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REMARKS ON A CONJECTURE OF BATYREV AND MANIN

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Abstract. We give several remarks on a conjecture of Batyrev and Manin on the distribution of rational points on algebraic varieties. We study the signature of the geometric invariant of Batyrev-Manin in relation to the value of the Kodaira dimension, and study the validity of the Batyrev-Manin conjecture under the assumption that there exists an unramified covering for which the Batyrev-Manin conjecture holds.

Introduction.

0–1. A conjecture of Batyrev and Manin. Let k be an algebraic number field of finite degree, V a non-singular projective variety defined over k, and K_V the canonical line bundle of V. Let $\mathrm{NS}(V)$ be the Néron-Severi group of $\bar{V} := V \times_k \bar{\boldsymbol{Q}}$, $\mathrm{N}^1(V) := \mathrm{NS}(V) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$, and $\mathrm{N}^1_{\mathrm{eff}}(V)$ the closed cone in $\mathrm{N}^1(V)$ generated by effective divisors on $\bar{V}$.

Let L be an ample line bundle on V defined over k. We defined a *geometric invariant* $\alpha(L)$ of L by

$$\alpha(L) := \inf\{\gamma \in \boldsymbol{R} \mid \gamma \cdot [L] + K_V \in \mathrm{N}^1_{\mathrm{eff}}(V)\} .$$

Since L and K_V are defined over k, it is easy to see that this invariant $\alpha(L)$ does not change even if we replace the Néron-Severi group of $\bar{V}$ by the Néron-Severi group of V. In this sense, geometric invariants are independent of a special choice of a field of definition.

Let $H_L : V(\bar{\boldsymbol{Q}}) \to \boldsymbol{R}_{\geq 0}$ be an exponential height of L. Hence H_L is replaced by $H_L^{[M:k]}$ if k is replaced by a finite extension M of k. For any non-empty Zariski open k-subset U of V, we define a function $Z_U(L; s)$ of s by

$$Z_U(L; s) := \sum_{x \in U(k)} H_L(x)^{-s} .$$

This Dirichlet series converges if $\mathrm{Re}(s)$ is sufficiently large. We define an *arithmetic invariant* $\beta_U(L)$ of L as the inferior limit of $\sigma \in \boldsymbol{R}$ such that this series $Z_U(L; s)$ converges for $s = \sigma$.

It is known that $\beta_U(L) \geq 0$ or $= -\infty$, and that $\beta_U(L) = -\infty$ holds if and only if $U(k)$ is a finite set. We note that this invariant $\beta_U(L)$ depends only on L (and U).

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Namely, though H_L is defined modulo bounded functions, $\beta_U(L)$ does not depend on a special choice of H_L from its class.

Batyrev and Manin introduced in [B-M] three conjectures on these invariants. The first conjecture is the following:

CONJECTURE (Batyrev-Manin). *For any positive number $\varepsilon > 0$, there would exist an open dense k-subset $U = U(L, \varepsilon)$ of V such that*

$$\beta_U(L) \leq \alpha(L) + \varepsilon .$$

In this paper, we give several remarks on this conjecture. We do not discuss other conjectures of [B-M] which describe the distribution of rational points on varieties with positive geometric invariants.

0–2. Results. In §1, we study the relationship between the signature of the geometric invariant $\alpha(L)$ and the value of the Kodaira dimension $\kappa(V)$ of the algebraic variety V.

We show:

(a) if $\kappa(V) \geq 0$, then $\alpha(L) \leq 0$;

(b) if $\kappa(V) = \dim V$, then $\alpha(L) < 0$.

Further, if V is a non-singular algebraic surface, and if L is an ample line bundle on V, then we prove the following (cf. Theorem 1):

(1) $\alpha(L) > 0$ if and only if $\kappa(V) = -\infty$;

(2) $\alpha(L) = 0$ if and only if $\kappa(V) = 0$ or 1;

(3) $\alpha(L) < 0$ if and only if $\kappa(V) = 2$.

In §2, we study the Batyrev-Manin conjectures for an unramified covering of algebraic varieties.

Let V be a non-singular projective variety defined over an algebraic number field k of finite degree, and let L be an ample line bundle on V.

We prove that, if $f: \tilde{V} \to V$ is an unramified covering of V defined over a finite extension M of k with $\alpha(L) \leq 0$, and if the Batyrev-Manin conjecture holds for the ample line bundle $f^*(L \times_k M)$ on $\tilde{V}$, then the Batyrev-Manin conjecture holds also for the ample line bundle L on V (cf. Proposition 5).

Further, if $f: \tilde{V} \to V$ is an unramified Galois covering of V defined over k with Galois group A, then we construct a $\deg(f)$-to-one correspondence of the union for all $\xi \in \mathrm{H}^1(G, A)$ of the sets $\tilde{V}_\xi(k)$ of all k-rational points on the twists $\tilde{V}_\xi$ of $\tilde{V}$ to the set $V(k)$ of all rational points on V (cf. Proposition 6):

$$f_*: \coprod_{\xi \in \mathrm{H}^1(G,A)} \tilde{V}_\xi(k) \longrightarrow V(k) .$$

By using this correspondence, we prove that if the Batyrev-Manin conjecture holds for all $\tilde{V}_\xi$, then the Batyrev-Manin conjecture holds for V (cf. Theorem 2).

