Tôhoku Math. J.
47 (1995), 391–403

ROOT STRINGS WITH TWO CONSECUTIVE REAL ROOTS

YULY BILLIG AND ARTURO PIANZOLA

(Received February 24, 1994, revised July 12, 1994)

Abstract. For a Kac-Moody Lie algebra we study pairs of real roots the sum of which is a real root. More precisely, we study in which way the existence of such pair of roots determines the existence of certain subroot system within the root system.

0. Introduction. The study of pairs of real roots $\{\gamma_1, \gamma_2\}$ of a Kac-Moody Lie algebra $\mathfrak{g}$ whose sum is a real root was initiated by Morita in [3] and [4] (though [4] contains a mistake as pointed out in [5]). Morita put this information to good use to derive information about K_2 in the case of Kac-Moody groups.

Morita looks at the case when $\langle\gamma_1, \gamma_2^\vee\rangle = -1$ and $\langle\gamma_2, \gamma_1^\vee\rangle = -a$ where $a=1, 2, 3$. (There are also some results if $a>3$ but only under some strong assumptions on the Cartan matrix.) Morita assumes that γ_1, γ_2 are positive and that $\gamma_1 - \gamma_2$ is not a root (a *Morita pair* in our terminology). His key observation is that a determines the existence of certain entries in the corresponding Cartan matrix A of $\mathfrak{g}$ (and hence that A somehow sheds information about the existence of such pairs of roots).

Our own interest in this problem came out from trying to understand the nilpotency degree of certain subalgebras of $\mathfrak{g}$ (Conjecture 1 below). We will deal with a above arbitrary and show how a determines a sequence of entries in A with certain properties.

1. Notation and some basic facts about root systems of Kac-Moody Lie algebra. We begin by recalling some well-known objects related to Kac-Moody Lie algebras. Our running reference for this will be [9, Ch. 4, 5]. Most of this material is also covered in [1].

$A = (A_{ij})_{i,j\in I}$ will throughout denote a generalized Cartan matrix. (The index set I is allowed to be infinite.) Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. Thus

$$\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*, \quad \Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h},$$
$$\langle\alpha_i, \alpha_j^\vee\rangle = A_{ij}, \qquad \forall i, j \in I.$$

As usual we set

$$W = \langle r_i \mid i \in I\rangle \text{ where } r_i := r_{\alpha_i},$$
$$^{\mathrm{re}}\Delta = W\Pi,$$

1991 *Mathematics Subject Classification*. Primary 17B67.

Δ the root string closure of ${}^{\mathrm{re}}\Delta$ (= the set of all roots, real and imaginary, of the corresponding Kac-Moody Lie algebra),

${}^{\mathrm{im}}\Delta = \Delta \setminus {}^{\mathrm{re}}\Delta$,

$Q_+ = \bigoplus_{i \in I} N\alpha_i$,

$\Delta_+ = \Delta \cap Q_+$, ${}^{\mathrm{re}}\Delta_+ = {}^{\mathrm{re}}\Delta \cap Q_+$.

Let $\beta \in \Delta$ and $\alpha \in {}^{\mathrm{re}}\Delta$. Recall the *$\alpha$-string through β*, defined by $S(\alpha, \beta) = \{\beta + k\alpha \mid k \in \boldsymbol{Z}\} \cap \Delta$. Then there exist $u, v \in \boldsymbol{N}$ such that

$$S(\alpha, \beta) = \beta - u\alpha, \dots, \beta, \dots, \beta + v\alpha .$$

Moreover $u - v = \langle \beta, \alpha^\vee \rangle$ and the reflection r_α flips $S(\alpha, \beta)$ about its midpoint $\beta - (\langle \beta, \alpha^\vee \rangle / 2)\alpha$. We refer to $\beta - u\alpha$ and $\beta + v\alpha$ as the *first* and *last* roots of $S(\alpha, \beta)$ respectively.

We intend to describe the real and imaginary nature of the roots in a root string. It is easy to do this visually by attaching to $S(\alpha, \beta)$ a series of nodes; black for real roots and white for imaginary. For example ● ○ ○ ● depicts a string $S(\alpha, \beta)$ with four roots where only the first and last roots are real.

Parts of the next proposition are exercises in [1] and are also implicitly used in [3]. For the sake of completeness and convenience we state and prove.

PROPOSITION 1. *Let $\beta \in \Delta$ and $\alpha \in {}^{\mathrm{re}}\Delta$. Let $r(\alpha, \beta)$ denote the number of real roots in $S(\alpha, \beta)$. Assume $r(\alpha, \beta) > 0$. Then*

(i) *The first and last roots of $S(\alpha, \beta)$ are real.*

(ii) *$r(\alpha, \beta) = 1, 2, 3,$ or 4. Moreover*

(a) *If $r(\alpha, \beta) = 1$, then $S(\alpha, \beta) = \{\beta\}$ and β is real.*

(b) *If $r(\alpha, \beta) = 2$, then $S(\alpha, \beta)$ is depicted by a diagram of shape* ● ○ ··· ○ ··· ○ ●.

(c) *If $r(\alpha, \beta) = 3$, then $S(\alpha, \beta)$ is depicted by the diagram* ● ● ● *and $\{\alpha, \beta\}$ generates a root system of type C_2.*

(d) *If $r(\alpha, \beta) = 4$, then $S(\alpha, \beta)$ is depicted by a diagram of shape* ● ● ○ ··· ○ ··· ○ ● ●. *Furthermore, if $S(\alpha, \beta)$ does not contain imaginary roots, then $\{\alpha, \beta\}$ generates a root system of type G_2.*

PROOF. There is no loss of generality in assuming that $\alpha \in {}^{\mathrm{re}}\Delta_+$ and that $S(\alpha, \beta)$ contains real roots. Moreover since $S(\alpha, \beta)$ is independent of $\beta \in S(\alpha, \beta)$ we may assume that $\beta \in {}^{\mathrm{re}}\Delta$, $\beta - \alpha \notin {}^{\mathrm{re}}\Delta$, and $\langle \beta, \alpha^\vee \rangle \leq 0$. (Choose β to be the real roots of smallest height in $S(\alpha, \beta)$.)

We begin by reducing the problem to the rank 2 case. Let W' be the subgroup of W generated by r_α and r_β and let ${}^{\mathrm{re}}\Delta' = W'\alpha \cup W'\beta$. With the terminology of [8] ${}^{\mathrm{re}}\Delta'$ is a closed subroot system of ${}^{\mathrm{re}}\Delta$ and $\{\alpha, \beta\}$ is a base of ${}^{\mathrm{re}}\Delta'$ [8, Proposition 8.1] and hence

$$(\boldsymbol{Z}\alpha+\boldsymbol{Z}\beta)\cap\Delta=\Delta', \quad (\boldsymbol{Z}\alpha+\boldsymbol{Z}\beta)\cap{}^{\mathrm{re}}\Delta={}^{\mathrm{re}}\Delta',$$

where Δ' denotes the root string closure of ${}^{\mathrm{re}}\Delta'$.

From this discussion it follows that it will suffice to establish the proposition in the case where ${}^{\mathrm{re}}\Delta$ is a rank 2 root system with base $\{\alpha, \beta\}$. We assume this for the remainder of this proof.

Let $(\cdot|\cdot)$ be a symmetric W-invariant bilinear form satisfying $\|\gamma\|:=(\gamma|\gamma)>0$ if $\gamma\in{}^{\mathrm{re}}\Delta$ and $\|\gamma\|\leq 0$ if $\gamma\in{}^{\mathrm{im}}\Delta$.

