$ does not converge to x_0 in the norm topology.

Two more important examples are shown in the following.

EXAMPLE 1 (cf. [5], [8]). Let $G(\cdot)$ and $H(\cdot)$ be semigroups of class C_0 on X. Then the family $\{T(t): t\ge 0\}$ defined by $T(t)C=H(t)CG(t)$ is a semigroup on the Banach space $L(X)$ of bounded operators. ($T(\cdot)$ is called the tensor product of $G(\cdot)$ and $H(\cdot)$). For $x\in X$, $x^*\in X^*$, define the linear functional on $L(X)$, by $\langle C, f_{x,x^*}\rangle = \langle Cx, x^*\rangle$. Let Y be the closed linear span of all such f_{x,x^*}. It was shown in [5] that $T(\cdot)$ is weakly Y-integrable. $T(\cdot)$ need not be strongly continuous. For instance (cf. [8]), let $G(t)=I$, the identity on X, and let $H(\cdot)$ be a C_0-semigroup which is not uniformly continuous. We have $T(t)I=H(t)$, $(t\ge 0)$. If $T(\cdot)$ is strongly continuous, then one is led to the contradiction that $H(\cdot)$ is uniformly continuous.

EXAMPLE 2. Let α be fixed with $0<\alpha<1$ and let X be the space of Lip α functions over $[0, 1]$ vanishing at 1. X endowed with the norm

$$\|x\|_\alpha = \sup\left\{\frac{|x(s')-x(s'')|}{|s'-s''|^\alpha} : s', s''\in[0, 1], s'\ne s''\right\}$$

is a nonseparable Banach space. The family $\{T(t): t\ge 0\}$ defined by

$$[T(t)x](s)=\begin{cases} x(s+t), & \text{if } 0\le s\le 1, 0\le s+t\le 1; \\ 0, & \text{if } 0\le s\le 1, 1<s+t \end{cases}$$

is not a strongly continuous semigroup. In fact, let $x_0(s)=(1-s)^\alpha$. Then $x_0\in X$ and for $t>0$ sufficiently small,

$$\|[T(t)-T(0)]x_0\|_\alpha=\sup\frac{|(x_0(s'+t)-x_0(s'))-(x_0(s''+t)-x_0(s''))|}{|s'-s''|^\alpha}$$

where the supremum is taken over $s', s''\in[0, 1]$, $s'\neq s''$. Let $s'=1-t$, $0<t<1$, $s''=1$. Then

$$x_0(s'+t)=0\,,\quad x_0(s')=t^\alpha\,,\quad x_0(s''+t)=x_0(s'')=0\,.$$

Here we agree on $x_0(s)=0$ for $s\geq 1$. Thus we have

$$\|[T(t)-T(0)]x_0\|_\alpha\geq 1\,.$$

But $T(\cdot)$ is weakly Y-integrable with Y suitably chosen. To this end, for each $x\in X$, let

$$z(s', s'')=\begin{cases}\dfrac{x(s')-x(s'')}{|s'-s''|^\alpha}\,, & \text{if } s'\neq s''\,;\ \ s', s''\in[0, 1]\,;\\ 0\,, & \text{if } s'=s''\,;\ \ s', s''\in[0, 1]\,.\end{cases}$$

Then, the mapping Φ: $x(\cdot)\to z(\cdot, \cdot)$ is an isometry from X onto a closed subspace of $Z=L^\infty([0, 1]\times[0, 1])$. With the help of the mapping Φ, we may identify X with a closed subspace of Z. Let $Y=L([0, 1]\times[0, 1])$. Then X and Y are reciprocal with the duality

$$\langle x, y\rangle=\int_0^1\int_0^1\frac{x(s')-x(s'')}{|s'-s''|^\alpha}\,y(s', s'')ds'ds''\,. \tag{32}$$

In fact, (32) is the restriction of the duality between Z and Y to the pair X, Y. X being nonseparable and Y separable, it follows that Y is a proper subspace of X^*. Finally, the $\sigma(X, Y)$-continuity of $T(\cdot)$ is an easy consequence of the $\sigma(Z, Y)$-continuity of the right translation. Now it is easy to verify that the conditions (W1)–(W4) hold for $T(\cdot)$.