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1. The signature of the geometric invariant $\alpha(L)$.

1–1. Results in the general case. Let k be an algebraic number field of finite degree, V a non-singular projective variety defined over k, and K_V the canonical line bundle of V. Let $R(V)$ denote the canonical ring:

$$R(V) := \boldsymbol{C} \oplus \sum_{m \geq 1} \Gamma(V, K_V^{\otimes m}).$$

We define the *Kodaira-dimension* $\kappa(V)$ of V by

$$\kappa(V) := \begin{cases} -\infty & \cdots \quad \text{if } R(V) = \boldsymbol{C} \\ \operatorname{trdeg}_{\boldsymbol{C}} R(V) - 1 & \cdots \quad \text{otherwise.} \end{cases}$$

Hence $\kappa(V) = -\infty$, or $0 \leq \kappa(V) \leq \dim(V)$. It is known that the Kodaira-dimension is a birational invariant (cf., e.g., [U]).

We assume that $\kappa(V) \geq 0$. Then there exists an integer $m \geq 1$ such that $\Gamma(V, K_V^{\otimes m})$ has a non-zero section. Hence $K_V^{\otimes m}$ is linearly equivalent to an effective divisor. It follows that $K_V \in \mathrm{N}^1_{\mathrm{eff}}(V)$ and hence $\alpha(L) \leq 0$. Hence we have proved the following proposition:

PROPOSITION 1. *If the Kodaira dimension of V is not $-\infty$, then for any ample line bundle L on V, the geometric invariant satisfies $\alpha(L) \leq 0$.*

We assume that V is of *general type*. Namely, we assume that $\kappa(V) = \dim(V)$. Then, by Theorem 8.1 of [U], there exist positive integers d, m_0 and positive real numbers a, b such that for any $m \geq m_0$, we have

$$a \cdot m^{\dim V} \leq l(K_V^{\otimes md}) \leq b \cdot m^{\dim V}.$$

Hence, by a result of K. Kodaira (cf. [C-S, p. 349, Lemma 4.1]), for any ample line bundle L on V, there is a positive integer n such that $\Gamma(V, K_V^{\otimes nd} \otimes L)$ has a non-zero section. Hence $K_V^{\otimes nd} \otimes L^{-1}$ is linearly equivalent to an effective divisor. It follows that $K_V^{\otimes nd} \otimes L^{-1} \in \mathrm{N}^1_{\mathrm{eff}}(V)$. Hence

$$-(nd)^{-1}[L] + K_V \in \mathrm{N}^1_{\mathrm{eff}}(V)$$

and hence $\alpha(L) \leq -(nd)^{-1} < 0$.

Now we assume that the Batyrev-Manin conjecture holds for V and L. Then there exists an open dense k-subset $U = U(L, -\alpha(L)/2)$ of V such that

$$\beta_U(L) \leq \alpha(L) - \alpha(L)/2 = \alpha(L)/2 < 0.$$

It follows that $U(k)$ is a finite set, because $\beta_U(L) < 0$. Hence the Batyrev-Manin conjecture implies the conjecture of E. Bombieri and P. Vojta (cf. [C-S, pp. 349–350]).

Therefore we have proved the following proposition:

PROPOSITION 2. *If V is a non-singular algebraic variety of general type, then for any ample line bundle L on V, the geometric invariant $\alpha(L)$ is negative. Hence, if the Batyrev-Manin conjecture holds for V and L, then there exists an open dense k-subset U*

of V such that U has only a finite number of k-rational points.

1–2. Results for non-singular algebraic surfaces. From now on until the end of §1, *we assume that V is a non-singular projective algebraic surface.* Hence $\kappa(V) = -\infty, 0, 1$, or 2.

Let $f : \tilde{V} \to V$ be a σ-transform (one point blow up) of a non-singular surface, P the center of f, and $E := f^{-1}(P)$ the exceptional curve of the first kind. Let K_V be the canonical line bundle, and L a very ample line bundle on V. Then the canonical bundle $K_{\tilde{V}}$ of $\tilde{V}$ satisfies $K_{\tilde{V}} = f^*(K_V) \otimes \mathcal{O}(E)$ (cf. [B-P-V] and [H]), and $\tilde{L} := f^*(L)^{\otimes 2} \otimes \mathcal{O}(-E)$ is ample on $\tilde{V}$ (cf. [H, p. 394, Ex. 3.3]).

For any real number γ, by taking f_*, we have

$$\gamma \cdot [\tilde{L}] + K_{\tilde{V}} \in \mathrm{N}^1_{\mathrm{eff}}(\tilde{V}) \Longrightarrow 2\gamma \cdot [L] + K_V \in \mathrm{N}^1_{\mathrm{eff}}(V).$$

Hence the inequality $\alpha(L)/2 \leq \alpha(\tilde{L})$ holds in any case.

