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BOUNDARY LAYERS IN A SEMILINEAR PARABOLIC PROBLEM

Dedicated to Professor Junji Kato on his sixtieth birthday

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Abstract. We study a singular perturbation problem for a certain type of reaction diffusion equation with a space-dependent reaction term. We compare the effect that the presence of boundary layers versus internal layers has on the existence and stability of stationary solutions. In particular, we show that the associated eigenvalues are of different orders of magnitude for the two kinds of layers.

1. Introduction. In [3], Hale and Sakamoto studied the parabolic equation

$$u_t = \varepsilon^2 u_{xx} + f(x, u)\,, \qquad -1 < x < 1\,, \quad t \geq 0\,, \quad \varepsilon > 0\,, \tag{1}$$

with homogeneous *Neumann boundary conditions*

$$u_x(-1, t) = u_x(1, t) = 0\,.$$

Under mild hypothesis on f and Robin Boundary conditions (see below), Zelenyak [8] proved that the ω-limit set of each solution is a stationary solution. Hale and Sakamoto's goal was to prove the existence and determine the stability of equilibrium solutions of (1) that exhibit n internal transition layers. To do that, they assumed that f verifies the following hypotheses:

H1. $f : \boldsymbol{R} \times [-1, 1] \to \boldsymbol{R}$ is a C^∞-function of (x, u) with $f(x, 0) = 0$, $f(x, 1) = 0$.

H2. There is a positive constant ζ such that

$$f_u(x, 0)\,, \quad f_u(x, 1) \leq -\zeta^2 \quad \text{for } x \in [-1, 1]\,.$$

H3. If $J(x) = \int_0^1 f(x, u)\mathrm{d}u$, $x \in [-1, 1]$, then there exist n points x_i in the interval $(-1, 1)$, $x_i < x_{i+1}$, such that

$$J(x_i) = 0\,, \quad \mathrm{d}J(x)/\mathrm{d}x\big|_{x=x_i} \neq 0 \quad \text{and} \quad \int_0^u f(x_i, \sigma)\mathrm{d}\sigma < 0 \quad \text{for } u \in (0, 1)\,.$$

Their main example (for the case $n = 1, x_1 = 0$) was

$$f(x, u) = u(1 - u)(u - c(x))\,, \tag{2}$$

where

$$c(0) = 1/2\,, \quad c'(0) \neq 0\,, \quad \text{and} \quad 0 < c(x) < 1 \quad \text{for } x \in [-1, 1]\,.$$

Assuming hypothesis H1, H2 and H3, they constructed approximate solutions that exhibit n internal transition layers from 0 to 1 or vice-versa, Using these approximations, they applied the Liapunov-Schmidt method, to obtain the existence of exact solutions of (1) with

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the same type of layers. Their method also produced results about the stability of these solutions and, in particular, they proved that the first n eigenvalues of a solution that exhibit n internal transition layers are of order ε.

Our goal in this article is to obtain analogous results on existence and stability of solutions of (1) that, in addition to possible internal layers, have boundary layers. We will work with the following *Robin Boundary conditions*: $0 \le \alpha_l, \alpha_r \le 1$; $\beta_l, \beta_r \in \boldsymbol{R}$,

$$\begin{cases} \alpha_l u(-1,t) - (1-\alpha_l)u'(-1,t) = \beta_l, \\ \alpha_r u(1,t) + (1-\alpha_r)u'(1,t) = \beta_r. \end{cases} \tag{3}$$

With these boundary conditions, we may have solutions that exhibit boundary layers at both, one or none of the endpoints of the interval $[-1, 1]$. If they exist, the layers may connect some value $u = u_l$ (resp. $u = u_r$) at the endpoints with $u = 0$ or $u = 1$ inside the interval $[-1, 1]$. Solutions with boundary layers exist if we assume the following hypothesis (in addition to H1 and H2):

