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ON THE DRESSING ACTION OF LOOP GROUPS ON CONSTANT MEAN CURVATURE SURFACES

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Abstract. We deal with the question of whether the Hopf differential on constant mean curvature surfaces parameterizes the dressing orbits of these surfaces. It is shown that in addition to the Hopf differential, there are infinitely many dressing invariants associated with an umbilic point of order larger than or equal to 2. Thus, when such an umbilic point is present, there are many dressing orbits sharing the same Hopf differential. We also give a procedure for computing all these dressing invariants associated with an umbilic point and they are used to show that the sizes of the dressing orbits in general depend on the topology on the loop group used.

The dressing action in soliton theory was first introduced into the study of harmonic maps by Uhlenbeck [7] and then clarified by Bergvelt and Guest [1] and Guest and Ohnita [6], among others. It was shown in [5] (also see [2]) that every constant mean curvature (CMC) torus in space, whose Gauss map is harmonic, is in the dressing orbit through the cylinder. This result implies that any two CMC tori are dressing equivalent. Naturally, one wonders if any two compact CMC surfaces of a fixed genus ≥ 2 can always be put in one dressing orbit and, if that is the case, what the "simplest" CMC surface in such an orbit is.

Since the dressing action is fairly complicated, not very much is known about it yet. For example, a very basic question has not been answered yet: does the Hopf differential $E(z)(dz)^2$ on CMC surfaces parameterize the dressing orbits? In this paper, we shall show that the answer to this question is negative.

The tool to be used in this paper is the Weierstrass type representation of the CMC surfaces given in [4]: every CMC surface can be constructed from two meromorphic differentials on its universal covering via two loop group factorizations. A certain 2×2 matrix formed by these two meromorphic differentials is called the normalized potential for the surface. We look at the action on normalized potentials for CMC surfaces corresponding to the dressing action on CMC surfaces, still called the dressing action. The formal integrability of the equation defining the corresponding transformations among normalized potentials, called dressing transformations, is investigated. It turns out that if $E(z)(dz)^2$ has a zero (i.e., the surfaces have an umbilic point) of order ≥ 2, then there are infinitely many dressing invariants associated with the zero. As a consequence of this fact, there are at least C^∞-many dressing orbits of CMC surfaces with an umbilic point of order ≥ 2. We also show that these dressing invariants are

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unique among the quantities of similar forms.

Actually, given two normalized potentials P and Q on which the dressing invariants take the same values, the equation on the dressing transformation from P to Q is formally solvable. One certainly wants to know when the formal solution converges. Even though this difficult question is not answered here, we address a relatively easy one: does the topology on loop groups affect the convergence of a formal dressing transformation?

The second purpose of this paper is to show that if $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on the loop group involved such that the dressing action is defined, the Weierstrass type representation [4] holds under both of them, and $\mathscr{T}_1$ is weaker than $\mathscr{T}_2$ when restricted to the "positive loops", then the dressing orbit under $\mathscr{T}_2$ through a CMC surface having an umbilic point is always smaller than the corresponding orbit under $\mathscr{T}_1$.

Since the Hopf differential does not parameterize the dressing orbits, it is interesting to know how simple the normalized potential for a CMC surface in a given dressing orbit can be. In a forthcoming publication [9], we will prove that normalized potentials with at most quadratic poles away from umbilic points are dense in each dressing orbit.

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1. The equations for dressing transformations. First, let us recall the dressing action on CMC surfaces and the Weierstrass type representation of CMC surfaces from [4]. Set

$$\sigma = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $D \subseteq \boldsymbol{C}$ be either the whole $\boldsymbol{C}$ or the open unit disc centered at the origin, $f\colon D \to \boldsymbol{R}^3$ a CMC immersion and $\Phi\colon D \to \Lambda SU(2)_\sigma$ the extended frame for the unit normal $\boldsymbol{n}$ of f (such that the corresponding mean curvature $H \equiv 1/2$) satisfying

$$\Phi(0, 0, \cdot) = I. \tag{1.1}$$

Here one uses the fact that the unit normal of f is a harmonic map to $S^2 = SU(2)/SU(1)$. An Iwasawa decomposition $SU(1)^{\boldsymbol{C}} = SU(1) \cdot B$ for $SU(1)^{\boldsymbol{C}}$ is given by

$$\left\{\begin{pmatrix} e^w & 0 \\ 0 & e^{-w} \end{pmatrix};\, w \in \boldsymbol{C}\right\} = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix};\, \theta \in \boldsymbol{R}\right\}\left\{\begin{pmatrix} e^{\rho} & 0 \\ 0 & e^{-\rho} \end{pmatrix};\, \rho \in \boldsymbol{R}\right\}. \tag{1.2}$$

The group $\Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$ acts on the CMC immersions f as follows. For $h_+ \in \Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$, the loop group Iwasawa decomposition $\Lambda SL(2, \boldsymbol{C})_\sigma = \Lambda SU(2)_\sigma \cdot \Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$ yields

$$h_+(\lambda)\Phi(z, \bar{z}, \lambda) = \Psi(z, \bar{z}, \lambda) H_+(z, \bar{z}, \lambda)\,, \tag{1.3}$$

where z is a global coordinate on D, and Ψ is an extended frame for the unit normal of a new CMC surface $g: D \to \boldsymbol{R}^3$, which can be computed from Ψ using the Sym-Bobenko formula, see [4]. $f \mapsto g = (h_+)_\bullet f$ is the so-called dressing transformation corresponding to h_+ on CMC surfaces. The loop group decomposition of the big cell [4] in $\Lambda SL(2, \boldsymbol{C})_\sigma$ as $\Lambda_*^- SL(2, \boldsymbol{C})_\sigma \cdot \Lambda^+ SL(2, \boldsymbol{C})_\sigma$ implies

$$\Phi(z, \bar{z}, \cdot) = \Phi_-(z, \cdot)\Phi_+(z, \bar{z}, \cdot) \quad \text{and} \quad \Psi(z, \bar{z}, \cdot) = \Psi_-(z, \cdot)\Psi_+(z, \bar{z}, \cdot) \tag{1.4}$$

for all $z \in D \setminus S$, where S is at most a discrete subset of $D \setminus \{0\}$. The normalized potential for f with reference point $z=0$ is the meromorphic $(1, 0)$-form

$$P(z) = \Phi_-(z, \lambda)^{-1} d\Phi_-(z, \lambda)\lambda \tag{1.5}$$

on D, whose poles are in S. Let Q be the normalized potential for g with reference point $z=0$. Setting $G_+ = \Phi_+ H_+^{-1} \Psi_+^{-1}$, we have $G_+ : D \setminus S \to \Lambda^+ SL(2, \boldsymbol{C})_\sigma$ with

$$G_+(0, 0, \cdot) = H_+(0, 0, \cdot)^{-1} = h_+^{-1} \in \Lambda_B^+ SL(2, \boldsymbol{C})_\sigma \tag{1.6}$$

and

$$\Psi_- = h_+ \Phi_- G_+\,. \tag{1.7}$$

From (1.7) we obtain

$$\Psi_-^{-1} d\Psi_- = G_+^{-1} \cdot \Phi_-^{-1} d\Phi_- \cdot G_+ + G_+^{-1} dG_+\,, \tag{1.8}$$

i.e., G_+ is holomorphic on $D \setminus S$ and

$$G_+ Q = P G_+ + dG_+ \lambda\,. \tag{1.9}$$

The initial condition (1.6) for G_+ now becomes

$$G_+(0, \cdot) = h_+^{-1}\,. \tag{1.10}$$

REMARK. The equation (1.9) has already appeared in [3].