On the other hand, we can express

$$\begin{aligned}\gamma \cdot [\tilde{L}] + K_{\tilde{V}} &= \gamma \cdot (2[f^*(L)] - E) + f^*(K_V) + E \\ &= f^*(2\gamma \cdot [L] + K_V) + (-\gamma + 1) \cdot E \\ &= f^*(\gamma \cdot [L] + K_V) + \gamma \cdot (f^*(L) - E) + E .\end{aligned}$$

If $\alpha(L) < 0$, then for any γ satisfying $\alpha(L) < 2\gamma \leq 0$, we have

$$f^*(2\gamma \cdot [L] + K_V) + (-\gamma + 1) \cdot E \in \mathrm{N}^1_{\mathrm{eff}}(\tilde{V}) .$$

Thus $\alpha(\tilde{L}) \leq \gamma$, and $\alpha(\tilde{L}) \leq \alpha(L)/2$. Here we note

$$\alpha(L) = 0 \Longrightarrow K_V \in \mathrm{N}^1_{\mathrm{eff}}(V) \Longrightarrow \alpha(\tilde{L}) \leq 0 ,$$

because $\mathrm{N}^1_{\mathrm{eff}}(V)$ is closed. Hence the inequality $\alpha(\tilde{L}) \leq \alpha(L)/2 \leq 0$ holds for any $\alpha(L) \leq 0$. Therefore we obtain

$$\alpha(L) \leq 0 \Longrightarrow \alpha(\tilde{L}) = \alpha(L)/2 \leq 0 .$$

If $\alpha(L) > 0$, let γ be a positive number such that $\gamma > \alpha(L)$. Then $\gamma \cdot [L] + K_V \in \mathrm{N}^1_{\mathrm{eff}}(V)$. Further, since L is very ample, there is an effective divisor D such that $\mathcal{O}(D) \cong L$ and the support of D contains P. Then $f^*(L) - E \in \mathrm{N}^1_{\mathrm{eff}}(\tilde{V})$. Hence

$$f^*(\gamma \cdot [L] + K_V) + \gamma \cdot (f^*(L) - E) + E \in \mathrm{N}^1_{\mathrm{eff}}(\tilde{V}) .$$

Hence $\alpha(\tilde{L}) \leq \alpha(L)$, and

$$\alpha(L) > 0 \Longrightarrow 0 < \alpha(L)/2 \leq \alpha(\tilde{L}) \leq \alpha(L) .$$

Therefore we have proved the following proposition:

PROPOSITION 3. *Let $f : \tilde{V} \to V$ be a σ-transform of a non-singular algebraic surface, and let L be a very ample line bundle on V. Then the geometric invariant $\alpha(L)$ of L and the geometric invariant $\alpha(\tilde{L})$ of the ample line bundle $\tilde{L} := f^*(L)^{\otimes 2} \otimes \mathcal{O}(-E)$ on $\tilde{V}$ have* *the same signature.*

REMARK. It is known that the signature of the geometric invariant $\alpha(L)$ does not depend on a special choice of an ample line bundle L (cf. [B-M]). Hence this proposition shows that the signature of the geometric invariants of ample line bundles on a non-singular algebraic surface does not change by any σ-transform.

It is well-known that for any birational map $f: U \to V$ of non-singular algebraic surfaces, there exist an algebraic surface W and birational morphisms $\pi_1: W \to U$ and $\pi_2: W \to V$ such that π_1 and π_2 are composites of σ-transforms, and $f \circ \pi_1 = \pi_2$ (cf., e.g., [B-P-V, Chap. III, Theorem 6.3]). Hence Proposition 3 and the above remark imply the following:

COROLLARY. *The signature of the geometric invariants of ample line bundles on a non-singular algebraic surface is a birational invariant.*

Let V be a (relatively) minimal non-singular algebraic surface, and L an ample line bundle on V.

If $\kappa(V) = 2$, then, by Proposition 2, $\alpha(L)$ is negative.

If $\kappa(V) = 1$, then V is a proper elliptic surface (cf., e.g, ibid., Chap. V). Hence there exists a minimal elliptic fibration $\pi: V \to C$ over a non-singular curve C. Further, by the canonical bundle formula (cf., ibid., p. 161, Theorem 12.1), there exist an integer $m > 0$ and an effective divisor D on C such that $K_V^{\otimes m} \cong f^*(D)$ (cf., ibid., pp. 161–164 also). Hence $K_V \in \mathrm{N}^1_{\mathrm{eff}}(V)$, and hence $\alpha(L) \leq 0$. Further, $K_V \cdot F = 0$ holds for any fibre F of π.

If $\alpha(L) < 0$, then there is a positive integer n such that $-[L] + K_V^{\otimes n} \in \mathrm{N}^1_{\mathrm{eff}}(V)$. Then, taking the intersection with the fibre F, we obtain $-L \cdot F \geq 0$. Since L is ample, we have $L \cdot F > 0$, a contradiction. Hence we obtain $\alpha(L) = 0$.