It is easily seen that the above examples are all locally Y-integrable semigroups satisfying (A5). It is however easy to construct an example of a weakly Y-integrable semigroup that does not satisfy (A5), (cf. [8, Example 4.4]).

Let $\{T(t): t>0\}$ be a quasi-weakly Y-integrable semigroup. Then

$$\tilde{\omega}_0=\inf\{t^{-1}\log\|T(t)\|: t>0\}$$

satisfies $-\infty\leq\tilde{\omega}_0<\infty$. By [8, Theorem 3.6],

$$\langle R(\lambda)x, y\rangle=\int_0^\infty e^{-\lambda t}\langle T(t)x, y\rangle dt\,,\qquad(x\in X, y\in Y)$$

defines a bounded linear operator $R(\lambda)$ on X, for each λ with $\operatorname{Re}\lambda>\tilde{\omega}_0$. $R(\lambda)$ also verifies the first resolvent equation and hence it is a pseudo-resolvent on $\{\lambda\in \boldsymbol{C}: \operatorname{Re}\lambda>\tilde{\omega}_0\}$ (cf. [8]).

One may also define the Laplace transform $R_Y(\cdot)$ of $T(\cdot)$ in a weaker sense. Set $R_Y(\lambda, t)$ to be the operator defined by

$$\langle R_Y(\lambda, t)x, y\rangle = \int_0^t e^{-\lambda u}\langle T(u)x, y\rangle du. \tag{33}$$

It follows from [8, Proposition 3.3] that $R_Y(\lambda, t)$ is a bounded linear operator on X, for each $t>0$ and $\lambda \in \boldsymbol{C}$. We shall consider those λ's for which $\lim_{t\to\infty}\langle R_Y(\lambda, t)x, y\rangle$ exists for all $x\in X$, $y\in Y$ and for which there exists a bounded linear operator $R_Y(\lambda)$ on X which satisfies the condition

$$\langle R_Y(\lambda)x, y\rangle = \lim_{t\to\infty}\langle R_Y(\lambda, t)x, y\rangle = \lim_{t\to\infty}\int_0^t e^{-\lambda u}\langle T(u)x, y\rangle du.$$

The following is an extension of [2, Proposition 1.1].

PROPOSITION 9. *If, for a number λ_0, $R_Y(\lambda_0)$ is a bounded linear operator on X, then for each λ with* $\operatorname{Re}\lambda > \operatorname{Re}\lambda_0$, *$R_Y(\lambda)$ exists and is a bounded linear operator on X.*

PROOF. Let $x\in X$, $y\in Y$ be fixed and assume that $\operatorname{Re}\lambda > \operatorname{Re}\lambda_0$. Use integration by parts to obtain

$$\int_0^t e^{-\lambda u}\langle T(u)x, y\rangle du = e^{-(\lambda-\lambda_0)t}\int_0^t e^{-\lambda_0 s}\langle T(s)x, y\rangle ds$$
$$+(\lambda-\lambda_0)\int_0^t e^{-(\lambda-\lambda_0)u}\left(\int_0^u e^{-\lambda_0 s}\langle T(s)x, y\rangle ds\right)du.$$

Since $\lim_{u\to\infty}\int_0^u e^{-\lambda_0 s}\langle T(s)x, y\rangle ds$ exists so does $\lim_{t\to\infty}\int_0^t e^{-\lambda u}\langle T(u)x, y\rangle du$. Hence going to the limit, we obtain

$$\int_0^\infty e^{-\lambda u}\langle T(u)x, y\rangle du = (\lambda-\lambda_0)\int_0^\infty e^{-(\lambda-\lambda_0)u}\langle R_Y(\lambda_0, u)x, y\rangle du. \tag{34}$$

Let $\{y_\alpha\}$ be a bounded net converging to $y\in Y$ in the $\sigma(Y, X)$-topology. Since $R_Y(\lambda_0, u)$ is continuous in the uniform operator topology for $u\in(0, \infty)$, it is bounded in a neighborhood of $u=0$ (see its definition (33)). Moreover, the limit

$$\lim_{u\to\infty}\langle R_Y(\lambda_0, u)x, y\rangle = \langle R_Y(\lambda_0)x, y\rangle$$

exists for all $x\in X$ and $y\in Y$. Lebesgue's dominated convergence theorem is applicable to the second integral of (34)