By [4] or [8], we can set

$$P(z) = \begin{pmatrix} 0 & p(z) \\ E(z)/p(z) & 0 \end{pmatrix} dz\,, \tag{1.11}$$

where $E(z)(dz)^2 = -\langle f_{zz}(z), \boldsymbol{n}(z)\rangle (dz)^2$ is the Hopf differential of f. By the format of G_+,

$$\begin{aligned} G_+(z, \lambda) = &\begin{pmatrix} a_0(z) & 0 \\ 0 & d_0(z) \end{pmatrix} + \begin{pmatrix} 0 & b_1(z) \\ c_1(z) & 0 \end{pmatrix}\lambda \\ &+ \begin{pmatrix} a_2(z) & 0 \\ 0 & d_2(z) \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & b_3(z) \\ c_3(z) & 0 \end{pmatrix}\lambda^3 + \cdots \end{aligned} \tag{1.12}$$

with $a_0 d_0 = 1$. Thus, by (1.9), we can write

$$Q(z) = \begin{pmatrix} 0 & q(z) \\ E(z)/q(z) & 0 \end{pmatrix} dz \,. \tag{1.13}$$

In particular, the Hopf differential $E(z)(dz)^2$ is a global invariant of the dressing action. From (1.11)–(1.13) we see that (1.9) is equivalent to

$$a_0 q = d_0 p \,, \tag{1.14}$$

$$b_n \cdot \frac{E}{q} = c_n p + a'_{n-1} \qquad \text{for odd} \quad n \geq 1 \,, \tag{1.15}$$

$$c_n q = b_n \cdot \frac{E}{p} + d'_{n-1} \qquad \text{for odd} \quad n \geq 1 \,, \tag{1.16}$$

$$a_n q = d_n p + b'_{n-1} \qquad \text{for even} \quad n \geq 2 \,, \tag{1.17}$$

$$d_n \cdot \frac{E}{q} = a_n \cdot \frac{E}{p} + c'_{n-1} \qquad \text{for even} \quad n \geq 2 \,. \tag{1.18}$$

For applications in the sequel, we rewrite (1.15)–(1.18) in the following two ways.

$$c_n = b_n \cdot \frac{E}{pq} - a'_{n-1} \cdot \frac{1}{p} \qquad \text{for odd} \quad n \geq 1 \,, \tag{1.19}$$

$$b'_n E + c'_n pq = 0 \qquad \text{for odd} \quad n \geq 1 \,, \tag{1.20}$$

$$a'_n q + d'_n p = 0 \qquad \text{for even} \quad n \geq 0 \,, \tag{1.21}$$

$$a_n q - d_n p = b'_{n-1} \qquad \text{for even} \quad n \geq 2 \,, \tag{1.22}$$

or

$$a'_n = b_{n+1} \cdot \frac{E}{q} - c_{n+1} p \qquad \text{for even} \quad n \geq 0 \,, \tag{1.23}$$

$$b'_n = a_{n+1} q - d_{n+1} p \qquad \text{for odd} \quad n \geq 1 \,, \tag{1.24}$$

$$c'_n = -a_{n+1} \cdot \frac{E}{p} + d_{n+1} \cdot \frac{E}{q} \qquad \text{for odd} \quad n \geq 1 \,, \tag{1.25}$$

$$d'_n = -b_{n+1} \cdot \frac{E}{p} + c_{n+1} q \qquad \text{for even} \quad n \geq 0 \,. \tag{1.26}$$

Finally, the condition $\det G_+ = 1$ is equivalent to

$$a_0 d_0 = 1 \,, \tag{1.27}$$

$$a_2 d_0 + a_0 d_2 = b_1 c_1 \,, \tag{1.28}$$

(1.29) $$a_0d_n+a_nd_0=\sum_{k=0}^{n/2-1} b_{2k+1}c_{n-2k-1}-\sum_{k=1}^{n/2-1} a_{2k}d_{n-2k} \quad \text{for even } n\geq 4\,.$$

Note that (1.14) and (1.27) together are equivalent to

(1.30) $$a_0^2=\frac{p}{q} \quad \text{and} \quad d_0=a_0\cdot\frac{q}{p}\,,$$

or

(1.31) $$d_0^2=\frac{q}{p} \quad \text{and} \quad a_0=d_0\cdot\frac{p}{q}\,.$$

Moreover, differentiating (1.19) and substituting the result into (1.20), we can rewrite (1.19)–(1.22) as

(1.32) $$c_n=b_n\cdot\frac{E}{pq}-a'_{n-1}\cdot\frac{1}{p} \qquad \text{for odd} \quad n\geq 1\,,$$

(1.33) $$2b'_nE+b_n\left[E'-\left(\frac{p'}{p}+\frac{q'}{q}\right)E\right]=\left(a''_{n-1}-a'_{n-1}\cdot\frac{p'}{p}\right)q \qquad \text{for odd} \quad n\geq 1\,,$$

(1.34) $$a'_nq+d'_np=0 \qquad \text{for even} \quad n\geq 0\,,$$

(1.35) $$a_nq-d_np=b'_{n-1} \qquad \text{for even} \quad n\geq 2\,.$$

When $z=0$ is a non-umbilic point, the solution to (1.33) is

(1.36) $$b_n(z)=b_n(0)\frac{\sqrt{E(0)p(z)q(z)}}{\sqrt{p(0)q(0)E(z)}} +\frac{\sqrt{p(z)q(z)}}{2\sqrt{E(z)}}\int_0^z\left(a''_{n-1}(w)-a'_{n-1}(w)\cdot\frac{p'(w)}{p(w)}\right)\frac{\sqrt{q(w)}}{\sqrt{E(w)p(w)}}\,dw\,.$$

When $z=0$ is an umbilic point, one only needs to solve (1.33) on a small sector $\{z=\rho e^{i\alpha};\ 0<\rho<1,\ -\varepsilon<\alpha<\varepsilon\}$ in D, where $\varepsilon>0$ is small enough, and obtains

(1.37) $$b_n(z)=\frac{\sqrt{p(z)q(z)}}{2\sqrt{E(z)}}\int_0^z\left(a''_{n-1}(w)-a'_{n-1}(w)\cdot\frac{p'(w)}{p(w)}\right)\frac{\sqrt{q(w)}}{\sqrt{E(w)p(w)}}\,dw$$

by integrating (1.33) from z/m to z and then letting $m\to+\infty$.

2. The first set of invariants on the dressing orbits. In this section, we give the first set of invariants on the dressing orbits of CMC surfaces associated with an umbilic point of order ≥ 2 on the surfaces.

Assume that $z=0$ is an umbilic point of order $k\geq 2$, i.e., $E(0)=E'(0)=\cdots=E^{(k-1)}(0)=0$ and $E^{(k)}(0)\neq 0$. Let n be an odd integer. Then, (1.33) implies that, at $z=0$,

$$(2.1)\qquad a''_{n-1}-a'_{n-1}\cdot\frac{p'}{p}=\left(a''_{n-1}-a'_{n-1}\cdot\frac{p'}{p}\right)'=\cdots=\left(a''_{n-1}-a'_{n-1}\cdot\frac{p'}{p}\right)^{(k-2)}=0\,.$$

Next, we spell out what (2.1) means for $n=1$. By (1.30),

$$(2.2)\qquad 2a_0a'_0=\frac{p'}{q}-\frac{pq'}{q^2}\,.$$

Multiplying (2.2) by a_0 yields

$$(2.3)\qquad a'_0=\frac{a_0}{2}\left(\frac{p'}{p}-\frac{q'}{q}\right).$$

Differentiating (2.3) one obtains

$$(2.4)\qquad a''_0=\frac{a_0}{2}\left(\frac{p''}{p}-\frac{q''}{q}-\frac{p'^2}{2p^2}-\frac{p'q'}{pq}+\frac{3q'^2}{2q^2}\right).$$

So,

$$(2.5)\qquad a''_0-a'_0\cdot\frac{p'}{p}=\frac{a_0}{2}\left(\frac{p''}{p}-\frac{q''}{q}-\frac{3p'^2}{2p^2}+\frac{3q'^2}{2q^2}\right).$$

By (2.1) and (2.5), we have

$$(2.6)\qquad \frac{p''}{p}-\frac{3p'^2}{2p^2}=\frac{q''}{q}-\frac{3q'^2}{2q^2}\,,\ \ldots,\ \left(\frac{p''}{p}-\frac{3p'^2}{2p^2}\right)^{(k-2)}=\left(\frac{q''}{q}-\frac{3q'^2}{2q^2}\right)^{(k-2)}$$

at $z=0$. Therefore, we have proved the following result.

THEOREM 2.7. *At an umbilic point $z=0$ of order $k\geq 2$, the quantities*

$$(2.8)\qquad \left(\frac{p''}{p}-\frac{3p'^2}{2p^2}\right)(0),\ \left(\frac{p''}{p}-\frac{3p'^2}{2p^2}\right)'(0),\ \ldots,\ \frac{1}{(k-2)!}\left(\frac{p''}{p}-\frac{3p'^2}{2p^2}\right)^{(k-2)}(0)$$

defined by using the normalized potentials

$$(2.9)\qquad \begin{pmatrix} 0 & p(z) \\ E(z)/p(z) & 0 \end{pmatrix}dz$$

for CMC *surfaces with reference point $z=0$ are invariant under the dressing action.*

Since the normalized potential for a CMC surface depends on the point chosen as the reference point for the loop group factorizations, it is necessary to express the quantities in (2.8) in terms of the geometric quantities of the surface. By [8], this can be done using only the holomorphic part of the induced metric on the surface.