If $\kappa(V) = 0$, then, by the result of the classification of minimal algebraic surfaces, the canonical line bundle K_V satisfies $K_V^{\otimes 12} \cong \mathcal{O}_V$ (cf., e.g., ibid., Chap. V). Hence $K_V = 0$ in $\mathrm{N}^1_{\mathrm{eff}}(V)$. It follows that $\alpha(L) = 0$ holds for any ample line bundle L.

If $\kappa(V) = -\infty$, then there exist a non-singular algebraic curve C and a birational map $f: V \to \boldsymbol{P}^1 \times C$. Further, there exist a non-singular algebraic surface $\tilde{V}$ and birational morphisms $\pi_1: \tilde{V} \to V$ and $\pi_2: \tilde{V} \to \boldsymbol{P}^1 \times C$ such that $f \circ \pi_1 = \pi_2$ (cf., e.g., ibid., Chap. V). In view of Corollary to Proposition 3, to study the signature of the geometric invariants of ample line bundles on V, it is sufficient to study the signature of the geometric invariant of an ample line bundle on $\boldsymbol{P}^1 \times C$.

Since the Néron-Severi group of $\boldsymbol{P}^1 \times C$ satisfies $\mathrm{NS}(\boldsymbol{P}^1 \times C) \cong \mathrm{NS}(\boldsymbol{P}^1) \times \mathrm{NS}(C) \cong \boldsymbol{Z} \times \boldsymbol{Z}$ (cf. [H. p. 146, Ex. 6.1]), we identify $\mathrm{NS}(V) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ with $\boldsymbol{R} \times \boldsymbol{R}$. We observe that a line bundle L on $\boldsymbol{P}^1 \times C$ is ample if and only if its image in $\boldsymbol{R} \times \boldsymbol{R}$ is a pair of positive integers. Further, it is easy to see that $\mathrm{N}^1_{\mathrm{eff}}(V) = \{(x, y) \in \boldsymbol{R} \times \boldsymbol{R};\, x, y \geq 0\}$. Since the image in $\boldsymbol{R} \times \boldsymbol{R}$ of the canonical bundle of $\boldsymbol{P}^1 \times C$ is a pair of -2 and an even integer, the geometric invariant $\alpha(L)$ is positive. It follows from Proposition 3 and the remark after it that the signature of the geometric invariant $\alpha(L)$ of an ample line bundle L on V is positive.

Note that we can prove $\alpha(L)>0$ also by using the classification of minimal surfaces with Kodaira dimension $-\infty$ (cf. [B-P-V, pp. 191–192]) and by studying the structure of ruled surfaces (cf. [H, p. 365, Theorem 1.10, p. 370, Proposition 2.3 and p. 374, Corollary 2.11]).

Since every non-singular algebraic surface has a (relatively) minimal model, and since the Kodaira dimension and the signature of the geometric invariants of ample line bundles are birational invariants (cf. Corollary to Proposition 3), we have proved the following result:

THEOREM 1. *Let V be a non-singular algebraic surface, and let L be an ample line bundle on V. Then the Kodaira dimension $\kappa(V)$ and the geometric invariant $\alpha(L)$ satisfies the following*:

(1) $\alpha(L)>0$ *if and only if* $\kappa(V)=-\infty$;
(2) $\alpha(L)=0$ *if and only if* $\kappa(V)=0$ or 1;
(3) $\alpha(L)<0$ *if and only if* $\kappa(V)=2$.

2. Study of the conjecture in unramified coverings.

2–1. The case of $\kappa(V)\geq 0$. Let k be an algebraic number field of finite degree, V a projective non-singular algebraic variety, K_V the canonical line bundle of V, and L an ample line bundle on V. Let $f:\tilde{V}\to V$ be an unramified covering of V defined over $\bar{\boldsymbol{Q}}$.

Let M be a finite extension of k such that $\tilde{V}$ and f are defined over M. Since f is an unramified covering, $f^*(K_V\times_k M)\cong K_{\tilde{V}}$ holds. Further, since f is a finite morphism, $f^*(L\times_k M)$ is an ample line bundle on $\tilde{V}$ (cf. [H, p. 232, Ex. 5.7]).

Since f^* preserves numerically equivalence classes (cf. [Ful, p. 376, Example 19.1.6]), we have a homomorphism $f^*:\mathrm{N}^1(V\times_k M)\to\mathrm{N}^1(\tilde{V})$. Since f^* maps effective divisors on $V\times_k M$ to effective divisors on $\tilde{V}$, we have $f^*(\mathrm{N}^1_{\mathrm{eff}}(V\times_k M))\subset\mathrm{N}^1_{\mathrm{eff}}(\tilde{V})$.