H4. Let p be either -1 or 1 depending on which endpoint of the interval we are considering. Let q be either 0 or 1 depending on what value we want u to reach inside the interval $(0, 1)$. Let G be defined as

$$G(p,q,u) = \int_q^u f(p,\sigma)\mathrm{d}\sigma\,.$$

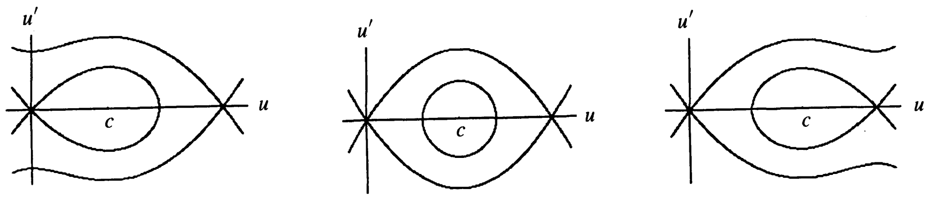
• If $\alpha_l \neq 1$ (resp. $\alpha_r \neq 1$) we will assume that there exists $\gamma \in (0, 1)$ such that $G(p, q, \gamma) = 0$, $f(p, \gamma) \neq 0$, and $G(p, q, u) < 0$ if u is between γ and q.

• Otherwise, we will assume that $f(-1, \beta_l) \neq 0$ and $G(-1, q, u) < 0$ if u is between q and β_l (resp. substitute -1 by 1 and β_l by β_r).

For the first case, the hypothesis guarantees that if $\alpha_l \neq 1$ (resp. $\alpha_r \neq 1$) there is a homoclinic orbit around q. For the second case, it guarantees that there is a piece of the stable manifold of q that extends at least until β_l (resp. β_r).

In particular, for the example (2), if $\alpha_l \neq 1$ (resp. $\alpha_r \neq 1$), there are solutions that exhibit a boundary layer on the left (resp. right) endpoint of the interval if $c(-1) < 1/2$ (resp. $c(1) > 1/2$)—see Figure 1. If $\alpha_l = 1$ (resp. $\alpha_r = 1$), there are solutions that exhibit a left (right) boundary layer that connects β_l (resp. β_r) with one (resp. zero) for all $\beta_l \in (0, 1)$ (resp. for all $\beta_r \in (0, 1)$). Also there are solutions that exhibit a left (resp. right) boundary layer that connects β_l (resp. β_r) with zero (resp. one) if $\beta_l \le \gamma(c(-1))$ (resp. $\beta_r \ge \gamma(c(1))$) where

$$\gamma(c) = \begin{cases} (2 + 2c - \sqrt{4 - 10c + 4c^2})/3\,, & \text{if } c < 1/2\,; \\ (-1 + 2c + \sqrt{2}\sqrt{-1 + c + 2c^2})/3\,, & \text{if } c > 1/2\,. \end{cases}$$



(a) $c < 1/2$ (b) $c = 1/2$ (c) $c > 1/2$

FIGURE 1. Phase portraits of example (2) for different values of c.

From a variational point of view, hypotheses H1–H4 represent the existence of minimizers of a certain functional. This approach is followed in [5], [2]. Their method produces existence of steady states of (1) that exhibit boundary and internal layers, but it does not tell us anything about their stability as solutions of (1). The approach followed in [3] has the advantage of providing existence as well as stability results.

To proceed with that method, we construct approximate solutions $u = u(x)$ to

$$\tag{4} \begin{cases} \varepsilon^2 u'' + f(x,u) = 0\,, \qquad -1 < x < 1\,, \quad \varepsilon > 0\,, \\ \alpha_l u(-1) - (1-\alpha_l)u_x(-1) = \beta_l\,, \\ \alpha_r u(1) + (1-\alpha_r)u_x(1) = \beta_r\,; \end{cases}$$

and study the linearization of (4) around these approximate solutions. Then, to prove the existence of exact solutions of (4) with the same layers and stability properties as our approximations, we just have to refer to [3] (see also [7]), as long as our approximate solutions verify (4) up to order ε^2.