THEOREM 2.10. *At an umbilic point $z=0$ of order $k\geq 2$, the quantities*

(2.11) $$2(\theta''-\theta'^2)(0)\,,\ 2(\theta''-\theta'^2)'(0)\,,\ \cdots,\ \frac{2}{(k-2)!}(\theta''-\theta'^2)^{(k-2)}(0)$$

defined by using the induced metric $4e^{2\omega(z,\bar z)}dzd\bar z$ *on* CMC *surfaces, where* $\theta(z)$ *is the holomorphic part in the Taylor expansion with respect to* $z=0$ *of* $\omega(z,\bar z)$ *as a real analytic function of* z *and* $\bar z$, *are invariant under the dressing action.*

Proof. From [8], if we choose $z=0$ as the reference point for the loop group factorizations, then the normalized potentials is

(2.12) $$\begin{pmatrix} 0 & p(z) \\ E(z)/p(z) & 0 \end{pmatrix} dz$$

with

(2.13) $$p(z)=e^{2\theta(z)-\theta(0)}\,.$$

Then, direct calculations using (2.13) yield the expressions given in (2.11) for the quantities in (2.8). ∎

Note that θ is holomorphic in a neighborhood of $z=0$. The values of the $k-1$ quantities in (2.8) or (2.11) depend on the choice of the complex coordinate on D. Actually, direct computations yield:

Proposition 2.14. *If* $z=z(w)$ *with* $z(0)=0$ *and* $z'(0)\neq 0$, *then the value of the first quantity under the new local coordinate* w *is*

(2.15) $$2(\theta''(0)-\theta'(0)^2)z'(0)^2+\frac{z'''(0)}{z'(0)}-\frac{3z''(0)^2}{2z'(0)^2}\,,$$

where $2(\theta''(0)-\theta'(0)^2)$ *is the value of the first quantity under the original coordinate* z.

For this reason, from now on we are going to use a local coordinate z around the umbilic point $z=0$ of order k such that

(2.16) $$E(z)=z^k\,.$$

Such a local coordinate will be called a *canonical coordinate*. There are exactly $k+2$ canonical coordinates around $z=0$: if z is one, then the others are $e^{2j\pi i/(k+2)}z$ for $j=1,2,\dots,k+1$. Moreover, from the discussions of Section 1, the dressing action respects every canonical coordinate.

Under a canonical coordinate z around $z=0$,

(2.17) $$I(z)=\theta''(z)-\theta'(z)^2$$

will be called the *dressing function* around $z=0$, and

(2.18) $$I(0)\,,\ I'(0)\,,\ \cdots,\ \frac{1}{(k-2)!}I^{(k-2)}(0)$$

the *first set of dressing invariants* associated with $z=0$. Note that under a change of canonical coordinates, the dressing function gets a constant factor and the first set of dressing invariants gain constant factors accordingly. In general, any continuous function in a finite number of coefficients of θ that is invariant under the dressing action will be called a dressing invariant associated with $z=0$. Note that if $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n$ are dressing invariants associated with $z=0$, then so is $f(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n)$ for any continuous function f. We will ignore this freedom. Since the first dressing invariant $I(0)$ is defined by using $\theta(0)$, $\theta'(0)$ and $\theta''(0)$, or equivalently using $p(0)$, $p'(0)$ and $p''(0)$, we will say that $I(0)$ is of *order* 2. Similarly, $I'(0)$ is of order 3, and so on.

We are going to call CMC surfaces $\boldsymbol{C}\to\boldsymbol{R}^3$ CMC planes. As a direct application of the first set of dressing invariants, we have the following examples.

COROLLARY 2.19. *Let $k\geq 2$ be an integer and $(I_0, I_1, \ldots, I_{k-2})$ an ordered $(k-1)$-tuple of complex numbers. Then there exist $k-1$ constants $\theta_2, \theta_3, \ldots, \theta_k$ such that the values of the first set of dressing invariants associated with $z=0$ for the* CMC *plane having normalized potential*

$$\begin{pmatrix} 0 & e^{2\theta(z)} \\ z^k e^{-2\theta(z)} & 0 \end{pmatrix} dz \tag{2.20}$$

with reference point $z=0$ are $I_0, I_1, \ldots, I_{k-2}$, respectively, where $\theta(z)=\theta_2 z^2+\theta_3 z^3+\cdots+\theta_k z^k$. Moreover, there are at least $\boldsymbol{C}^{k-1}$-many dressing orbits of CMC *planes with $z^k(dz)^2$ as their Hopf differentials.*

PROOF. First we note that since the potential in (2.20) is holomorphic on $\boldsymbol{C}$, it is the normalized potential for some CMC plane. Set $\theta_1=0$. For any $k-1$ constants $\theta_2, \theta_3, \ldots, \theta_k$, let $\theta(z)=\theta_1 z+\theta_2 z^2+\cdots+\theta_k z^k$. Then,

$$\theta''(z)-\theta'(z)^2=\sum_{j=0}^{k-2}\left[(j+2)(j+1)\theta_{j+2}-\sum_{l=0}^{j}(l+1)(j-l+1)\theta_{l+1}\theta_{j-l+1}\right]z^j + \text{higher order terms}\,. \tag{2.21}$$

By Theorem 2.10, in order to obtain the desired values $I_0, I_1, \ldots, I_{k-2}$ of the first set of dressing invariants associated with $z=0$ using such a $\theta(z)$, one only needs to choose the coefficients of $\theta(z)$ by

$$\theta_{j+2}=\frac{1}{(j+2)(j+1)}\left[I_j+\sum_{l=0}^{j}(l+1)(j-l+1)\theta_{l+1}\theta_{j-l+1}\right] \tag{2.22}$$

for $j=0, 1, \ldots, k-2$.

If two distinct ordered $(k-1)$-tuples $(I_0, I_1, \ldots, I_{k-2})$ and $(J_0, J_1, \ldots, J_{k-2})$ of complex numbers satisfy

$$(e^{4j\pi i/(k+2)}I_0, e^{6j\pi i/(k+2)}I_1, \ldots, e^{2kj\pi i/(k+2)}I_{k-2})\neq(J_0, J_1, \ldots, J_{k-2}) \tag{2.23}$$

for $j=1, 2, \dots, k+1$, then the corresponding two CMC planes as constructed above are in different dressing orbits. Thus, the CMC planes constructed from $(k-1)$-tuples of complex numbers with small arguments are in mutually different dressing orbits. ■

More dressing invariants will be given in §4.

3. Basic effects of one-parameter subgroups. In this section, we work out the basic effects of the canonical one-parameter subgroups of $\Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$ on the normalized potentials. Here by the canonical one-parameter subgroups of $\Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$ we mean the subgroups

$$\left\{\begin{pmatrix} e^\rho & 0 \\ 0 & e^{-\rho} \end{pmatrix}; \rho \in \boldsymbol{R}\right\},$$

$$\left\{\begin{pmatrix} e^{t\lambda^n} & 0 \\ 0 & e^{-t\lambda^n} \end{pmatrix}; t \in \boldsymbol{C}\right\} \quad \text{for even} \quad n \geq 2\,,$$

$$\left\{\begin{pmatrix} 1 & 0 \\ t\lambda^n & 1 \end{pmatrix}; t \in \boldsymbol{C}\right\} \quad \text{and} \quad \left\{\begin{pmatrix} 1 & t\lambda^n \\ 0 & 1 \end{pmatrix}; t \in \boldsymbol{C}\right\} \quad \text{for odd} \quad n \geq 1\,.$$

The results are used to determine the orders of dressing invariants associated with points on CMC surfaces and to prove that there is at most one such dressing invariant of a given order.

First of all, from the definition of the dressing action in §1, one easily derives the formula

$$\tag{3.1} \begin{pmatrix} e^\rho & 0 \\ 0 & e^{-\rho} \end{pmatrix} \bullet \begin{pmatrix} 0 & p(z) \\ E(z)/p(z) & 0 \end{pmatrix} dz = \begin{pmatrix} 0 & e^{2\rho}p(z) \\ E(z)/(e^{2\rho}p(z)) & 0 \end{pmatrix} dz$$

for $\rho \in \boldsymbol{R}$, which implies the following result.

LEMMA 3.2. *Any dressing invariant associated with $z=0$ has the form*

$$\tag{3.3} J\left(\frac{p'(0)}{p(0)}, \frac{p''(0)}{p(0)}, \dots, \frac{p^{(m)}(0)}{p(0)}\right)$$

for some positive integer m and function $J\colon \boldsymbol{C}^m \to \boldsymbol{C}$.