Hence, for any $\gamma\in\boldsymbol{R}$,

$$\gamma\cdot[L]+K_V\in\mathrm{N}^1_{\mathrm{eff}}(V)\Longrightarrow\gamma\cdot[f^*(L\times_k M)]+K_{\tilde{V}}\in\mathrm{N}^1_{\mathrm{eff}}(\tilde{V})$$

holds. Hence $\alpha(L)\geq\alpha(f^*(L\times_k M))$.

We repeat a similar argument for the direct image functor f_*, and obtain $\alpha(L)\leq\alpha(f^*(L\times_k M))$ (cf., ibid., p. 376, Example 19.1.6). Hence $\alpha(L)=\alpha(f^*(L\times_k M))$.

PROPOSITION 4. *Let $f:\tilde{V}\to V$ be an unramified covering of a non-singular projective algebraic variety, L an ample line bundle on V, and $\tilde{L}:=f^*(L)$. Then we have*

$$\alpha(\tilde{L})=\alpha(L).$$

Let k, V, and $f:\tilde{V}\to V$ be as before, and let M be a finite extension of k such that $\tilde{V}$ and f are defined over M. Since $f:\tilde{V}\to V$ is an unramified covering, there exists a finite extension M' of M such that $f^{-1}(V(k))\subset\tilde{V}(M')$ holds (cf. [MW, p. 50, Chevalley-Weil Theorem]).

By replacing the extension M of k with this extension M' if necessary, *we assume that M satisfies $f^{-1}(V(k)) \subset \tilde{V}(M)$. We also assume that the Batyrev-Manin conjecture holds for $\tilde{V}$ and $f^*(L \times_k M)$*. Then, for any $\varepsilon > 0$, there is an open dense M-subset $\tilde{U}$ of $\tilde{V}$ such that

$$\beta_{\tilde{U}}(f^*(L \times_k M)) \leq \alpha(f^*(L \times_k M)) + [M:k]^{-1}\varepsilon .$$

Since $f: \tilde{V} \to V$ is an unramified covering, f is a proper map. Hence the image $f(\tilde{V} \setminus \tilde{U}) \subset V \times_k M$ of the closed set $\tilde{V} \setminus \tilde{U}$ is a closed M-subset of $V \times_k M$. Let U_0 be the complement in $V \times_k M$ of all conjugates of this closed M-subset of $V \times_k M$. Then U_0 has the form $U \times_k M$ with a certain open dense k-subset U of V.

Since

$$H_{f^*(L \times_k M)} = \mathrm{O}(1) \cdot (H_{L \times_k M} \circ f) = \mathrm{O}(1)^{[M:k]} \cdot (H_L \circ f)^{[M:k]} ,$$

there exists a positive number $C > 0$ such that

$$(H_L \circ f)^{-[M:k]} \leq C^{-[M:k]} \cdot (H_{f^*(L \times_k M)})^{-1} .$$

Since $f: \tilde{V} \to V$ induces a deg(f)-to-one correspondence of geometric points, for any $\sigma > 0$. we have

$$\deg(f) \cdot Z_U(L; [M:k]\sigma) \leq C^{-[M:k]\sigma} \cdot Z_{\tilde{U} \times_k M}(f^*(L \times_k M); \sigma) .$$

Hence, if the Dirichlet series which defines the right side of this inequality converges for σ, then the Dirichlet series which defines the left side of this inequality also converges for σ. Hence

$$\beta_U(L) \leq [M:k] \cdot \beta_{\tilde{U} \times_k M}(f^*(L \times_k M)) .$$

Now *we assume* $\alpha(L) \leq 0$. Then we have $[M:k] \cdot \alpha(L) \leq \alpha(L)$. It follows that

$$\begin{aligned}\beta_U(L) &\leq [M:k] \cdot \beta_{\tilde{U} \times_k M}(f^*(L \times_k M)) \\ &\leq [M:k] \cdot \alpha(f^*(L \times_k M)) + \varepsilon = [M:k] \cdot \alpha(L) + \varepsilon \leq \alpha(L) + \varepsilon .\end{aligned}$$

Hence we have proved the following proposition:

PROPOSITION 5. *Let k, V, and L be as before. We assume $\alpha(L) \leq 0$. Let $f: \tilde{V} \to V$ be an unramified covering of V defined over a finite extension M of k. We assume that M is large enough so that the condition $f^{-1}(V(k)) \subset \tilde{V}(M)$ is satisfied. Then the Batyrev-Manin conjecture holds for an ample line bundle L on V, if the Batyrev-Manin conjecture holds for the ample line bundle $f^*(L \times_k M)$ on $\tilde{V}$.*

By the result of [N], the Batyrev-Manin conjecture holds for any abelian variety. Since the canonical line bundle of an abelian variety is trivial, $\alpha(L) = 0$. Hence we obtain the following corollary:

COROLLARY. *Let V be a non-singular algebraic variety defined over k. We assume* *that three exist an abelian variety A and an unramified covering $f: A \to V$ defined over $\bar{Q}$. Then the Batyrev-Manin conjecture holds for any ample line bundle L on V.*

REMARK. This corollary can be applied to hyperelliptic surfaces. Further, by Proposition 5, the Batyrev-Manin conjecture for Enriques surfaces can be reduced to the Batyrev-Manin conjecture for $K3$-surfaces.