Consider the function $F_{\alpha,\beta}\colon \boldsymbol{R}\to\boldsymbol{R}$ defined by

$$F_{\alpha,\beta}\colon t\mapsto\|\beta+t\alpha\|=\|\alpha\|t^2+2(\alpha|\beta)t+\|\beta\|^2 .$$

We now prove the proposition.

(i) If the first root $\beta-u\alpha$ of $S(\alpha, \beta)$ is imaginary, so is the last $\beta+v\alpha=r_\alpha(\beta-u\alpha)$. By assumption there exists $-u<k<v$ such that $\beta+k\alpha\in{}^{\mathrm{re}}\Delta$. Then $F_{\alpha,\beta}(-u)\leq 0$, $F_{\alpha,\beta}(k)>0$ and $F_{\alpha,\beta}(v)\leq 0$, which contradicts the fact that the graph of $F_{\alpha,\beta}(t)$ is a concave up parabola.

(ii) For each $c\in\boldsymbol{R}$ the equation $F_{\alpha,\beta}(t)=c$ has at most two solutions. Now if $\beta+k\alpha\in{}^{\mathrm{re}}\Delta$, then $\beta+k\alpha$ is W-conjugate to either α or β and hence $\|\beta+k\alpha\|\in\{\|\alpha\|, \|\beta\|\}$. Thus k is a solution of either $F_{\alpha,\beta}(t)=\|\alpha\|$ or $F_{\alpha,\beta}(t)=\|\beta\|$, so at most four real roots appear in $S(\alpha, \beta)$. The statements of (a), (b), (c) and (d) now follow from the symmetry of $S(\alpha, \beta)$ about its midpoint and the concave up nature of the graph of $F_{\alpha,\beta}(t)$. □

2. Morita pairs. We begin by looking at root strings with two consecutive real roots up to conjugation by the Weyl group and sign. To this end we define a non-ordered pair of positive real roots $\{\alpha, \beta\}$ to be a *Morita pair* if

MP1. $\alpha-\beta\notin\Delta$

MP2. $\alpha+\beta\in\Delta^{\mathrm{re}}$

MP3. (Minimality condition) $\mathrm{ht}(\alpha+\beta)\leq\mathrm{ht}(w(\alpha+\beta))$ for all $w\in W$ such that $w(\alpha)$, $w(\beta)\in{}^{\mathrm{re}}\Delta_+$.

PROPOSITION 2. (i) *Let $\alpha_i, \alpha_j\in\Pi$. A pair $\{\alpha_i, \alpha_j\}$ is a Morita pair if and only if $\langle\alpha_i, \alpha_j^\vee\rangle=-1$ or $\langle\alpha_j, \alpha_i^\vee\rangle=-1$.*

(ii) *Every Morita pair is of the form $\{\alpha_i, \beta\}$ for some $i\in I$. Moreover if $\beta\notin\Pi$, then $\langle\beta, \alpha_i^\vee\rangle=-1$.*

(iii) *If $\{\alpha_i, \beta\}$ is a Morita pair with $\beta\notin\Pi$, then $\langle\beta+\alpha_i, \alpha_j^\vee\rangle\leq 0$ for all $j\neq i$.*

PROOF. (i) If $\langle\alpha_i, \alpha_j^\vee\rangle=-1$, then $r_j(\alpha_i)=\alpha_i-\langle\alpha_i, \alpha_j^\vee\rangle=\alpha_i+\alpha_j$, hence $\alpha_i+\alpha_j\in{}^{\mathrm{re}}\Delta$. Since $\alpha_i-\alpha_j\notin\Delta$ and the minimality condition obviously holds, we have that $\{\alpha_i, \alpha_j\}$ is a Morita pair. If $\langle\alpha_i, \alpha_j^\vee\rangle<-1$ and $\langle\alpha_j, \alpha_i^\vee\rangle<-1$, then $\alpha_i+\alpha_j\notin{}^{\mathrm{re}}\Delta$, and therefore $\{\alpha_i, \alpha_j\}$ is not a Morita pair.

(ii) Let $\{\alpha, \beta\}$ be a Morita pair. Then $\alpha+\beta\in{}^{\mathrm{re}}\Delta_+$ so that there exists $i\in I$ such that $\langle\alpha+\beta, \alpha_i^\vee\rangle>0$. As $\mathrm{ht}(r_i(\alpha+\beta))<\mathrm{ht}(\alpha+\beta)$ then by MP3 either $r_i(\alpha)\in\Delta_-$ or $r_i(\beta)\in\Delta_-$, so that either $\alpha=\alpha_i$ or $\beta=\alpha_i$. If $\beta\notin\Pi$ then $\alpha=\alpha_i$. As $\langle\beta+\alpha_i, \alpha_i^\vee\rangle>0$ we have $\langle\beta, \alpha_i^\vee\rangle>-2$, while $\langle\beta, \alpha_i^\vee\rangle<0$ since $\beta-\alpha_i$ is not a root by MP1. Thus $\langle\beta, \alpha_i^\vee\rangle=-1$.

(iii) If $\langle\beta+\alpha_i, \alpha_j^\vee\rangle>0$ for some $j\neq i$, then $\mathrm{ht}(r_j(\beta+\alpha_i))<\mathrm{ht}(\beta+\alpha_i)$ and $r_j(\beta)$, $r_j(\alpha_i)\in{}^{\mathrm{re}}\Delta_+$ as $\beta\notin\Pi$ and $i\neq j$. This contradicts MP3. $\square$

The main theorem of this paper describes in which way the value $\langle\alpha_i, \alpha^\vee\rangle$ of a Morita pair $\{\alpha, \alpha_i\}$ determines the existence of a sequence of fundamental roots with certain properties.

The next two results show in which way two real roots whose sum is a real root determines the existence of a Morita pair $\{\alpha, \alpha_i\}$ with a certain $\langle\alpha_i, \alpha^\vee\rangle$.

PROPOSITION 3. *Let $\gamma_1, \gamma_2\in{}^{\mathrm{re}}\Delta$ be such that $\gamma_1+\gamma_2\in{}^{\mathrm{re}}\Delta$.*

(i) *If $\gamma_1, \gamma_2\in\Delta_+$ and $\gamma_2-\gamma_1\notin\Delta$, then there exists $w\in W$ such that $\{w\gamma_1, w\gamma_2\}$ is a Morita pair.*

(ii) *There exists $\sigma\in\pm W$ such that $\sigma\gamma_1\in\Delta_+$ and $S(\sigma\gamma_1, \sigma\gamma_2)\subset\Delta_+$.*

PROOF. (i) Among all $w\in W$ for which $\{w\gamma_1, w\gamma_2\}\subset\Delta_+$ choose one minimizing $\mathrm{ht}(w\gamma_1+w\gamma_2)$.

(ii) We may assume that $\gamma_1\in\Pi$. In this case either $S(\gamma_1, \gamma_2)\subset\Delta_+$ (in which case $\gamma_1, \gamma_2\in{}^{\mathrm{re}}\Delta_+$) or $S(\gamma_1, \gamma_2)\subset\Delta_-$. Now if $S(\gamma_1, \gamma_2)\subset\Delta_-$, then $r_{\gamma_1}S(\gamma_1, \gamma_2)\subset\Delta_-$ (since $-\gamma_1\notin S(\gamma_1, \gamma_2)$). But $r_{\gamma_1}S(\gamma_1, \gamma_2)=S(-\gamma_1, r_{\gamma_1}\gamma_2)$ where both $-\gamma_1$ and $r_{\gamma_1}\gamma_2$ belong to Δ_-. Now (ii) follows if we set $\sigma=-r_{\gamma_1}$.