PROOF. If $J=J(p(0), p'(0), \dots, p^{(m)}(0))$ is a dressing invariant associated with $z=0$, then

$$\tag{3.4} J(p(0), p'(0), \dots, p^{(m)}(0)) = J\left(1, \frac{p'(0)}{p(0)}, \dots, \frac{p^{(m)}(0)}{p(0)}\right)$$

since the fact that $p(0)$ is positive implies that

$$\tag{3.5} \begin{pmatrix} 0 & p(z)/p(0) \\ p(0)E(z)/p(z) & 0 \end{pmatrix} dz = \begin{pmatrix} e^\rho & 0 \\ 0 & e^{-\rho} \end{pmatrix} \bullet \begin{pmatrix} 0 & p(z) \\ E(z)/p(z) & 0 \end{pmatrix} dz\,,$$

where $\rho=-(\ln p(0))/2$. ■

Next, we compute the basic effects of the other canonical one-parameter subgroups under the assumption that the origin is a non-umbilic point.

LEMMA 3.6. *If the origin is a non-umbilic point, then for any odd* $m\geq 1$,

$$\begin{pmatrix} 1 & 0 \\ t\lambda^m & 1 \end{pmatrix} \quad and \quad \begin{pmatrix} 1 & t\lambda^m \\ 0 & 1 \end{pmatrix} \tag{3.7}$$

leave $p(0), p'(0), \ldots, p^{(m-1)}(0)$ *invariant and change* $p^{(m)}(0)$ *by* $-2^m tp(0)^2 E(0)^{(m-1)/2}$ *and* $2^m tE(0)^{(m+1)/2}$, *respectively; for any even* $m\geq 2$,

$$\begin{pmatrix} e^{t\lambda^m} & 0 \\ 0 & e^{-t\lambda^m} \end{pmatrix} \tag{3.8}$$

leaves $p(0), p'(0), \ldots, p^{(m-1)}(0)$ *invariant and changes* $p^{(m)}(0)$ *by* $2^{m+1}tp(0)E(0)^{m/2}$.

PROOF. From (1.26) with $n=0$ we obtain

$$d_0'=-b_1\cdot\frac{E}{p}+c_1q\,. \tag{3.9}$$

Starting from (3.9) and repeatedly using (1.23)–(1.26), we derive

$$d_0^{(2l)}=2^{2l-1}\left(-a_{2l}\cdot\frac{qE^l}{p}+d_{2l}E^l\right) \tag{3.10}$$
$$+\text{terms involving } b_1, c_1, \ldots, b_{2l-1} \text{ or } c_{2l-1}\,,$$

$$d_0^{(2l+1)}=2^{2l}\left(-b_{2l+1}\cdot\frac{E^{l+1}}{p}+c_{2l+1}qE^l\right) \tag{3.11}$$
$$+\text{terms involving } b_1, c_1, \ldots, a_{2l} \text{ or } d_{2l}\,,$$

for $l\geq 1$. If

$$h_+(\lambda)=\begin{pmatrix} 1 & 0 \\ t\lambda^m & 1 \end{pmatrix} \tag{3.12}$$

with $m\geq 1$ odd, then

$$G_+(0,\lambda)=\begin{pmatrix} 1 & 0 \\ -t\lambda^m & 1 \end{pmatrix}, \tag{3.13}$$

hence,

$$\begin{gathered} a_0(0)=d_0(0)=1\,,\quad c_m(0)=-t\,, \\ b_1(0)=c_1(0)=a_2(0)=d_2(0)=\cdots=b_m(0)=0\,, \end{gathered} \tag{3.14}$$

and (3.9)–(3.11) together with (3.14) imply

$$d_0'(0)=d_0''(0)=\cdots=d_0^{(m-1)}(0)=0\,,\quad d_0^{(m)}(0)=-2^{m-1}tq(0)E(0)^{(m-1)/2}\,. \tag{3.15}$$

From (1.31) and (3.15) we obtain

$$\begin{aligned} &q^{(j)}(0)=p^{(j)}(0) \qquad \text{for} \quad j=0,1,\dots,m-1\,,\\ &q^{(m)}(0)=-2^m tp(0)^2E(0)^{(m-1)/2}+p^{(m)}(0)\,. \end{aligned} \tag{3.16}$$

Similarly, if

$$h_+(\lambda)=\begin{pmatrix} 1 & t\lambda^m \\ 0 & 1 \end{pmatrix} \tag{3.17}$$

with $m\geq 1$ odd, then

$$\begin{aligned} &q^{(j)}(0)=p^{(j)}(0) \qquad \text{for} \quad j=0,1,\dots,m-1\,,\\ &q^{(m)}(0)=2^m tE(0)^{(m+1)/2}+p^{(m)}(0)\,; \end{aligned} \tag{3.18}$$

if $h_+(\lambda)$ is given by (3.8) with $m\geq 2$ even, then

$$\begin{aligned} &q^{(j)}(0)=p^{(j)}(0) \qquad \text{for} \quad j=0,1,\dots,m-1\,,\\ &q^{(m)}(0)=2^{m+1}tp(0)E(0)^{m/2}+p^{(m)}(0)\,. \end{aligned} \tag{3.19}$$

This proves the lemma. ∎

A direct consequence of Lemma 3.6 is the following result.

PROPOSITION 3.20. *There is no dressing invariant associated with any non-umbilic point.*

PROOF. Assume that the origin is a non-umbilic point. By Lemma 3.6, one can change the value of $p'(0)$ arbitrarily while keeping the value of $p(0)$ fixed. Hence, there is no dressing invariant of the form

$$J\left(\frac{p'(0)}{p(0)}\right). \tag{3.21}$$

Let $m\geq 2$ be an integer. Also by Lemma 3.6, we can change the value of

$$\frac{p^{(m)}(0)}{p(0)} \tag{3.22}$$

arbitrarily while keeping the values of

$$\frac{p'(0)}{p(0)}\,,\ \frac{p''(0)}{p(0)}\,,\ \dots,\ \frac{p^{(m-1)}(0)}{p(0)} \tag{3.23}$$

fixed. So, there is no dressing invariant of the form

$$J\left(\frac{p'(0)}{p(0)},\frac{p''(0)}{p(0)},\dots,\frac{p^{(m)}(0)}{p(0)}\right). \tag{3.24}$$

Therefore, the conclusion follows from Lemma 3.2. ■

Finally, we give the basic effects of the other canonical one-parameter subgroups under the assumption that the origin is an umbilic point.

LEMMA 3.25. *If the origin is an umbilic point of order k, then for any odd $m \geq 1$,*

$$\begin{pmatrix} 1 & 0 \\ t\lambda^m & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & t\lambda^m \\ 0 & 1 \end{pmatrix} \tag{3.26}$$

leave $p(0), p'(0), \ldots, p^{(l-1)}(0)$ invariant and change $p^{(l)}(0)$ by $-2\gamma tp(0)^2E^{(k)}(0)^{(m-1)/2}$ and $2\beta tE^{(k)}(0)^{(m+1)/2}$, respectively, where

$$l=\frac{1}{2}(m-1)k+m \quad \text{and} \quad l=\frac{1}{2}(m+1)k+m\,, \tag{3.27}$$

respectively, and $\gamma=\gamma_{l,m}$, $\beta=\beta_{l,m}$ are positive integers; for any even $m \geq 2$,

$$\begin{pmatrix} e^{t\lambda^m} & 0 \\ 0 & e^{-t\lambda^m} \end{pmatrix} \tag{3.28}$$

leaves $p(0), p'(0), \ldots, p^{(l-1)}(0)$ invariant and changes $p^{(l)}(0)$ by $4\alpha tp(0)E^{(k)}(0)^{m/2}$, where

$$l=\frac{1}{2}mk+m \tag{3.29}$$

and $\alpha=\alpha_{l,m}$ is a positive integer.