2–2. The case of unramified Galois coverings. Let k and V be as before, and let $f: \tilde{V} \to V$ be an unramified covering of V defined over k. *We assume that $f: \tilde{V} \to V$ is a Galois covering* with Galois group A. Hence A is a finite subgroup of the biregular automorphism group $\mathrm{Aut}(\tilde{V})$ of $\tilde{V}$, and $f \circ a = f$ holds for any $a \in A$. Further, for any geometric point P of V, and for any two geometric points Q, R of $\tilde{V}$ such that $f(Q) = f(R) = P$, there exists a unique element $a \in G$ such that $a(Q) = R$. We denote this element of A by a_R^Q.

Let M be a finite Galois extension of k such that $f^{-1}(V(k)) \subset \tilde{V}(M)$ holds (cf. [MW, p. 50, Chevalley-Weil Theorem]) and such that all $a \in A$ are defined over M. Hence, for any k-rational point P of V, all geometric points in $f^{-1}(P)$ are M-rational. Let G be the Galois group $\mathrm{Gal}(M/k)$ of M/k.

Let P be a k-rational point of V, and Q a geometric point of $\tilde{V}$ contained in $f^{-1}(P)$. Let σ be any element of G. Then $f(\sigma(Q)) = \sigma(f(Q)) = \sigma(P) = P = f(Q)$. Hence there exists a unique element $a \in A$ such that $a(Q) = \sigma(Q)$. We denote this $a \in A$ by $a(\sigma) = a(Q; \sigma)$.

Let σ, τ be elements of G. Then $a(\sigma \cdot \tau)(Q) = (\sigma \cdot \tau)(Q) = \sigma(\tau(Q)) = \sigma(a(\tau)(Q)) = \sigma(a(\tau))(\sigma(Q)) = \sigma(a(\tau))(a(\sigma)(Q))$. Hence

$$\sigma(a(\tau)) \circ a(\sigma) = a(\sigma \cdot \tau)$$

holds for any σ, $\tau \in G$. Hence $\xi_Q: G \ni \sigma \mapsto a(Q; \sigma) \in A$ is a 1-cocycle, and satisfies

$$\xi_Q(\sigma)(Q) = \sigma(Q)$$

for any $\sigma \in G$.

If we replace Q by another element R of $f^{-1}(P)$, then $\xi_R = a(R; \sigma)$ satisfies $a(R; \sigma)(a_R^Q(Q)) = a(R; \sigma)(R) = \sigma(R) = \sigma(a_R^Q(Q)) = \sigma(a_R^Q)(\sigma(Q)) = \sigma(a_R^Q)(a(Q; \sigma)(Q))$. Hence we have

$$a(R; \sigma) = \sigma(a_R^Q) \circ a(Q; \sigma) \circ (a_R^Q)^{-1} .$$

In other words, ξ_R and ξ_Q determine the same element of $\mathrm{H}^1(G, A)$. Hence we denote this element of $\mathrm{H}^1(G, A)$ by ξ_P.

Since G and A are finite groups, $\mathrm{H}^1(G, A)$ is also a finite set. Let ξ be an element of $\mathrm{H}^1(G, A)$. Then we can twist $\tilde{V}$ by this 1-cocycle. Namely, there are a non-singular algebraic variety $\tilde{V}_\xi$ defined over k and an M-isomorphism $g = g_\xi: \tilde{V} \to \tilde{V}_\xi$ such that $\xi(\sigma) = \sigma(g)^{-1} \circ g$.

Let P be any k-rational point of V, Q a geometric point of $\tilde{V}$ satisfying $f(Q) = P$, and $\xi_Q \in \mathrm{H}^1(G, A)$ as before. Let $\tilde{V}_{\xi_Q}$ be the twist of $\tilde{V}$ with respect to this 1-cocycle, and $g_Q: \tilde{V} \to \tilde{V}_\xi$ the M-isomorphisms such that $\xi_Q(\sigma) = \sigma(g_Q)^{-1} \circ g_Q$. Put $R := g_Q(Q)$. Since

$\xi_Q(\sigma)(Q)=\sigma(Q)$,

$$R=g_Q(Q)=(\sigma(g_Q)\circ\xi_Q(\sigma))(Q)=\sigma(g_Q)(\xi_Q(\sigma)(Q))=\sigma(g_Q)(\sigma(Q))=\sigma(g_Q(Q))=\sigma(R)$$

holds for any $\sigma\in G$. Hence R is a k-rational point of $\tilde{V}_{\xi_Q}$.