PROPOSITION 4. *Let $\gamma_1, \gamma_2\in{}^{\mathrm{re}}\Delta_+$ be such that $\gamma_1+\gamma_2\in{}^{\mathrm{re}}\Delta$. Then there exists an integer $N\geq-1$ and a Morita pair $\{\alpha, \alpha_i\}$ such that exactly one of the following holds:*

(i) *$\gamma_1-\gamma_2\in\Delta$ and $\{\gamma_1, \gamma_2\}$ is not conjugate by $\pm W$ to a Morita pair, $\langle\gamma_1, \gamma_2^\vee\rangle=\langle\gamma_2, \gamma_1^\vee\rangle=N$, and $\langle\alpha_i, \alpha^\vee\rangle=-(2+|N|)$.*

(ii) *$\gamma_1-\gamma_2\notin\Delta$ and $\{\gamma_1, \gamma_2\}$ is conjugate by $\pm W$ to $\{\alpha, \alpha_i\}$. In particular, after interchanging γ_1 and γ_2 if necessary, we have $\langle\gamma_1, \gamma_2^\vee\rangle=-1$ and $\langle\alpha_i, \alpha^\vee\rangle=\langle\gamma_2, \gamma_1^\vee\rangle$.*

PROOF. By Proposition 3(ii) we may assume that γ_1 and $S(\gamma_1, \gamma_2)$ lie inside Δ_+.

Assume $\gamma_2-\gamma_1\in\Delta$. If $\langle\gamma_2, \gamma_1^\vee\rangle<0$, then γ_1 and γ_2 generate a G_2-type root system (Proposition 1). By direct inspection we find that $\langle\gamma_2, \gamma_1^\vee\rangle=\langle\gamma_1, \gamma_2^\vee\rangle=-1$. Furthermore, any base of this subroot system can be conjugated to a Morita pair (Proposition 3(i)). Thus (i) holds with $N=-1$.

If $\langle\gamma_2, \gamma_1^\vee\rangle=0$, then γ_1 and γ_2 generate a C_2-type root system and, just as above, we see that (i) holds with $N=0$.

If $\langle\gamma_2, \gamma_1^\vee\rangle=N>0$, then $\gamma:=r_{\gamma_1}(\gamma_2+\gamma_1)=\gamma_2-(N+1)\gamma_1$ and $r_{\gamma_1}(\gamma_2)=\gamma_2-N\gamma_1$ are the first two roots of $S(\gamma_1, \gamma_2)$. By Proposition 3(i) we can conjugate $\{\gamma, \gamma_1\}$ to a Morita pair. This pair is as desired since $\langle\gamma, \gamma_1^\vee\rangle=\langle\gamma_2+\gamma_1, -\gamma_1^\vee\rangle=-2-N$. Furthermore, we must have $\langle\gamma_1, \gamma^\vee\rangle=-1$ (Proposition 2). Thus $r_{\gamma_1}r_\gamma(\gamma_1)=r_{\gamma_1}(\gamma_1+\gamma)=-\gamma_1+r_{\gamma_1}\gamma=\gamma_2$ and therefore

$$\langle \gamma_1, \gamma_2^\vee \rangle = \langle \gamma_1, r_{\gamma_1} r_\gamma \gamma_1^\vee \rangle = \langle r_{\gamma_1}\gamma_1, r_\gamma \gamma_1^\vee \rangle = -\langle \gamma_1, r_\gamma \gamma_1^\vee \rangle = -\langle r_\gamma \gamma_1, \gamma_1^\vee \rangle$$
$$= \langle r_\gamma \gamma_1, r_{\gamma_1}\gamma_1^\vee \rangle = \langle r_{\gamma_1} r_\gamma \gamma_1, \gamma_1^\vee \rangle = \langle \gamma_2, \gamma_1^\vee \rangle = N .$$

Finally if $\gamma_2 - \gamma_1 \notin \Delta$, then $\{\gamma_1, \gamma_2\}$ is conjugate to a Morita pair by Proposition 3(ii) and clearly (ii) holds. □

3. Submatrices attached to Morita pairs. We begin by stating (with proofs when necessary) five lemmas that will be used in the proof of the main result. The first three lemmas are in [3] and are here restated with our present notation for the reader's convenience.

LEMMA 1. *Let $i, j \in I$, $i \neq j$, and let $\alpha \in \Delta_+$. Suppose $\langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_j, \alpha_i^\vee \rangle = -2$. Then we have*:

(i) $\langle \alpha, \alpha_i^\vee \rangle + \langle \alpha, \alpha_j^\vee \rangle \leq 0$.

(ii) *If* $\langle \alpha, \alpha_i^\vee \rangle + \langle \alpha, \alpha_j^\vee \rangle = 0$, *then* $\langle \alpha, \alpha_i^\vee \rangle = -\langle \alpha, \alpha_j^\vee \rangle \equiv 0 \pmod 2$.

LEMMA 2. *Let $i, j \in I$, $i \neq j$, and let $\alpha \in \Delta_+$. Suppose $\langle \alpha_i, \alpha_j^\vee \rangle = -4$ and $\langle \alpha_j, \alpha_i^\vee \rangle = -1$. Then $\langle \alpha, 2\alpha_i^\vee + \alpha_j^\vee \rangle \leq 0$.*

LEMMA 3. *Let $i, j \in I$, $i \neq j$, and let $\alpha \in \Delta_+$. Suppose $\langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle > 4$. If $\langle \alpha, \alpha_i^\vee \rangle = m > 0$, then $\langle r_j \alpha, \alpha_i^\vee \rangle \leq -(m+1)$.*

LEMMA 4. *Let $i, j \in I$, $i \neq j$, and let $\alpha \in {}^{\mathrm{re}}\Delta_+$. Suppose $\langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle > 4$. If $\langle \alpha, \alpha_i^\vee \rangle = -1$ and $\langle \alpha, \alpha_j^\vee \rangle > 0$, then $\langle \alpha_j, \alpha_i^\vee \rangle = -1$ and either $\langle \alpha, \alpha_j^\vee \rangle = 1$ or $\alpha = \alpha_j$.*

PROOF (due to J. Morita). Consider $\beta := r_j \alpha = \alpha - \langle \alpha, \alpha_j^\vee \rangle \alpha_j$. If $\alpha \neq \alpha_j$ then $\beta \in {}^{\mathrm{re}}\Delta_+$. Note that

$$\langle \beta, \alpha_i^\vee \rangle = \langle \alpha, \alpha_i^\vee \rangle - \langle \alpha, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle = -1 - \langle \alpha, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle \geq 0 .$$

By Lemma 3 if $\langle \beta, \alpha_i^\vee \rangle \geq 1$, then $\langle \alpha, \alpha_i^\vee \rangle < -1$, which will contradict the assumption $\langle \alpha, \alpha_i^\vee \rangle = -1$. It follows that $\langle \beta, \alpha_i^\vee \rangle = 0$ and hence $\langle \alpha, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle = -1$. From this last equality we deduce that $\langle \alpha, \alpha_j^\vee \rangle = 1$ and $\langle \alpha_j, \alpha_i^\vee \rangle = -1$. □

LEMMA 5. *Let J be a finite subset of I such that the submatrix A_J is indecomposable. Let $\mu \in Q_+$ be such that $\langle \mu, \alpha_j^\vee \rangle \geq 0$ for all $j \in J$. Assume* $\mathrm{supp}(\mu) \cap J \neq \emptyset$. *Then we have*:

(i) *A_J is either of finite or affine type.*

(ii) *If $\langle \mu, \alpha_j^\vee \rangle > 0$ for some $j \in J$, then A_J is of finite type.*

(iii) *If A_J is affine and* $\mathrm{supp}(\mu)$ *is connected, then μ is a null-root of the affine subsystem generated by $\{\alpha_j\}_{j \in J}$.*

PROOF. Let $\mu = \sum_{i \in I} c_i \alpha_i$. As $\mathrm{supp}(\mu) \cap J \neq \emptyset$ there exists $k \in J$ such that $c_k > 0$. Consider $\beta = \sum_{k \in J} c_k \alpha_k \neq 0$. Then $\langle \beta, \alpha_j^\vee \rangle \geq \langle \mu, \alpha_j^\vee \rangle \geq 0$ for every $j \in J$. By a result of Vinberg it then follows that A_J is of finite or affine type (cf. [9, Proposition 3.6.5]). If in addition $\langle \mu, \alpha_j^\vee \rangle > 0$ for some $j \in J$, then $\langle \beta, \alpha_j^\vee \rangle > 0$ and A_J is of finite type (ibid.).