PROOF. By (3.9) and (1.23)–(1.26),

$$d_0''=-2a_2\cdot\frac{Eq}{p}+2d_2E-b_1\cdot\left(\frac{E'}{p}+\cdots\right)+c_1q'\,, \tag{3.30}$$

$$\begin{aligned} d_0'''=&-4b_3\cdot\frac{E^2}{p}+4c_3Eq-a_2\cdot\left(\frac{3E'q}{p}+\cdots\right) \\ &+d_2\cdot(3E'+\cdots)-b_1\cdot\left(\frac{E''}{p}+\cdots\right)+c_1q''\,, \end{aligned} \tag{3.31}$$

where the terms involving only lower orders of derivatives of E are abbreviated as the dots. In general, for $n \geq 2$, we have

$$\begin{aligned} d_0^{(2n)}=&-\sum_{j=1}^{n} a_{2j}\cdot\left(\frac{\alpha_{2n,2j}E_{j,2n-2j}q}{p}+\cdots\right)+\sum_{j=1}^{n} d_{2j}\cdot(\alpha_{2n,2j}E_{j,2n-2j}+\cdots) \\ &-\sum_{j=2}^{n} b_{2j-1}\cdot\left(\frac{\beta_{2n,2j-1}E_{j,2n-2j+1}}{p}+\cdots\right) \\ &+\sum_{j=2}^{n} c_{2j-1}\cdot(\gamma_{2n,2j-1}E_{j-1,2n-2j+1}q+\cdots) \end{aligned} \tag{3.32}$$

$$-b_1\cdot\left(\frac{E^{(2n-1)}}{p}+\cdots\right)+c_1q^{(2n-1)},$$

$$\begin{aligned}d_0^{(2n+1)}=&-\sum_{j=2}^{n+1}b_{2j-1}\cdot\left(\frac{\beta_{2n+1,2j-1}E_{j,2n-2j+2}}{p}+\cdots\right)\\&+\sum_{j=2}^{n+1}c_{2j-1}\cdot(\gamma_{2n+1,2j-1}E_{j-1,2n-2j+2}q+\cdots)\\&-b_1\cdot\left(\frac{E^{(2n)}}{p}+\cdots\right)+c_1q^{(2n)}\\&-\sum_{j=1}^{n}a_{2j}\cdot\left(\frac{\alpha_{2n+1,2j}E_{j,2n-2j+1}q}{p}+\cdots\right)\\&+\sum_{j=1}^{n}d_{2j}\cdot(\alpha_{2n+1,2j}E_{j,2n-2j+1}+\cdots),\end{aligned}\tag{3.33}$$

where $\alpha_{r,2j}$, $\beta_{r,2j-1}$ and $\gamma_{r,2j-1}$ are positive integers,

$$\begin{aligned}&E_{1,0}=E,\quad E_{1,1}=E',\quad E_{1,2}=E'',\quad\cdots,\\&E_{2,0}=E^2,\quad E_{2,1}=EE',\quad E_{2,2}=E'^2,\quad E_{2,3}=E'E'',\\&E_{2,4}=E''^2,\quad E_{2,5}=E''E''',\quad\cdots,\\&E_{3,0}=E^3,\quad E_{3,1}=E^2E',\quad E_{3,2}=EE'^2,\\&E_{3,3}=E'^3,\quad E_{3,4}=E'^2E'',\quad E_{3,5}=E'E''^2,\\&E_{3,6}=E''^3,\quad E_{3,7}=E''^2E''',\quad E_{3,8}=E''E'''^2,\quad\cdots,\end{aligned}\tag{3.34}$$

and each set of dots in (3.32) and (3.33) denotes the remaining terms therein, each of which has the same number of derivatives of E (of orders ≥ 0) as the corresponding $E_{r,j}$ does, and has either derivatives of E of orders differing at least by 2 or a lower sum of the orders of the derivatives of E. If

$$h_+(\lambda)=\begin{pmatrix}1&0\\t\lambda^m&1\end{pmatrix}\tag{3.35}$$

with $m\geq 1$ odd, then, as in the proof of Lemma 3.6,

$$\begin{aligned}&a_0(0)=d_0(0)=1,\quad c_m(0)=-t,\\&b_1(0)=c_1(0)=a_2(0)=\cdots=b_m(0)=0,\end{aligned}\tag{3.36}$$

and (3.9), (3.30)–(3.33) and (3.36) imply

$$d_0'(0)=\cdots=d_0^{(l-1)}(0)=0,\quad d_0^{(l)}(0)=-\gamma_{l,m}tq(0)E^{(k)}(0)^{(m-1)/2},\tag{3.37}$$

and hence,

$$q^{(j)}(0)=p^{(j)}(0) \qquad \text{for} \quad j=0,1,\dots,l-1\,, \tag{3.38}$$
$$q^{(l)}(0)=-2\gamma_{l,m}tp(0)^2E^{(k)}(0)^{(m-1)/2}+p^{(l)}(0)\,,$$

where

$$l=\frac{1}{2}(m-1)k+m\,. \tag{3.39}$$

Similarly, if

$$h_+(\lambda)=\begin{pmatrix} 1 & t\lambda^m \\ 0 & 1 \end{pmatrix} \tag{3.40}$$

with $m\geq 1$ odd, then

$$q^{(j)}(0)=p^{(j)}(0) \qquad \text{for} \quad j=0,1,\dots,l-1\,, \tag{3.41}$$
$$q^{(l)}(0)=2\beta_{l,m}tE^{(k)}(0)^{(m+1)/2}+p^{(l)}(0)\,,$$

where

$$l=\frac{1}{2}(m+1)k+m\,; \tag{3.42}$$

if $h_+(\lambda)$ is given by (3.28) with $m\geq 2$ even, then

$$q^{(j)}(0)=p^{(j)}(0) \qquad \text{for} \quad j=0,1,\dots,l-1\,, \tag{3.43}$$
$$q^{(l)}(0)=4\alpha_{l,m}tp(0)E^{(k)}(0)^{m/2}\,,$$

where l is given by (3.29). This completes the proof. ■

The following results are direct consequences of Lemma 3.25. To better understand the first of them, we remark that by the Weierstrass type representation in [4], any holomorphic 2-form on D is the Hopf differential of a CMC surface $D\to \boldsymbol{R}^3$ and hence there are many CMC surfaces with simple umbilic points.

PROPOSITION 3.44. *There is no dressing invariant associated with any simple umbilic point.*

PROPOSITION 3.45. *Let $k\geq 2$ be an integer and $j\in\{1, k+1, k+2, k+3, 2k+3, 2k+4, 2k+5, 3k+5, 3k+6, 3k+7, \dots\}$. Then, there is no dressing invariant of order j associated with any umbilic point of order k.*

Moreover, combining Theorem 2.7 and Lemma 3.25 yields the following result.

PROPOSITION 3.46. *Let $k\geq 2$ be an integer and $j\in\{2,3,\dots,k\}$. Then, there is exactly one dressing invariant of order j associated with any umbilic point of order k.*

PROOF. By Lemma 3.2, any order-2 dressing invariant associated with the umbilic point, $z=0$, is a function of $p'(0)/p(0)$ and $p''(0)/p(0)$. Fix a dressing orbit and a canonical coordinate z around $z=0$ on the surfaces in the orbit. By Theorem 2.7, there is a constant c such that for each surface in the orbit, the normalized potential

$$\begin{pmatrix} 0 & p(z) \\ E(z)/p(z) & 0 \end{pmatrix} dz \tag{3.47}$$

with reference point $z=0$ satisfies

$$\frac{p''(0)}{p(0)} = \frac{3}{2}\left(\frac{p'(0)}{p(0)}\right)^2 + c\,. \tag{3.48}$$

Now, by Lemma 3.25, one can change $p'(0)/p(0)$ arbitrarily using dressing transformations. Therefore, there is no more relations between $p'(0)/p(0)$ and $p''(0)/p(0)$, i.e., there is no more order-2 dressing invariants on the orbit associated with the umbilic point $z=0$. Similarly, there is no more dressing invariants of order 3 or ... or k on the orbit associated with the umbilic point $z=0$. ■

4. The other sets of invariants on the dressing orbits. In this section, we prove the existence of dressing invariants of the remaining orders, i.e., the orders $k+4$, $k+5, \dots, 2k+2, 2k+6, 2k+7, \dots, 3k+4, \dots$, associated with an umbilic point of order $k \geq 2$.

We fix a canonical coordinate z around the umbilic point, $z=0$, of order $k\geq 2$, i.e., a local coordinate z around $z=0$ such that

$$E(z)=z^k\,. \tag{4.1}$$

LEMMA 4.2. *Assume $p'(0)=0$. For each odd integer $m\geq 1$ and any constant $t \in \boldsymbol{C}$,*

$$\begin{pmatrix} 1 & t\lambda^m \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

does not change $p^{(l+1)}(0)$, where

$$l=\frac{1}{2}(m+1)k+m\,, \tag{4.4}$$

and

$$\begin{pmatrix} 1+t\lambda^{m+1} & t\lambda^m \\ -t\lambda^{m+2} & 1-t\lambda^{m+1} \end{pmatrix} \tag{4.5}$$

leaves $p(0), p'(0), \dots, p^{(l-1)}(0)$ invariant and changes $p^{(l)}(0)$ and $p^{(l+1)}(0)$ by βt and $\alpha t p(0)$, *respectively, where β and α are positive integers.*

PROOF. If $h_+(\lambda)$ is given by (4.3), then

$$G_+(0,\lambda)=\begin{pmatrix}1 & -t\lambda^m\\ 0 & 1\end{pmatrix}, \tag{4.6}$$

and hence,

$$\begin{aligned}&a_0(0)=d_0(0)=1\,,\quad b_m(0)=-t\,,\\ &b_1(0)=c_1(0)=a_2(0)=d_2(0)=\cdots=d_{m-1}(0)\\ &\quad=c_m(0)=a_{m+1}(0)=d_{m+1}(0)=0\,.\end{aligned} \tag{4.7}$$