Further, if R is a k-rational point of the twist $\tilde{V}_\xi$ ($\xi\in \mathrm{H}^1(G, A)$), let $g=g_\xi\colon \tilde{V}\to\tilde{V}_\xi$ be the M-isomorphism such that $\xi(\sigma)=\sigma(g)^{-1}\circ g$ holds for any $\sigma\in G$. Then

$$\sigma(g^{-1}(R))=\sigma(g^{-1})(\sigma(R))=\sigma(g^{-1})(R)=(\xi(\sigma)\circ g^{-1})(R)=\xi(\sigma)(g^{-1}(R))\,.$$

Since $f\circ\xi(\sigma)=f$,

$$\sigma(f(g^{-1}(R)))=f(\sigma(g^{-1}(R)))=f(\xi(\sigma)(g^{-1}(R)))=f(g^{-1}(R))$$

holds for any $\sigma\in G$. Hence $f(g^{-1}(R))$ is a k-rational point of V.

Hence we have proved the following proposition:

PROPOSITION 6. *Let k and V be as before, and $f\colon\tilde{V}\to V$ an unramified covering of V defined over k. We assume that $f\colon\tilde{V}\to V$ is a Galois covering with Galois group A. Let M be a finite Galois extension of k such that $f^{-1}(V(k))\subset\tilde{V}(M)$ and such that all $a\in A$ are defined over M. Let G be the Galois group* $\mathrm{Gal}(M/k)$. *Then we have a* $\deg(f)$*-to-one correspondence*

$$f_*\colon \coprod_{\xi\in\mathrm{H}^1(G,A)}\tilde{V}_\xi(k)\ni g_\xi(Q)\mapsto P\in V(k)\,,$$

where Q is an M-rational point of $\tilde{V}$ satisfying $f(Q)=P$, $\tilde{V}_\xi$ is the twist of $\tilde{V}$ with respect to ξ, and $g_\xi\colon\tilde{V}\to\tilde{V}_\xi$ is an M-isomorphism such that $\xi(\sigma)=\sigma(g_\xi)^{-1}\circ g_\xi$ holds for any $\sigma\in G$. Further, $Q\in\tilde{V}_\xi(k)$ holds if and only if $\xi\in\mathrm{H}^1(G, A)$ satisfies $\xi(\sigma)(Q)=\sigma(Q)$ for any $\sigma\in G$.

Let k, V, $f\colon\tilde{V}\to V$, A, M, and G be as above, let K_V be the canonical line bundle on V, and let L be an ample line bundle on V. Since f is unramified, $f^*(K_V)\cong K_{\tilde{V}}$, and $f^*(L)$ is an ample line bundle on $\tilde{V}$.

Let $\xi\in\mathrm{H}^1(G, A)$, $\tilde{V}_\xi$ the twist of $\tilde{V}$ with respect to ξ, and $g=g_\xi\colon\tilde{V}\to\tilde{V}_\xi$ an M-isomorphism such that $\xi(\sigma)=\sigma(g)^{-1}\circ g$.

Since $f\circ\xi(\sigma)\circ f^{-1}=\mathrm{id}_V$ and $\xi(\sigma)=\sigma(g)^{-1}\circ g$ hold for any $\sigma\in G$, the composite

$$f_\xi=f\circ g^{-1}\colon\tilde{V}_\xi\to V$$

of two M-morphisms is an M-morphism, and satisfies

$$\begin{aligned}\sigma(f_\xi)(\sigma(x))&=\sigma(f_\xi(x))=\sigma(f(g^{-1}(x)))=f(\sigma(g^{-1}(x)))=f(\sigma(g^{-1})(\sigma(x)))\\&=(f\circ\xi(\sigma)\circ g^{-1})(\sigma(x))=(f\circ g^{-1})(\sigma(x))=f_\xi(\sigma(x))\end{aligned}$$

for any $\sigma\in G$ and for any geometric points $x\in\tilde{V}_\xi$. Hence $f_\xi\colon\tilde{V}_\xi\to V$ is an unramified covering defined over k, and satisfies $f=f_\xi\circ g$.

Since $f_\xi\colon\tilde{V}_\xi\to V$ is an unramified covering, the inverse image $f_\xi^*(K_V)$ of the canonical line bundle of V is isomorphic to the canonical line bundle $K_{\tilde{V}_\xi}$ of $\tilde{V}_\xi$. Further, $L_\xi=f_\xi^*(L)$

is an ample line bundle on $\tilde{V}_\xi$, and satisfies $g^*(L_\xi) \cong f^*(L)$.

Let $H_L: V(\bar{\mathbf{Q}}) \to \mathbf{R}$ be the exponential height of the ample line bundle L, and let $H_{L_\xi}: \tilde{V}_\xi(\bar{\mathbf{Q}}) \to \mathbf{R}$ be the exponential height of L_ξ. Then, for any $R \in \tilde{V}_\xi(\bar{\mathbf{Q}})$, we have

$$H_L(f_\xi(R)) = \mathrm{O}(1) \cdot (H_{L_\xi}(R)) .$$

Hence the deg(f)-to-one correspondence in Proposition 6 preserves heights up to a non-zero bounded function $\mathrm{O}(1) > 0$.