From what has been said it follows that if A_J is affine then $\langle\mu, \alpha_j^\vee\rangle = \langle\beta, \alpha_j^\vee\rangle = 0$ for all $j \in J$. Hence for every vertex $i \notin J$ of the Dynkin diagram of A such that i is connected by an edge to some vertex from J we have $c_i = 0$. As $\mathrm{supp}(\mu)$ is connected and $\mathrm{supp}(\mu) \cap J \neq \varnothing$ it follows that $\mathrm{supp}(\mu) \subset J$ and $\mu = \beta$ is by definition a null-root. □

PROPOSITION 5. *Let $\{\alpha_i, \beta\}$ be a Morita pair with $\beta \notin \Pi$. Then we have*:

(i) $\langle\beta+\alpha_i, \alpha_j^\vee\rangle \leq 0$ *for all $j \in I$, $j \neq i$. Furthermore if $\langle\beta+\alpha_i, \alpha_j^\vee\rangle = 0$, then either*
  (a) $\langle\alpha_i, \alpha_j^\vee\rangle = 0$ *or*
  (b) *$\{\alpha_i, \alpha_j\}$ generates a subsystem of type G_2, with $\langle\alpha_i, \alpha_j^\vee\rangle = -3$, $\langle\alpha_j, \alpha_i^\vee\rangle = -1$; and $\beta = r_j(\alpha_i) = \alpha_i + 3\alpha_j$.*

(ii) *Assume* (i)(b) *above is not the case. If $j \in I$ is such that $\langle\beta, \alpha_j^\vee\rangle > 0$, then either*
  (a) $\langle\beta, \alpha_j^\vee\rangle = 1$, $\langle\alpha_i, \alpha_j^\vee\rangle < -1$, $\langle\alpha_j, \alpha_i^\vee\rangle = -1$ *or*
  (b) *$\{\alpha_i, \alpha_j\}$ generates a subsystem of type $BC_1^{(2)}$, with $\langle\alpha_i, \alpha_j^\vee\rangle = -4$, $\langle\alpha_j, \alpha_i^\vee\rangle = -1$. Moreover $\beta = \alpha_j + n\delta$, where $n \in N$ and $\delta = 2\alpha_i + \alpha_j$ is a null-root of the affine subroot system in question.*

(iii) *There exists a unique $j \in I$ such that $\langle\beta, \alpha_j^\vee\rangle > 0$. Moreover $j \neq i$.*

PROOF. (i) We have seen that $\langle\beta+\alpha_i, \alpha_j^\vee\rangle \leq 0$ in Proposition 2(iii). Furthermore, if $\langle\beta+\alpha_i, \alpha_j^\vee\rangle = 0$, then $\mathrm{ht}(r_i r_j(\beta+\alpha_i)) < \mathrm{ht}(\beta+\alpha_i)$, hence $r_i r_j \alpha_i \in \Delta_-$ or $r_i r_j \beta \in \Delta_-$. Consequently, either $\langle\alpha_i, \alpha_j^\vee\rangle = \langle\alpha_j, \alpha_i^\vee\rangle = 0$ or $\beta = r_j \alpha_i$. In the latter case $\beta + \alpha_i$ is a real root of the subsystem generated by $\{\alpha_i, \alpha_j\}$. Since $\langle\beta+\alpha_i, \alpha_j^\vee\rangle = 0$ and $\langle\beta+\alpha_i, \alpha_i^\vee\rangle = 1$, Lemma 5(ii) shows that $\{\alpha_i, \alpha_j\}$ generate a subsystem of finite type for which $\beta+\alpha_i$ is a dominant root. In the C_2 case $\beta+\alpha_i$ is the highest root (because $\beta \notin \Pi$). But then $\beta - \alpha_i \in \Delta$, which contradicts $\{\alpha_i, \beta\}$ being a Morita pair.

(ii) From (i) and the assumption $\langle\beta, \alpha_j^\vee\rangle > 0$ it follows that $\langle\alpha_i, \alpha_j^\vee\rangle < -1$ and $\langle\beta+\alpha_i, \alpha_j^\vee\rangle < 0$. Let us consider the cases where $\{\alpha_i, \alpha_j\}$ generates a subsystem of hyperbolic, affine or finite type separately.

Case 1. Suppose that $\{\alpha_i, \alpha_j\}$ generates a hyperbolic system, i.e., $\langle\alpha_i, \alpha_j^\vee\rangle\langle\alpha_j, \alpha_i^\vee\rangle > 4$. Then by Lemma 4, $\langle\alpha_j, \alpha_i^\vee\rangle = -1$, $\langle\alpha_i, \alpha_j^\vee\rangle < -4$ and $\langle\beta, \alpha_j^\vee\rangle = 1$.

Case 2. Suppose that the subsystem generated by $\{\alpha_i, \alpha_j\}$ is affine. Since $\langle\beta, \alpha_i^\vee\rangle = -1$ and $\langle\beta, \alpha_j^\vee\rangle > 0$, Lemma 1 rules out the case $A_1^{(1)}$, i.e., $\langle\alpha_i, \alpha_j^\vee\rangle = \langle\alpha_j, \alpha_i^\vee\rangle = -2$. Since $\langle\alpha_i, \alpha_j^\vee\rangle < -1$, the only possible case is $\langle\alpha_i, \alpha_j^\vee\rangle = -4$, $\langle\alpha_j, \alpha_i^\vee\rangle = -1$. Applying Lemma 2 we get that $\langle\beta, 2\alpha_i^\vee + \alpha_j^\vee\rangle \leq 0$, hence $0 < \langle\beta, \alpha_j^\vee\rangle \leq 2$. If $\langle\beta, \alpha_j^\vee\rangle = 1$ then (ii)(a) holds. If $\langle\beta, \alpha_j^\vee\rangle = 2$ then by Lemma 5(iii) $\beta - \alpha_j$ is a null-root of the affine system of type $BC_1^{(2)}$ generated by $\{\alpha_i, \alpha_j\}$. Then $\beta - \alpha_j = n\delta$ as prescribed by (ii)(b).

Case 3. Suppose that $\{\alpha_i, \alpha_j\}$ generate a finite subsystem. Since $\langle\alpha_i, \alpha_j^\vee\rangle < -1$, it follows that $\langle\alpha_j, \alpha_i^\vee\rangle = -1$ and $\langle\alpha_i, \alpha_j^\vee\rangle$ may equal either -2 or -3. If $\langle\alpha_i, \alpha_j^\vee\rangle = -2$ then $\langle\beta, \alpha_j^\vee\rangle = 1$, because $\langle\beta, \alpha_j^\vee\rangle > 0$ and $\langle\beta+\alpha_i, \alpha_j^\vee\rangle < 0$. Similarly if $\langle\alpha_i, \alpha_j^\vee\rangle = -3$, then $\langle\beta, \alpha_j^\vee\rangle \in \{1, 2\}$. Now if $\langle\beta, \alpha_j^\vee\rangle = 2$, then $\langle\beta+\alpha_i, \alpha_j^\vee\rangle = -1$, and hence

$(r_ir_j)^2(\beta+\alpha_i)=\beta$, whereas $(r_ir_j)^2(\alpha_i)=\alpha_i+3\alpha_j$. From the minimality of $\{\beta, \alpha_i\}$ it then follows that $(r_ir_j)^2(\beta)\in\Delta_-$, so that β belongs to the finite root system generated by $\{\alpha_i, \alpha_j\}$ and either $\beta=\alpha_j\in\Pi$ or $\beta=\alpha_i+3\alpha_j$, both of which are ruled out by assumption. Consequently, $\langle\beta, \alpha_j^\vee\rangle=1$ as desired.