By a more detailed analysis than that in the proof of Lemma 3.25, from (4.7) one deduces that

$$\begin{aligned}&d_0'(0)=d_0''(0)=\cdots=d_0^{(l-1)}(0)=0\,,\\ &d_0^{(l)}(0)=\beta_{l,m}(k!)^{(m+1)/2}t/p(0)\,,\\ &d_0^{(l+1)}(0)=(k!)^{(m+1)/2}t\left(-\frac{\tilde{\beta}_{l+1,m}p'(0)}{p(0)^2}+\frac{\hat{\beta}_{l+1,m}q'(0)}{p(0)q(0)}\right),\end{aligned} \tag{4.8}$$

where $\beta_{l,m}$ and $\tilde{\beta}_{l+1,m}$ are positive integers, while $\hat{\beta}_{l+1,m}$ is a non-negative integer. Since $l>1$, (1.31), (4.7) and (4.8) together with the assumption $p'(0)=0$ imply that

$$q'(0)=p'(0)=0\,,\qquad q^{(l+1)}(0)=p^{(l+1)}(0)\,. \tag{4.9}$$

If $h_+(\lambda)$ is given by (4.5), then

$$G_+(0,\lambda)=\begin{pmatrix}1-t\lambda^{m+1} & -t\lambda^m\\ t\lambda^{m+2} & 1+t\lambda^{m+1}\end{pmatrix}, \tag{4.10}$$

and hence,

$$\begin{aligned}&a_0(0)=d_0(0)=1\,,\quad b_m(0)=-t\,,\quad a_{m+1}(0)=-t\,,\quad d_{m+1}(0)=t\,,\\ &b_1(0)=c_1(0)=a_2(0)=d_2(0)=\cdots=d_{m-1}(0)=c_m(0)=0\,.\end{aligned} \tag{4.11}$$

By the first equations in (1.31) and (4.11), there holds

$$q(0)=p(0)\,. \tag{4.12}$$

As in the previous case, from (4.11) and (4.12) we obtain

$$\begin{aligned}&d_0'(0)=d_0''(0)=\cdots=d_0^{(l-1)}(0)=0\,,\\ &d_0^{(l)}(0)=\beta_{l,m}(k!)^{(m+1)/2}\,t/p(0)\,,\\ &d_0^{(l+1)}(0)=(k!)^{(m+1)/2}t\left(2\alpha_{l+1,m+1}-\frac{\tilde{\beta}_{l+1,m}p'(0)}{p(0)^2}+\frac{\hat{\beta}_{l+1,m}q'(0)}{p(0)q(0)}\right),\end{aligned} \tag{4.13}$$

where $\beta_{l,m}$, $\alpha_{l+1,m+1}$, and $\tilde{\beta}_{l+1,m}$ are positive integers, while $\hat{\beta}_{l+1,m}$ is a non-negative integer. (1.31), (4.12) and (4.13) together with the assumption $p'(0)=0$ imply that

$$
\begin{aligned}
q^{(j)}(0) &= p^{(j)}(0) \qquad \text{for} \quad j=1,2,\ldots,l-1\,, \\
q^{(l)}(0) &= 2\beta_{l,m}(k!)^{(m+1)/2}t+p^{(l)}(0)\,, \\
q^{(l+1)}(0) &= 4\alpha_{l+1,m+1}(k!)^{(m+1)/2}tp(0)+p^{(l+1)}(0)\,.
\end{aligned} \tag{4.14}
$$

This completes the proof. ∎

Write the dressing function I around $z=0$ as

$$
I(z)=I_0+I_1z+I_2z^2+\cdots\,. \tag{4.15}
$$

Now we are ready to prove the main result of this section.

THEOREM 4.16. *Let $k\geq 2$ be an integer, and $j\in\{k+4, k+5, \ldots, 2k+2, 2k+6, 2k+7, \ldots, 3k+4, \ldots\}$. Then there is exactly one dressing invariant of order j associated with any umbilic point of order k. Moreover, this invariant is a differential polynomial in the dressing function I evaluated at the umbilic point.*

PROOF. Here we only sketch a proof. Let

$$
P(z)=\begin{pmatrix} 0 & p(z) \\ z^k/p(z) & 0 \end{pmatrix}dz \tag{4.17}
$$

be the normalized potential with reference point $z=0$ for a CMC surface with the induced metric $4e^{2\omega(z,\bar{z})}dzd\bar{z}$. Then, by [8],

$$
p(z)=e^{2\theta(z)-\theta(0)}\,, \tag{4.18}
$$

where $\theta(z)$ is the holomorphic part in the Taylor expansion with respect to $z=0$ of $\omega(z,\bar{z})$ as a real analytic function of z and $\bar{z}$. By Lemma 3.2, we may assume that $\theta(0)=0$. For each odd integer $n\geq 1$ and any constant $t\in \boldsymbol{C}$, set

$$
\begin{aligned}
T_n(t) &= \begin{pmatrix} 1 & 0 \\ t\lambda^n & 1 \end{pmatrix}, \\
U_n(t) &= \begin{pmatrix} 1 & t\lambda^n \\ 0 & 1 \end{pmatrix}, \\
V_{n+1}(t) &= \begin{pmatrix} 1+t\lambda^{n+1} & t\lambda^n \\ -t\lambda^{n+2} & 1-t\lambda^{n+1} \end{pmatrix}.
\end{aligned} \tag{4.19}
$$

By (1.23)–(1.26) and an argument using (1.33) with $n=1$ which is similar to that for the existence of the first set of dressing invariants, $T_1(t)$ does not change I_{k-1}, $I_k, \ldots, I_{2k}$. From (1.17) with $n=2$ and (1.28) one can solve for a_2, while $c_1(0)$ is given by (1.19) with $n=1$. By (1.25) with $n=1$, there hold $c_1'(0)=\cdots=c_1^{(k)}(0)=0$. Moreover, $b_1(0), b_1'(0), \ldots, b_1^{(k+1)}(0)$ can be expressed in terms of $I_{k-1}, I_k, \ldots, I_{2k}$ using (1.37) with $n=1$. Then, it is not difficult to show that (1.33) with $n=3$ implies the invariance of some quantities

$$
(4.20)\qquad
\begin{aligned}
&c_{2,1}I_{k-1}+d_{2,1}I_k+e_{2,1}I_{k+1}+I_{k+2}\,,\\
&c_{2,2}I_{k-1}+d_{2,2}I_k+e_{2,2}I_{k+1}+I_{k+3}\,,\\
&\qquad\ldots,\\
&c_{2,k-2}I_{k-1}+d_{2,k-2}I_k+e_{2,k-2}I_{k+1}+I_{2k-1}\,,\\
&c_{2,k-1}I_{k-1}+d_{2,k-1}I_k+e_{2,k-1}I_{k+1}^2+I_{2k}
\end{aligned}
$$

when $\theta'(0)=0$, where the constants $c_{r,s}$, $d_{r,s}$, and $e_{r,s}$ are polynomials in the first set of dressing invariants $I_0, I_1, \ldots, I_{k-2}$. By Lemma 3.25 and (4.18), $T_1(t)$ with an appropriate t changes $\theta'(0)$ to 0. Therefore, the quantities in (4.20) are the second set of dressing invariants on any dressing orbit.

Similarly, by (1.23)–(1.26) and an argument using (1.33) with $n=3$ and some other identities in §1 that is similar to that above for the existence of the second set of dressing invariants, there are polynomials $\mathscr{P}_{2k+1}, \mathscr{P}_{2k+2}, \ldots, \mathscr{P}_{3k+2}$ without constant terms such that $U_1(t)$, $V_2(t)$ and $T_3(t)$ do not change

$$
(4.21)\qquad
\begin{aligned}
&\mathscr{P}_{2k+1}(I_{k-1}, I_k, I_{k+1})+I_{2k+1}\,,\\
&\mathscr{P}_{2k+2}(I_{k-1}, I_k, I_{k+1})+I_{2k+2}\,,\\
&\ldots,\\
&\mathscr{P}_{3k+2}(I_{k-1}, I_k, I_{k+1})+I_{3k+2}\,,
\end{aligned}
$$

where some of the coefficients of the polynomials $\mathscr{P}_{2k+1}, \mathscr{P}_{2k+2}, \ldots, \mathscr{P}_{3k+2}$ depend on the orbit. It is not difficult to prove that (1.33) with $n=5$ and some other identities in §1 imply the invariance of some quantities