Hence we have proved the following propositon:

PROPOSITION 7. *Let the notation and assumptions be as in Proposition* 6. *Then there is a k-morphism* $f_\xi: \tilde{V}_\xi \to V$ *satisfying* $f = f_\xi \circ g_\xi$, *and* $\tilde{V}_\xi$ *becomes an unramified covering of* V *by this morphism. Let* K_V *be the canonical line bundle on* V, *and let* L *be an ample line bundle on* V. *Then* $f_\xi^*(K_V)$ *is isomorphic to the canonical line bundle of* $\tilde{V}_\xi$, *and* $L_\xi := f_\xi^*(L)$ *is an ample line bundle on* $\tilde{V}_\xi$. *Furthermore, we have*

$$H_L \circ f_\xi = \mathrm{O}(1) \cdot H_{L_\xi}$$

as functions on $\tilde{V}_\xi(\bar{\mathbf{Q}})$, *and the* deg($f$)*-to-one correspondence in Proposition* 6 *preserves heights up to a non-zero bounded function.*

Let the notation and assumptions be as in Proposition 7. *We assume that the Batyrev-Manin conjecture holds for* $\tilde{V}_\xi$ *and* $L_\xi := f_\xi^*(L)$ *for any* $\xi \in \mathrm{H}^1(G, A)$. Then, for any positive number ε, there is an open dense k-subset U_ξ of $\tilde{V}_\xi$ such that

$$\beta_{U_\xi}(L_\xi) \leq \alpha(L_\xi) + \varepsilon .$$

By Proposition 4, geometric invariants do not change in unramified covering. Hence $\alpha(L_\xi) = \alpha(L)$. It follows that

$$\beta_{U_\xi}(L_\xi) \leq \alpha(L) + \varepsilon .$$

Since $f_\xi: \tilde{V}_\xi \to V$ is a proper morphism, the image $f_\xi(\tilde{V}_\xi \setminus U_\xi)$ of the closed set $\tilde{V}_\xi \setminus U_\xi$ is a proper closed k-subset of V. We define a subset U of V by

$$U := \bigcap_{\xi \in \mathrm{H}^1(G, A)} (V \setminus f_\xi(\tilde{V}_\xi \setminus U_\xi)) .$$

Then U is an open dense k-subset of V.

Let P be a k-rational point of U, and let $R = g_\xi(Q)$ be a k-rational point of $\tilde{V}_\xi$ that corresponds to P by Proposition 6. Then R is a k-rational point of U_ξ. Further, by Proposition 7, there is a non-zero bounded function $\mathrm{O}(1) > 0$ on $\tilde{V}_\xi(\bar{\mathbf{Q}})$ such that the equality

$$\mathrm{H}_L(f_\xi(R)) = \mathrm{O}(1) \cdot (H_{L_\xi}(R))$$

holds for any $R \in \tilde{V}_\xi(\bar{\mathbf{Q}})$. Hence, for any $\sigma \in \mathbf{R}$, we have

$$\deg(f)\cdot Z_U(L:\sigma)=\deg(f)\cdot \sum_{P\in U(k)} H_L(x)^{-\sigma}$$

$$\leq C^{-\sigma}\cdot \sum_{\xi\in \mathrm{H}^1(G,A)} \sum_{R\in U_\xi(k)} H_{L_\xi}(R)^{-\sigma}=C^{-\sigma}\cdot \sum_{\xi\in \mathrm{H}^1(G,A)} Z_{U_\xi}(L_\xi;\sigma),$$

for a positive constant $C>0$.

By the definition of the arithmetic invariant $\beta(L_\xi)$, the zeta function $Z_{U_\xi}(L_\xi;\sigma)$ converges for any $\sigma>\beta(L_\xi)$. Since $\alpha(L)+\varepsilon\geq\beta(L_\xi)$, the zeta function $Z_U(L;\sigma)$ converges for any $\sigma>\alpha(L)+\varepsilon$. Hence we have

$$\beta_U(L)\leq\alpha(L)+\varepsilon,$$

and the Batyrev-Manin conjecture holds for V and L.

Therefore we have proved the following theorem:

THEOREM 2. *Let V be a non-singular projective algebraic variety defined over k, L an ample line bundle over V, and $f:\tilde{V}\to V$ an unramified Galois covering of V defined over k with Galois group A. Let M be a finite Galois extension of k such that $f^{-1}(V(k))\subset\tilde{V}(M)$ and such that all $a\in A$ are defined over M, and let G be the Galois group* $\mathrm{Gal}(M/k)$. *For any $\xi\in\mathrm{H}^1(G,A)$, let $\tilde{V}_\xi$, $f_\xi:\tilde{V}_\xi\to V$, and $L_\xi:=f_\xi^*(L)$ be the twists of $\tilde{V}$ with respect to ξ, the unramified covering of V, and the ample line bundle on $\tilde{V}_\xi$, respectively* (*cf. Proposition* 7). *We assume that the Batyrev-Manin conjecture holds for $\tilde{V}_\xi$ and L_ξ for any $\xi\in\mathrm{H}^1(G,A)$. Then the Batyrev-Manin conjecture holds for V and L.*

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