(iii) Since $\beta\in{}^{\mathrm{re}}\Delta_+$ there exists $j\in I$ such that $\langle\beta, \alpha_j^\vee\rangle>0$. Moreover, $j\neq i$ since $\langle\beta, \alpha_i^\vee\rangle=-1$ by Proposition 2(iii). We show j to be unique by way of contradiction. To this end let us assume that $\langle\beta, \alpha_j^\vee\rangle>0$ and $\langle\beta, \alpha_k^\vee\rangle>0$, where $j\neq k$.

Note that $\langle\alpha_i, \alpha_j^\vee\rangle\neq 0$ and $\langle\alpha_i, \alpha_k^\vee\rangle\neq 0$ by (i). Since $\langle\beta, \alpha_k^\vee\rangle>0$, we have $\beta-\alpha_k\in\Delta_+$ and

$$\langle\beta-\alpha_k, \alpha_i^\vee\rangle=-\langle\alpha_k, \alpha_i^\vee\rangle-1\geq 0\,,$$

$$\langle\beta-\alpha_k, \alpha_j^\vee\rangle=\langle\beta, \alpha_j^\vee\rangle-\langle\alpha_k, \alpha_j^\vee\rangle>0\,.$$

By Lemma 5(ii) the root system generated by $\{\alpha_i, \alpha_j\}$ is finite. Mutatis mutandi for $\{\alpha_i, \alpha_k\}$. By (ii) then $\langle\alpha_j, \alpha_i^\vee\rangle=\langle\alpha_k, \alpha_i^\vee\rangle=-1$ and $\{\langle\alpha_i, \alpha_j^\vee\rangle, \langle\alpha_i, \alpha_k^\vee\rangle\}\subset\{-2, -3\}$. Write $\beta=\sum_{s\in I}c_s\alpha_s$ with $c_s\geq 0$. Then

$$(1)\qquad -1=\langle\beta, \alpha_i^\vee\rangle=\sum_{s\in I}c_s\langle\alpha_s, \alpha_i^\vee\rangle\leq c_i\langle\alpha_i, \alpha_i^\vee\rangle+c_j\langle\alpha_j, \alpha_i^\vee\rangle+c_k\langle\alpha_k, \alpha_i^\vee\rangle=2c_i-c_j-c_k\,.$$

On the other hand

$$(2)\qquad 0<\langle\beta, \alpha_j^\vee\rangle\leq c_i\langle\alpha_i, \alpha_j^\vee\rangle+c_j\langle\alpha_j, \alpha_j^\vee\rangle\leq -2c_i+2c_j\,;$$

and mutatis mutandi

$$(3)\qquad 0<\langle\beta, \alpha_k^\vee\rangle\leq -2c_i+2c_k\,.$$

From (2) and (3) we get that $c_i\leq c_j-1$, $c_i\leq c_k-1$, thereby contradicting (1). □

PROPOSITION 6. *Let $n\geq 2$ and let $\{i_1, \ldots, i_n\}$, $n\geq 2$, be (necessarily distinct) elements of I satisfying the following conditions*

$$\text{IND 1}\qquad \begin{cases}\langle\alpha_{i_{k+1}}, \alpha_{i_k}^\vee\rangle=-1 & \text{for}\quad k=1, \ldots, n-1\\ \langle\alpha_{i_k}, \alpha_{i_m}^\vee\rangle=0 & \text{for}\quad |k-m|>1\,;\\ \langle\alpha_{i_1}, \alpha_{i_2}^\vee\rangle\leq -2\,. & \end{cases}$$

Let $\beta\in{}^{\mathrm{re}}\Delta_+$ be such that

$$\text{IND 2}\qquad \begin{cases}\langle\beta, \alpha_{i_k}^\vee\rangle=0\quad \text{for } k=1, \ldots, n-2\,;\quad \langle\beta, \alpha_{i_{n-1}}^\vee\rangle=-1\,;\quad \langle\beta, \alpha_{i_n}^\vee\rangle=1\,,\\ \langle\beta, \alpha_j^\vee\rangle\leq 0\quad \text{for all } j\neq i_n\,.\end{cases}$$

Let $\alpha=r_{i_n}\beta=\beta-\alpha_{i_n}\in{}^{\mathrm{re}}\Delta_+$. Assume $j\in I$ satisfies $\langle\alpha, \alpha_j^\vee\rangle>0$. Then we have:

(i) *$\langle\alpha, \alpha_{i_k}^\vee\rangle=0$ for all $1\leq k<n$, and $\langle\alpha, \alpha_{i_n}^\vee\rangle=-1$. In particular $j\notin\{i_1, \ldots, i_n\}$.*

(ii) *Either*

(a) *$\alpha=\alpha_j$, $\langle\alpha_{i_n}, \alpha_j^\vee\rangle\leq -2$, or*

(b) $\langle \alpha, \alpha_j^\vee \rangle = 1$ *or*

(c) *the submatrix* A_J *of the generalized Cartan matrix* A *corresponding to the subset of indices* $J = \{i_1, \dots, i_n, j\}$ *is of affine type* $BC_n^{(2)}$ *and* $\alpha = \alpha_j + n\delta$, $n \in \boldsymbol{N}$, *where* δ *is a null-root of this subsystem.*

(iii) $\langle \alpha_j, \alpha_{i_n}^\vee \rangle = -1$.

(iv) *j is the unique element of I with the property* $\langle \alpha, \alpha_j^\vee \rangle > 0$.

(v) $\langle \alpha_j, \alpha_{i_k}^\vee \rangle = 0$ *for all* $k = 1, \dots, n-1$.

PROOF. (i) This follows easily from the assumptions.

(ii) If $\alpha = \alpha_j$, then $\langle \alpha_{i_n}, \alpha_j^\vee \rangle \le -2$ as $\langle \alpha + \alpha_{i_n}, \alpha_j^\vee \rangle \le 0$ and $\langle \alpha, \alpha_j^\vee \rangle = 2$. If neither (a) nor (b) hold, then $\langle \alpha, \alpha_j^\vee \rangle > 1$ and $\alpha \ne \alpha_j$ so that $\alpha - \alpha_j \in \Delta_+$ and $\langle \alpha_{i_n}, \alpha_j^\vee \rangle < -1$ (this last since $\langle \alpha + \alpha_{i_n}, \alpha_j^\vee \rangle \le 0$). Recall that $\langle \alpha_{i_n}, \alpha_j^\vee \rangle < -1$ implies $\langle \alpha_j, \alpha_{i_n}^\vee \rangle < 0$. We then have

$$\langle \alpha - \alpha_j, \alpha_{i_k}^\vee \rangle \ge 0 \quad \text{for} \quad k = 1, \dots, n-1\,;$$

$$\langle \alpha - \alpha_j, \alpha_{i_n}^\vee \rangle = -1 - \langle \alpha_j, \alpha_{i_n}^\vee \rangle \ge 0 \quad \text{and}$$

$$\langle \alpha - \alpha_j, \alpha_j^\vee \rangle = \langle \alpha, \alpha_j^\vee \rangle - 2 \ge 0\,.$$

By Lemma 5(i) the submatrix A_J corresponding to the subset of indices $J = \{i_1, \dots, i_n, j\}$ is of finite or affine type. But since $\langle \alpha_{i_1}, \alpha_{i_2}^\vee \rangle < -1$ and $\langle \alpha_{i_n}, \alpha_j^\vee \rangle < -1$ we get that A_J is of type $BC_n^{(2)}$ (Figure 1). By Lemma 5(iii) $\alpha - \alpha_j$ is a null-root of this subsystem and we are in case (b).