$$
(4.22)\qquad
\begin{aligned}
&\mathscr{Q}_{2k+4}(I_{2k+1}, I_{2k+2}, I_{2k+3})+I_{2k+4}\,,\\
&\mathscr{Q}_{2k+5}(I_{2k+1}, I_{2k+2}, I_{2k+3})+I_{2k+5}\,,\\
&\ldots,\\
&\mathscr{Q}_{3k+2}(I_{2k+1}, I_{2k+2}, I_{2k+3})+I_{3k+2}
\end{aligned}
$$

when $\theta'(0)=I_{k-1}=I_k=I_{k+1}=0$, where some of the coefficients of the polynomials $\mathscr{Q}_{2k+4}, \mathscr{Q}_{2k+5}, \ldots, \mathscr{Q}_{3k+2}$ also depend on the orbit. By Lemmas 3.25 and 4.2, $T_3(t_4)V_2(t_3)U_1(t_2)T_1(t_1)$ with appropriate t_1, t_2, t_3, and t_4 changes all of $\theta'(0)$, I_{k-1}, I_k and I_{k+1} to 0. Hence, the quantities

$$
(4.23)\qquad
\begin{aligned}
&\mathscr{Q}_{2k+4}(\mathscr{P}_{2k+1}(I_{k-1}, I_k, I_{k+1})+I_{2k+1}, \mathscr{P}_{2k+2}(I_{k-1}, I_k, I_{k+1})+I_{2k+2},\\
&\qquad \mathscr{P}_{2k+3}(I_{k-1}, I_k, I_{k+1})+I_{2k+3})+\mathscr{P}_{2k+4}(I_{k-1}, I_k, I_{k+1})+I_{2k+4},\\
&\mathscr{Q}_{2k+5}(\mathscr{P}_{2k+1}(I_{k-1}, I_k, I_{k+1})+I_{2k+1}, \mathscr{P}_{2k+2}(I_{k-1}, I_k, I_{k+1})+I_{2k+2},\\
&\qquad \mathscr{P}_{2k+3}(I_{k-1}, I_k, I_{k+1})+I_{2k+3})+\mathscr{P}_{2k+5}(I_{k-1}, I_k, I_{k+1})+I_{2k+5},\\
&\ldots,\\
&\mathscr{Q}_{3k+2}(\mathscr{P}_{2k+1}(I_{k-1}, I_k, I_{k+1})+I_{2k+1}, \mathscr{P}_{2k+2}(I_{k-1}, I_k, I_{k+1})+I_{2k+2},\\
&\qquad \mathscr{P}_{2k+3}(I_{k-1}, I_k, I_{k+1})+I_{2k+3})+\mathscr{P}_{3k+2}(I_{k-1}, I_k, I_{k+1})+I_{3k+2}
\end{aligned}
$$

form the third set of dressing invariants on any dressing orbit.

Then, one uses an induction to prove the existence of the other sets of dressing invariants on the dressing orbits.

Finally, the uniqueness of dressing invariants of these orders is a consequence of Lemmas 3.2, 3.25, and 4.2 and the formats of the dressing invariants produced above (see the proof of Proposition 3.46). ■

We note that the proof sketched above is actually an algorithm for computing all the dressing invariants. For example, direct computations following the algorithm and using the identities in §1 yield that the second dressing invariant is

$$\frac{2}{3}I_1^2+\frac{4}{3}I_0I_2+I_4 \tag{4.24}$$

if $k=2$, and that the first dressing invariant in the second set is

$$\frac{k+6}{3k}I_1I_{k-1}+\frac{2(k+6)}{3(k+2)}I_0I_k+I_{k+2} \tag{4.25}$$

if $k>2$.

REMARK. By direct calculations, under a local coordinate around an umbilic point of order ≥ 2 that does satisfy (4.1), we may miss some of the dressing invariants.

5. Integrability and topology on $\Lambda_B^+SL(2,\mathbf{C})_\sigma$. By §2 and §4, if two normalized potentials P and Q are in the same dressing orbit, then each dressing invariant associated with the umbilic points of orders ≥ 2 takes the same value at P and Q. In this section, we show that the equation (1.9) on the dressing transformation G_+ from P to Q is always formally integrable if each dressing invariant takes the same value at P and Q. Then, we look at how the topology on the loop group affects the actual integrability of (1.9) in the case where there is an umbilic point on the corresponding CMC surfaces.

Let

$$P(z)=\begin{pmatrix} 0 & p(z) \\ E(z)/p(z) & 0 \end{pmatrix}dz \quad\text{and}\quad Q(z)=\begin{pmatrix} 0 & q(z) \\ E(z)/q(z) & 0 \end{pmatrix}dz \tag{5.1}$$

be two normalized potentials on D. Then, P and Q are in the same dressing orbit if and only if there exists a holomorphic map $G_+: D\setminus S\to\Lambda^+SL(2,\mathbf{C})_\sigma$ satisfying (1.9) with initial value $G_+(0,\cdot)\in\Lambda_B^+SL(2,\mathbf{C})_\sigma$, where S consists of the poles of P and Q. First, we look at the formal integrability of (1.9), i.e., the question of how to find the functions a_n, b_n, c_n, d_n in the coefficient matrices of G_+.

Since $a_0(0)$, $p(0)$ and $q(0)$ are all positive, both (1.30) and (1.31) mean

$$a_0=\sqrt{\frac{p}{q}} \quad\text{and}\quad d_0=\sqrt{\frac{q}{p}}, \tag{5.2}$$

where the radical satisfies $\sqrt{1}=1$. So, the functions a_n, b_n, c_n and d_n are determined by (5.2), (1.19)–(1.22), (1.28) and (1.29), or equivalently, by (5.2), (1.32)–(1.35), (1.28) and (1.29).

LEMMA 5.3. *The equation* (1.21) *is a consequence of* (5.2), (1.19), (1.20), (1.22), (1.28) *and* (1.29), *or equivalently, of* (5.2), (1.32), (1.33), (1.35), (1.28) *and* (1.29).

PROOF. By (5.2), we can rewrite (1.21) and (1.22) as

$$a_n'd_0+d_n'a_0=0 \qquad \text{for even} \quad n\geq 0 \tag{5.4}$$

and

$$a_nd_0-a_0d_n=\frac{b_{n-1}'d_0}{q} \qquad \text{for even} \quad n\geq 2\,, \tag{5.5}$$

respectively. Now, we proceed by induction on n in (5.4). From (5.2) one obtains

$$a_0'd_0+d_0'a_0=\frac{1}{2}\left(\frac{p'}{p}-\frac{q'}{q}\right)+\frac{1}{2}\left(\frac{q'}{q}-\frac{p'}{p}\right)=0\,, \tag{5.6}$$

i.e., (5.4) for $n=0$ is a consequence of (5.2). From (1.28) and (5.5) with $n=2$ we get

$$a_2d_0=\frac{1}{2}b_1c_1+\frac{b_1'd_0}{2q} \quad \text{and} \quad a_0d_2=\frac{1}{2}b_1c_1-\frac{b_1'd_0}{2q}\,, \tag{5.7}$$

which together with (1.19) for $n=1$, (1.20) for $n=1$ and (5.4) or (1.21) for $n=0$ imply

$$\begin{aligned} a_2'd_0+a_0d_2' &= (a_2d_0+a_0d_2)'-a_2d_0'-a_0'd_2 \\ &= (b_1c_1)'-\left(\frac{1}{2}a_0b_1c_1+\frac{b_1'}{2q}\right)d_0'-a_0'\left(\frac{1}{2}b_1c_1d_0-\frac{b_1'}{2p}\right) \\ &= b_1'c_1+b_1c_1'-\frac{1}{2}b_1c_1(a_0d_0'+a_0'd_0)-\frac{b_1'd_0'}{2q}+\frac{a_0'b_1'}{2p} \\ &= b_1'\left(\frac{b_1E}{pq}-\frac{a_0'}{p}\right)+b_1\left(-\frac{b_1'E}{pq}\right)+\frac{a_0'b_1'}{2p}+\frac{a_0'b_1'}{2p} \\ &= 0\,, \end{aligned} \tag{5.8}$$

i.e., (5.4) holds for $n=2$. For any integer $l\geq 2$, from (1.29) and (5.5) one obtains

$$\begin{aligned} a_{2l}d_0 &= \frac{1}{2}\sum_{j=0}^{l-1} b_{2j+1}c_{2l-2j-1}-\frac{1}{2}\sum_{j=1}^{l-1} a_{2j}d_{2l-2j}+\frac{b_{2l-1}'d_0}{2q}\,, \\ a_0d_{2l} &= \frac{1}{2}\sum_{j=0}^{l-1} b_{2j+1}c_{2l-2j-1}-\frac{1}{2}\sum_{j=1}^{l-1} a_{2j}d_{2l-2j}-\frac{b_{2l-1}'d_0}{2q}\,, \end{aligned} \tag{5.9}$$