$$\lim_\alpha\int_0^\infty e^{-\lambda u}\langle T(u)x, y_\alpha\rangle du = (\lambda-\lambda_0)\lim_\alpha\int_0^\infty e^{-(\lambda-\lambda_0)u}\langle R_Y(\lambda_0, u)x, y_\alpha\rangle du$$
$$=(\lambda-\lambda_0)\int_0^\infty e^{-(\lambda-\lambda_0)u}\langle R_Y(\lambda_0, u)x, y\rangle du$$

$$= \int_0^\infty e^{-\lambda u} \langle T(u)x, y \rangle du .$$

It follows that the integral on the left hand side of (34) is bounded and $\sigma(Y, X)$-continuous with respect to $y \in Y$ for each fixed $x \in X$. Then, there exists an element $x_\lambda \in X$ such that

$$\langle x_\lambda, y \rangle = \int_0^\infty e^{-\lambda u} \langle T(u)x, y \rangle du .$$

X and Y being reciprocal, x_λ is uniquely determined. Set $R_Y(\lambda)x = x_\lambda$. Then $R_Y(\lambda)$ is linear and it is a routine work to verify that $R_Y(\cdot)$ is bounded. □

We may also define σ, σ_a for $R_Y(\cdot)$. To do this it suffices to replace $R_s(\cdot)$ by $R_Y(\cdot)$. To distinguish these two cases: $R_s(\cdot)$ and $R_Y(\cdot)$, we shall use the notation $\tilde{\sigma}$, $\tilde{\sigma}_a$ for $R_Y(\cdot)$, instead of σ, σ_a. Moreover, we also write $R(\lambda)$ for $R_Y(\lambda)$, whenever $\operatorname{Re} \lambda > \tilde{\sigma}_a$.

For the quasi-weakly Y-integrable semigroup $T(\cdot)$, the operator $S(t)$, defined by

$$\langle S(t)x, y \rangle = \int_0^t \langle T(u)x, y \rangle du$$

is also bounded on X by [7, Theorem 2.3]. The Cesàro-ergodicity and Abel-ergodicity for the present $T(\cdot)$, are defined in the same way as for locally integrable semigroups.

The following analogue of Theorem 7 is true.

THEOREM 10. *Let $T(\cdot)$ be a quasi-weakly Y-integrable semigroup. If $\tilde{\sigma}_a \le 0$, then* (i), (ii) *and* (iii) *of Theorem* 7 *are equivalent for the present $T(\cdot)$.*

PROOF. We only sketch the proof. (i) ⇒ (ii): We have the following analogue of (28):

$$(28') \qquad \langle \lambda R(\lambda)x, y \rangle = \int_0^\infty \lambda^2 e^{-\lambda u} \langle S(u)x, y \rangle du , \qquad x \in X , \quad y \in Y , \quad \lambda > 0 .$$

With the help of (28′), one obtains an analogue of (29):

$$(29') \quad \|\lambda R(\lambda)x - Q_s x\| \le \left(\sup_{0 \le u \le N} \|S(u)\| + N\|Q_s\| \right) \lambda^2 N \|x\| + \sup_{u \ge N} \|u^{-1}S(u)x - Q_s x\| .$$

Now the relation $\lim_{u \to \infty} \|u^{-1}S(u)x - Q_s x\| = 0$ combined with (29′) infers that $T(\cdot)$ is strongly Abel-ergodic.

(ii) ⇒ (iii): clear.

(iii) ⇒ (i): identical to that of Theorem 7. □

As an application of Theorem 10, consider the semigroup $T(\cdot)$ in Example 2. Since

$\|T(t)\| \leq 1$ for all $t \geq 0$, (26) and (27) are automatically satisfied. On the other hand, $T(t)=0$ for $t \geq 1$. Thus, for $\lambda > 0$, $x \in X$, $y \in Y$, one has

$$\langle R(\lambda)x, y \rangle = \int_0^1 e^{-\lambda t} \langle T(t)x, y \rangle dt\,, \tag{35}$$

where X and Y are the spaces in Example 2. The relation (35) asserts that $R(\lambda)$ is bounded as $\lambda \to 0$. Therefore,

$$s\text{-}\lim_{\lambda \to 0} \lambda R(\lambda)x = 0 \qquad \text{for all} \quad x \in X$$

and hence $T(\cdot)$ is strongly Cesàro-ergodic by Theorem 10.

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