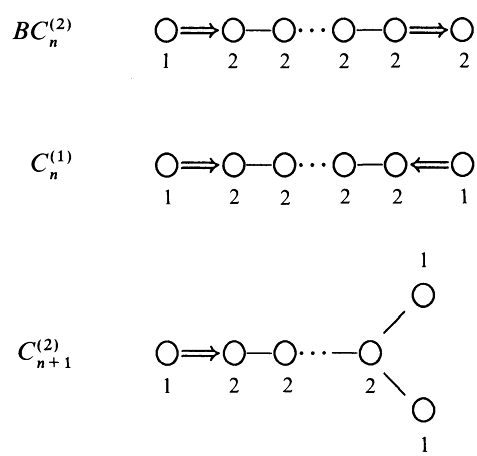

FIGURE 1.

(iii) Note that in case ii(a) and ii(c) we have $\langle \alpha_j, \alpha_{i_n}^\vee \rangle = -1$ (given that $\langle \alpha, \alpha_{i_n}^\vee \rangle = -1$). Assume that ii(b) holds, i.e., $\langle \alpha, \alpha_j^\vee \rangle = -1$. Suppose, by way of contradiction, that (iii) fails. Then $\langle \alpha_j, \alpha_{i_n}^\vee \rangle \le -2$. Note that $j \in \operatorname{supp}(\alpha)$ since $\langle \alpha, \alpha_j^\vee \rangle > 0$. Thus $2\alpha - \alpha_j \in Q_+$, $\operatorname{supp}(2\alpha - \alpha_j) = \operatorname{supp}(\alpha)$ is connected, and

$$\langle 2\alpha-\alpha_j, \alpha_{i_k}^\vee\rangle \geq 0 \qquad \text{for} \quad k=1,\dots,n-1\,;$$

$$\langle 2\alpha-\alpha_j, \alpha_{i_n}^\vee\rangle = -2-\langle\alpha_j, \alpha_{i_n}^\vee\rangle \geq 0 \quad \text{and}$$

$$\langle 2\alpha-\alpha_j, \alpha_j^\vee\rangle = 2\langle\alpha, \alpha_j^\vee\rangle - 2 = 0\,.$$

Hence, by Lemma 5(i) the submatrix A_J, where $J=\{i_1,\dots,i_n,j\}$, is of finite or affine type. Then it is of type $C_n^{(1)}$ in Figure 1, because $\langle\alpha_{i_1}, \alpha_{i_2}^\vee\rangle < -1$ and $\langle\alpha_j, \alpha_{i_n}^\vee\rangle < -1$. By Lemma 5(iii) $2\alpha-\alpha_j$ is a multiple of the null-root of $C_n^{(1)}$. Thus $2\alpha-\alpha_j = m(\alpha_{i_1}+2\alpha_{i_2}+\cdots+2\alpha_{i_n}+\alpha_j)$, but this equality is impossible as the left hand side has α_{i_1} with even multiplicity and α_j with odd multiplicity.

(iv) Let us prove the uniqueness of $j\in I$ such that $\langle\alpha, \alpha_j^\vee\rangle > 0$. Suppose, by way of contradiction, that $\langle\alpha, \alpha_j^\vee\rangle > 0$, $\langle\alpha, \alpha_k^\vee\rangle > 0$ for some $k\in I$, $k\neq j$. From (i) $k, j\notin\{i_1,\dots,i_n\}$ because $\langle\alpha, \alpha_{i_m}^\vee\rangle \leq 0$, $m=1,\dots,n$. From (ii) it follows that $\langle\alpha, \alpha_j^\vee\rangle = \langle\alpha, \alpha_k^\vee\rangle = 1$ while (iii) gives us $\langle\alpha_j, \alpha_{i_n}^\vee\rangle = \langle\alpha_k, \alpha_{i_n}^\vee\rangle = -1$. As $\langle\alpha, \alpha_j^\vee\rangle > 0$ and $\langle\alpha, \alpha_k^\vee\rangle > 0$ we have $2\alpha-\alpha_j-\alpha_k\in Q_+$. Now

$$\langle 2\alpha-\alpha_j-\alpha_k, \alpha_{i_m}^\vee\rangle \geq 0 \qquad \text{for} \quad m=1,\dots,n-1\,;$$

$$\langle 2\alpha-\alpha_j-\alpha_k, \alpha_{i_n}^\vee\rangle = -2+1+1=0\,,$$

$$\langle 2\alpha-\alpha_j-\alpha_k, \alpha_j^\vee\rangle \geq 2-2=0\,,$$

$$\langle 2\alpha-\alpha_j-\alpha_k, \alpha_k^\vee\rangle \geq 2-2=0\,,$$

so that by Lemma 5(i) the submatrix A_J corresponding to the set $J=\{i_1,\dots,i_n,j,k\}$ is of finite or affine type. The only possible type for A_J is $C_{n+1}^{(2)}$ in Figure 1, then by Lemma 5(iii) $2\alpha-\alpha_j-\alpha_k$ is a multiple of the null-root of $C_{n+1}^{(2)}$ and consequently, $2\alpha-\alpha_j-\alpha_k = m(\alpha_{i_1}+2\alpha_{i_2}+\cdots+2\alpha_{i_n}+\alpha_j+\alpha_k)$ for some $m\in\mathbf{N}$. However, this equality is impossible as can be seen by comparing parities as above. This finishes the proof that j is unique.

(iv) It remains to be shown that $\langle\alpha_{i_k}, \alpha_j^\vee\rangle = 0$ for all $1\leq k<n$. Suppose not. Then neither (ii)(a) nor (ii)(c) can hold (because otherwise $\langle\alpha_{i_k}, \alpha_j^\vee\rangle = 0$ as can be seen by (i)). We may therefore assume that $\langle\alpha, \alpha_j^\vee\rangle = 1$. Let $m<n$ be the maximal index with the property $\langle\alpha_{i_m}, \alpha_j^\vee\rangle \neq 0$. Let $\langle\alpha_{i_m}, \alpha_j^\vee\rangle = -x$, $\langle\alpha_j, \alpha_{i_m}^\vee\rangle = -y$, and note that $x, y>0$. Since $\langle\alpha, \alpha_j^\vee\rangle = 1$, we have $\alpha-\alpha_j\in\Delta_+$. Then by (i)

$$\langle\alpha-\alpha_j, \alpha_{i_m}^\vee\rangle = y\,;$$

$$\langle\alpha-\alpha_j, \alpha_{i_k}^\vee\rangle \geq 0 \qquad \text{for} \quad k=1,\dots,n-1\,;$$

while also

$$\langle\alpha-\alpha_j, \alpha_{i_n}^\vee\rangle = -1+1=0\,; \quad \text{and} \quad \langle\alpha-\alpha_j, \alpha_j^\vee\rangle = 1-2=-1\,.$$

Since $y>0$, we have $\alpha-\alpha_j-\alpha_{i_m}\in\Delta_+$. If we assume that $y\geq 2$, then

$$\langle \alpha-\alpha_j-\alpha_{i_m}, \alpha_{i_k}^\vee \rangle \geq 0 \qquad \text{for} \quad k=1, \ldots, n;$$

$$\langle \alpha-\alpha_j-\alpha_{i_m}, \alpha_j^\vee \rangle = -1-\langle \alpha_{i_m}, \alpha_j^\vee \rangle \geq 0$$

and it then follows from Lemma 5(i) that the submatrix A_J, where $J=\{i_1, \ldots, i_n, j\}$, is of finite or affine type. But this is impossible as $\langle \alpha_{i_1}, \alpha_{i_2}^\vee \rangle < -1$ and the Dynkin diagram of A_J contains a cycle (namely $j, i_m, i_{m+1}, \ldots, i_n$). If $y=1$ and $x\geq 2$, then the same argument works for $2(\alpha-\alpha_j)-\alpha_{i_m}$.