which together with (1.19) for $n\leq 2l-1$, (1.20) for $n\leq 2l-1$, (1.22) for $n\leq 2l-2$ and the induction assumption, i.e., (5.4) or (1.21) for $n\leq 2l-2$, imply

$$(5.10)\qquad \begin{aligned} a_{2l}'d_0+a_0d_{2l}' &= (a_{2l}d_0+a_0d_{2l})'-a_{2l}d_0'-a_0'd_{2l} \\ &= \sum_{j=0}^{l-1}(b_{2j+1}'c_{2l-2j-1}+b_{2j+1}c_{2l-2j-1}')-\sum_{j=1}^{l-1}(a_{2j}'d_{2l-2j}+a_{2j}d_{2l-2j}') \\ &\quad -\left(\frac{1}{2}\sum_{j=0}^{l-1}b_{2j+1}c_{2l-2j-1}-\frac{1}{2}\sum_{j=1}^{l-1}a_{2j}d_{2l-2j}+\frac{b_{2l-1}'d_0}{2q}\right)a_0d_0' \\ &\quad -\left(\frac{1}{2}\sum_{j=0}^{l-1}b_{2j+1}c_{2l-2j-1}-\frac{1}{2}\sum_{j=1}^{l-1}a_{2j}d_{2l-2j}-\frac{b_{2l-1}'d_0}{2q}\right)a_0'd_0 \\ &= \sum_{j=0}^{l-1}\left[b_{2j+1}'\left(\frac{b_{2l-2j-1}E}{pq}-\frac{a_{2l-2j-2}'}{p}\right)+b_{2j+1}\left(-\frac{b_{2l-2j-1}'E}{pq}\right)\right] \\ &\quad -\sum_{j=1}^{l-1}\left[a_{2j}'\left(\frac{a_{2l-2j}q}{p}-\frac{b_{2l-2j-1}'}{p}\right)+a_{2j}\left(-\frac{a_{2l-2j}'q}{p}\right)\right] \\ &\quad -\frac{b_{2l-1}'d_0'}{2q}+\frac{a_0'b_{2l-1}'}{2p} \\ &= -\frac{a_0'b_{2l-1}'}{p}+\frac{a_0'b_{2l-1}'}{2p}+\frac{a_0'b_{2l-1}'}{2p} \\ &= 0\,, \end{aligned}$$

i.e., (5.4) holds for $n=2l$. ■

Therefore, the functions a_n, b_n, c_n and d_n only need to satisfy (5.2), (1.28), (1.29), (1.32), (1.33) and (1.35). We collect these equations here with only evident modifications:

$$(5.11)\qquad a_0=\sqrt{\frac{p}{q}}\quad\text{and}\quad d_0=\sqrt{\frac{q}{p}}\,,$$

$$(5.12)\qquad 2b_n'E+b_n\left[E'-\left(\frac{p'}{p}-\frac{q'}{q}\right)E\right]=\left(a_{n-1}''-a_{n-1}'\cdot\frac{p'}{p}\right)q\quad\text{for odd } n\geq 1\,,$$

$$(5.13)\qquad c_n=b_n\cdot\frac{E}{pq}-a_{n-1}'\cdot\frac{1}{p}\qquad\text{for odd}\quad n\geq 1\,,$$

$$(5.14)\qquad a_2=\frac{1}{2}a_0b_1c_1+\frac{b_1'}{2b}\quad\text{and}\quad d_2=\frac{1}{2}d_0b_1c_1-\frac{b_1'}{2p}\,,$$

$$(5.15)\qquad a_n=\frac{1}{2}a_0\sum_{j=0}^{n/2-1}b_{2j+1}c_{n-2j-1}-\frac{1}{2}a_0\sum_{j=1}^{n/2-1}a_{2j}d_{n-2j}+\frac{b_{n-1}'}{2q}\quad\text{for even } n\geq 4\,,$$

$$(5.16)\qquad d_n=\frac{1}{2}d_0\sum_{j=0}^{n/2-1}b_{2j+1}c_{n-2j-1}-\frac{1}{2}d_0\sum_{j=1}^{n/2-1}a_{2j}d_{n-2j}-\frac{b_{n-1}'}{2p}\quad\text{for even } n\geq 4\,.$$

THEOREM 5.17. *If the origin is a non-umbilic point, then for any P, Q, and initial* *conditions* $b_1(0), b_3(0), \dots$, *the equation* (1.9) *on* $G_+ : D \setminus S \to \Lambda^+ SL(2, \boldsymbol{C})_\sigma$ *is formally integrable. If the origin is a simple umbilic point, then for any P and Q, the equation* (1.9) *on* $G_+ : D \setminus S \to \Lambda^+ SL(2, \boldsymbol{C})_\sigma$ *is formally integrable and the formal solution is unique. If the origin is an umbilic point of order* $k \geq 2$ *and the coordinate z satisfies* (4.1), *then for any P and Q such that* $\mathscr{I}(P) = \mathscr{I}(Q)$ *for each dressing invariant* $\mathscr{I}$ *associated with the origin, the equation* (1.9) *on* $G_+ : D \setminus S \to \Lambda^+ SL(2, \boldsymbol{C})_\sigma$ *is formally integrable and the formal solution is unique.*

COROLLARY 5.18. *If* $f: D \to \boldsymbol{R}^3$ *is a* CMC *immersion with an umbilic point, then the dressing action of* $\Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$ *on the dressing orbit through f is free, and hence, the orbit is diffeomorphic to* $\Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$.

REMARK. Josef Dorfmeister and Franz Pedit told us that they had obtained the results of Corollary 5.18 in a direct way.

PROOF. The first statement directly follows from (5.11)–(5.16). The idea for our proofs of the other two statements is: (5.12) determines $b_n(0)$ when $z = 0$ is an umbilic point, which can be seen from (1.37). For example, if the origin is a simple umbilic point, then (5.12) implies

$$b_n(0) = \left(a''_{n-1}(0) - a'_{n-1}(0) \cdot \frac{p'(0)}{p(0)} \right) \frac{q(0)}{E'(0)} \qquad \text{for odd} \quad n \geq 1\,, \tag{5.19}$$

and hence, (5.12) uniquely determines b_n from a_{n-1}, p, q and E. Therefore, all the functions a_n, b_n, c_n and d_n are uniquely determined by p, q and E. ■

So far, we have no knowledge as to if a formal solution to (1.9) actually converges under any topology on $\Lambda SL(2, \boldsymbol{C})_\sigma$. However, we turn to the question of whether the topology on $\Lambda SL(2, \boldsymbol{C})_\sigma$ affects the convergence of a formal solution to (1.9). In order to indicate the topology $\mathscr{T}$ used on $\Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$, we are goint to use the notation $\Lambda^+_{\mathscr{T}, B} SL(2, \boldsymbol{C})_\sigma$.

THEOREM 5.20. *Let* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *be two topologies on* $\Lambda SL(2, \boldsymbol{C})_\sigma$ *such that the dressing action is defined and the Weierstrass type representation of* CMC *surfaces holds,* $f: D \to \boldsymbol{R}^3$ *a* CMC *surface with an umbilic point. If* $\mathscr{T}_1$ *is weaker than* $\mathscr{T}_2$ *when restricted to* $\Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$, *then the dressing orbit through f under the action of* $\Lambda^+_{\mathscr{T}_1, B} SL(2, \boldsymbol{C})_\sigma$ *contains the dressing orbit through f under the action of* $\Lambda^+_{\mathscr{T}_2, B} SL(2, \boldsymbol{C})_\sigma$ *as a proper subset.*

PROOF. Since $\Lambda^+_{\mathscr{T}_2, B} SL(2, \boldsymbol{C})_\sigma$ is a proper subset of $\Lambda^+_{\mathscr{T}_1, B} SL(2, \boldsymbol{C})_\sigma$, there is an element $h_+ \in \Lambda^+_{\mathscr{T}_1, B} SL(2, \boldsymbol{C})_\sigma \setminus \Lambda^+_{\mathscr{T}_2, B} SL(2, \boldsymbol{C})_\sigma$. Then, by Corollary 5.18, $(h_+)_\bullet f$ is not in the dressing orbit through f under the action of $\Lambda^+_{\mathscr{T}_2, B} SL(2, \boldsymbol{C})_\sigma$. ■

Hence, if one can weaken the topology on $\Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$ while keeping the dressing action defined and the Weierstrass type representation valid, then there are more than one dressing orbit with any given Hopf differential $E(z)(dz)^2$ on D having a zero.

In summary, the equations (5.11)–(5.16) only tells the format of the normalized potentials for the elements in a dressing orbit of CMC surfaces with an umbilic point, and the topology on $\Lambda_B^+ SL(2, \boldsymbol{C})_\sigma$ determines how big the orbit is.

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