We may therefore assume that $x=y=1$. Then (i) yields $r_{i_m}r_j(\alpha)=\alpha-\alpha_j-\alpha_{i_m}$. We consider two cases:

Case 1. $\langle \alpha_{i_m}, \alpha_{i_{m+1}}^\vee \rangle = -1$. Then $m>1$ and $\langle \alpha-\alpha_j-\alpha_{i_m}, \alpha_{i_{m-1}}^\vee \rangle > 0$. Thus $\mu := \alpha-\alpha_j-\alpha_{i_{m-1}}-\alpha_{i_m} \in \Delta_+$ satisfies the conditions of Lemma 5(i) for $J=\{i_m, \ldots, i_n, j\}$. Indeed, $\langle \mu, \alpha_{i_{m+1}}^\vee \rangle > 0$ and hence A_J is of finite type which contradicts the fact that A_J has a cycle.

Case 2. $\langle \alpha_{i_m}, \alpha_{i_{m+1}}^\vee \rangle < -1$. Consider $\mu := \alpha-\alpha_j-\alpha_{i_m}-\alpha_{i_{m+1}} \in \Delta_+$. Then μ satisfies the conditions of Lemma 5(i) for $J=\{i_m, \ldots, i_n, j\}$. Thus A_J is, on the one hand of finite or affine type, while on the other, having an entry less than -1 and a cycle is of indefinite type. This contradiction completes the proof of the proposition.

4. The main theorem.

THEOREM 1. *Let $\{\alpha, \beta\}$ be a Morita pair with $\langle \alpha, \beta^\vee \rangle = -a$ and $\langle \beta, \alpha^\vee \rangle = -1$. Then exactly one of the following holds.*

Case F. (*Finite case.*) $a=1, 2,$ *or* 3 *and either*

(i) $\alpha, \beta \in \Pi$ *or*

(ii) $a=1$ *and there exist* $\alpha_i, \alpha_j \in \Pi$ *such that* $\langle \alpha_i, \alpha_j^\vee \rangle = -3$, $\langle \alpha_j, \alpha_i^\vee \rangle = -1$, *and* $\{\alpha, \beta\} = \{\alpha_i, \alpha_i + 3\alpha_j\}$.

Case A. (*Affine case.*) $a=4$ *and there exists a sequence of distinct fundamental roots* $\alpha_{i_1}, \ldots, \alpha_{i_l} \in \Pi$, $l \geq 2$, *which generate an affine subsystem of type* $BC_{l-1}^{(2)}$ *and which furthermore satisfy*

$$\begin{array}{ll}
\langle \alpha_{i_{k+1}}, \alpha_{i_k}^\vee \rangle = -1 & \textit{for} \quad k=1, \ldots, l-1, \\
\langle \alpha_{i_k}, \alpha_{i_m}^\vee \rangle = 0 & \textit{if} \quad |k-m| \geq 2, \\
\langle \alpha_{i_1}, \alpha_{i_2}^\vee \rangle = \langle \alpha_{i_{l-1}}, \alpha_{i_l}^\vee \rangle = -2 & \textit{if} \quad l>2, \\
\langle \alpha_{i_1}, \alpha_{i_2}^\vee \rangle = -4 & \textit{if} \quad l=2, \\
\alpha = \alpha_{i_1}, \quad \textit{and} & \\
\beta = r_{i_2} \cdots r_{i_{l-1}}(\alpha_{i_l}) + n\delta, &
\end{array}$$

where δ is a null-root of $BC_{l-1}^{(2)}$ and $n \geq 0$ (by convention $r_{i_2} \cdots r_{i_{l-1}} = 1$ if $l=2$).

Case I. (*Indefinite case.*) $a \geq 5$ *and there exists a sequence of distinct fundamental roots* $\alpha_{i_1}, \ldots, \alpha_{i_l} \in \Pi$, $l \geq 2$ *such that*

$$\langle \alpha_{i_{k+1}}, \alpha_{i_k}^\vee \rangle = -1 \qquad \textit{for} \quad k=1,\ldots,l-1\,;$$
$$\langle \alpha_{i_k}, \alpha_{i_m}^\vee \rangle = 0 \qquad \textit{if} \quad |k-m| \geq 2\,,$$
$$\langle \alpha_{i_1}, \alpha_{i_2}^\vee \rangle < -1\,, \quad \langle \alpha_{i_{l-1}}, \alpha_{i_l}^\vee \rangle < -1\,;$$
$$\left| \prod_{k=1}^{l-1} \langle \alpha_{i_k}, \alpha_{i_{k+1}}^\vee \rangle \right| = a\,,$$
$$\alpha = \alpha_{i_1}\,, \quad \textit{and}$$
$$\beta = r_{i_2} \cdots r_{i_{l-1}}(\alpha_{i_l})\,.$$

PROOF. Before going into the main proof we note for future reference that if α and β are as stated in Case A then $\langle \alpha, \beta^\vee \rangle = -4$. To see this first use a positive semidefinite W-invariant bilinear form on the affine system in question to see that $\langle \alpha, (\beta + n\delta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle$ for all $n \in \boldsymbol{Z}$. Thus

$$\langle \alpha, \beta^\vee \rangle = \langle \alpha_{i_1}, r_{i_2} \cdots r_{i_{l-1}} \alpha_{i_l}^\vee \rangle = \langle r_{i_{l-1}} \cdots r_{i_2} \alpha_{i_1} \alpha_{i_l}^\vee \rangle$$
$$= \begin{cases} \langle \alpha_{i_1}, \alpha_{i_2}^\vee \rangle = -4 & \text{if } l=2 \\ \langle \alpha_{i_1} + 2\alpha_{i_2} + \cdots + 2\alpha_{i_{l-1}}, \alpha_{i_l}^\vee \rangle = -4 & \text{if } l>2\,. \end{cases}$$

If $a=1$, then the assertion of the theorem (namely Case F(i) or (ii)) remains true if we interchange α and β. It follows then by Proposition 2 that we may henceforth assume that $\langle \beta, \alpha^\vee \rangle = -1$ and that $\alpha = \alpha_i$ for some $i \in I$.

By Proposition 2(ii) and Proposition 5 there exists $j \in I$, $j \neq i$ such that one of the following holds:

(a) $\beta = \alpha_j \in \Pi$, and $\langle \alpha_j, \alpha_i^\vee \rangle = -1$.

(b) $\beta = \alpha_i + 3\alpha_j$, where $\{\alpha_i, \alpha_j\}$ generates a subsystem of type G_2.

(c) $\beta = \alpha_j + n\delta$, where $\{\alpha_i, \alpha_j\}$ generates an affine subsystem of type $BC_1^{(2)}$. Moreover $\langle \alpha_j, \alpha_i^\vee \rangle = -1$, $\langle \alpha_i, \alpha_j^\vee \rangle = -4$ and δ is a null-root of this subsystem.

(d) $\langle \beta, \alpha_j^\vee \rangle > 0$ and α_j is the unique element of Π with this property (i.e., $\langle \beta, \alpha_k^\vee \rangle \leq 0$ for all $k \neq j$). Moreover, $\langle \alpha_i, \alpha_j^\vee \rangle < -1$, $\langle \alpha_j, \alpha_i^\vee \rangle = -1$ and $\langle \beta, \alpha_j^\vee \rangle = 1$.

If (a) holds then we are either in Case F(i), Case A, or Case I according to whether $a \in \{1, 2, 3\}$, $a=4$ or $a>4$, respectively.

If (b) holds, then Case F(ii) holds.

If (c) holds, then Case A holds with $l=2$ as can be seen by setting $i_1 = i$, $i_2 = j$.

Assume (d) holds. We first note that β does not belong to the subroot system Δ' generated by α_i and α_j. Otherwise $r_j\beta = \beta - \alpha_j \in {}^{\mathrm{re}}\Delta'_+$, which is not possible given that $\langle r_j\beta, \alpha_i^\vee \rangle = 0$ and $\langle r_j\beta, \alpha_j^\vee \rangle = -1$. Let $i_1 = i$, $i_2 = j$, and $\beta_2 = \beta$. Then $J_2 := \{i_1, i_2\}$ satisfy Ind 1 while β satisfies Ind 2 of Proposition 6. Thus if we let $\beta_3 = r_{i_2}(\beta_2) = \beta_2 - \alpha_{i_2} \in {}^{\mathrm{re}}\Delta_+$, then there exists a unique $i_3 \in I \setminus J_2$ such that with $J_3 := J_2 \cup \{i_3\}$ either

(1.3) A_{J_3} is affine of type $BC_2^{(2)}$ and $\beta_3=\alpha_{i_3}+n\delta$, where δ is a null root of this subroot system or

(2.3) $\beta_3=\alpha_{i_3}$ and $\langle\alpha_{i_2},\alpha_{i_3}^\vee\rangle<-1$ or

(3.3) $\langle\beta_3,\alpha_{i_3}^\vee\rangle=1\,,\quad\langle\beta_3,\alpha_{i_2}^\vee\rangle=-1$

$\langle\alpha_{i_3},\alpha_{i_2}^\vee\rangle=1\,,\quad\langle\alpha_{i_2},\alpha_{i_1}^\vee\rangle=-1$

$\langle\alpha_{i_3},\alpha_{i_1}^\vee\rangle=0$ (Proposition 6(iii))

$\langle\alpha_{i_1},\alpha_{i_2}^\vee\rangle<-1\,,$ and

$\langle\beta_3,\alpha_{i_k}^\vee\rangle\le 0$ for all $k\notin J_3$ Proposition 6(v) .

If (1.3) holds, then we are in Case A of the theorem with α_{i_1}, α_{i_2}, α_{i_3}.

If (2.3) holds then Case A with $n=0$ or Case I of the theorem holds.

Assume (3.3) holds. Then J_3 and β_3 satisfy the assumptions Ind 1 and Ind 2 of Proposition 6. Thus there exists a unique i_4 and $\beta_4:=r_{i_3}\beta_3=r_{i_3}r_{i_2}\beta$ such that $J_4=J_3\cup\{i_4\}$ and β_4 satisfy the assumptions of Proposition 6.

What we have is an algorithm that creates in step $l\ge 3$ a sequence of distinct indices $J_l=\{i_1,\dots,i_l\}$ and of positive roots $\beta_2,\beta_3,\dots,\beta_l$ of *decreasing* height such that $\beta_l=r_{i_{l-1}}\cdots r_{i_2}\beta=\beta_{l-1}-\alpha_{i_{l-1}}$ and either

(1.l) Case A holds for the sequence $\alpha_{i_1},\dots,\alpha_{i_l}$

(2.l) $\beta_l=\alpha_{i_l}$ and $\langle\alpha_{i_{l-1}},\alpha_{i_l}^\vee\rangle<-1$ or

(3.l) J_l and β_l satisfy the assumptions Ind 1 and Ind 2 of Proposition 6.

It follows that for some $3\le l\le\mathrm{ht}(\beta)+2$ it is the case that (3.$l-1$) and either (1.l) and (2.l) hold. We then have

$$r_{i_{l-1}}\cdots r_{i_2}\alpha_{i_1}=\alpha_{i_1}-\langle\alpha_{i_1},\alpha_{i_2}^\vee\rangle\alpha_{i_2}-\cdots-\prod_{k=1}^{l-2}\langle\alpha_{i_k},\alpha_{i_{k+1}}^\vee\rangle\alpha_{i_{l-1}} \tag{4}$$

(because of (3.$l-1$)), while by Proposition 6(i) applied to $\alpha:=\alpha_{i_l}=\beta_{i_{l-1}}-\alpha_{i_{l-1}}$

$$\langle\alpha_{i_k},\alpha_l^\vee\rangle=0\qquad\text{for all}\quad 1\le k<l\,. \tag{5}$$

Thus by (4) and (5)

$$a=|\langle\alpha_{i_1},\beta^\vee\rangle|=|\langle r_{i_{l-1}}\cdots r_{i_2}\alpha_{i_1},\alpha_{i_l}^\vee\rangle|=\left|\prod_{k=1}^{l-1}\langle\alpha_{i_k},\alpha_{i_{k+1}}^\vee\rangle\right|.$$

□

COROLLARY. *If $\{\alpha,\beta\}$ generate a subsystem of type*

$$\begin{pmatrix}2 & -p\\ -1 & 2\end{pmatrix}$$

where p is a prime number, then there exists $w \in W$ such that $w\alpha$, $w\beta \in \Pi$.

REMARK 1. Case F of this theorem was proved in [5].

REMARK 2. The results of this paper hold also for root systems of a *set of root data* (cf. [8] and [9, Ch. 5]).

We now state a conjecture which is related to this work: Let $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a Kac-Moody algebra corresponding to a generalized Cartan matrix A, and let $\mathfrak{s}_w := \mathfrak{n}_+ \cap w(\mathfrak{n}_-)$ for $w \in W$. The subalgebra $\mathfrak{s}_w$ is nilpotent (since it is finite-dimensional and $\mathfrak{n}_+$ is residually nilpotent).

CONJECTURE 1. *The degree of nilpotency of $\mathfrak{s}_w$ is bounded by a constant which depends on A but not on w.*

REFERENCES

[1] V. G. KAC, Infinite-Dimensional Lie Algebras, Birkhauser, Boston, 1983.
[2] V. G. KAC AND D. H. PETERSON, Defining relations of infinite-dimensional groups, Proceedings of the E. Cartan conference, Lyon, 1984.
[3] J. MORITA, Root strings with three or four real roots in Kac-Moody root systems, Tôhoku Math. J. (4) 40 (1988), 645–650.
[4] J. MORITA, Certain rank two subsystems of Kac-Moody root systems, Infinite-Dimensional Lie Algebras and Groups, Proceedings of the conference held at CIRM, Luminy, Marseille, 1988.
[5] J. MORITA, Kac-Moody groups over fields with trivial K_2, Comm. Algebra (5) 19 (1991), 1541–1544.
[6] J. HURRELBRINK, J. MORITA AND U. REHMANN, On the homological π_0 of Kac-Moody groups over fields, Cont. Math. 126 (1992), 71–77.
[7] J. MORITA, Converings, Schur multipliers and K_2 of Kac-Moody groups (to appear).
[8] R. V. MOODY AND A. PIANZOLA, On infinite root systems, Trans. Amer. Math. Soc. 315 (1989), 661–696.
[9] R. V. MOODY AND A. PIANZOLA, Lie Algebras with Triangular Decomposition, John Wiley and Sons, 1995.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA
CANADA T6G 2G1