We are interested only in solutions that stay in [0, 1]. Therefore, when we deal with a Dirichlet boundary condition ($\alpha_l = 1$ or $\alpha_r = 1$) we will always assume that $\beta_l \in [0,1]$ or $\beta_r \in [0,1]$, whichever corresponds.

2. An approximate solution. We are now going to construct an approximation $U = U(x,\varepsilon)$ to the equilibrium solution u of equation (1) with boundary conditions (3); that is, U will be an approximate solution to (4).

We construct U by piecing together asymptotic approximations to each of the boundary and internal layers. Approximations to the internal layers were given in [3]. We prove next that we can apply the same procedure for the construction of the boundary layers and obtain an approximation of order ε^2.

2.1. The boundary layers. We present in detail only the construction of the boundary layer that verifies the left boundary condition at $x = 0$ and goes down to zero, since the construction of the other boundary layers in very similar. We first study the case $\alpha_l \neq 1$ and take care of any possible *Dirichlet boundary condition* afterward.

If we let $s = (x+1)/\varepsilon$, $s \in (0, 1/\varepsilon)$, and define $Z(s,\varepsilon) = u(-1+\varepsilon s)$, $\dot{} = \mathrm{d}/\mathrm{d}s$, then Z will satisfy:

$$
(5) \qquad \begin{cases} \ddot{Z} + f(-1+\varepsilon s, Z) = 0\,, & 0 < s < 1/\varepsilon\,, \\ \varepsilon\alpha_l Z(0,\varepsilon) - (1-\alpha_l)\dot{Z}(0,\varepsilon) = \varepsilon\beta_l\,, & Z(1/\varepsilon) = 0\,. \end{cases}
$$

If we formally write $Z(s,\varepsilon) = z_0(s) + \varepsilon z_1(s) + O(\varepsilon^2)$ and equate the coefficients of the powers of ε, then z_0 and z_1 satisfy

$$
(6) \qquad \begin{cases} \ddot{z}_0 + f(-1, z_0) = 0, & 0 < s < +\infty\,, \\ \dot{z}_0(0) = 0\,, & z_0(+\infty) = 0\,. \end{cases}
$$

$$
(7) \qquad \begin{cases} \ddot{z}_1 + f_u(-1, z_0)z_1 + f_x(-1, z_0)s = 0\,, & 0 < s < +\infty\,, \\ \dot{z}_1(0) = [\alpha_l z_0(0) - \beta_l]/(1-\alpha_l) =: d_l(\alpha_l, \beta_l)\,, & z_1(+\infty) = 0\,. \end{cases}
$$

Since $f_u(0,0) < 0$ by H2, the origin in the $(z_0, \dot{z}_0)$ plane for (6) is a hyperbolic critical point. Therefore, having a solution of (6) is equivalent to saying that the stable manifold of $(0,0)$ intersects the z-axis, which is guaranteed by H4 if we set $z_0(0) = \gamma$. Then $z_0(s)$ is uniquely determined and there is a constant $k_0 > 0$ such that

$$
(8) \qquad \max\{|z_0(s)|, |\dot{z}_0(s)|\} \le k_0 e^{-\zeta s}\,, \quad s \ge 0\,.
$$

Let us now write the equation

$$
(9) \qquad \ddot{\eta} + f_u(-1, z_0(s))\eta = 0\,, \quad 0 < s < +\infty\,,
$$

as the first order system $\dot{Y} = A(s)Y$ where

$$
Y = \begin{pmatrix} \eta \\ \dot{\eta} \end{pmatrix}, \quad A(s) = \begin{pmatrix} 0 & 1 \\ -f_u(-1, z_0(s)) & 0 \end{pmatrix}.
$$

Since zero is not an eigenvalue of $A(+\infty)$ (again by H2), we conclude that equation (9) has an exponential dichotomy in $(0, +\infty)$ (see [1]). Consequently, equation (7) has at least one bounded solution $\varphi(s)$. We can construct infinitely many other bounded solutions of the form $z_1(s) = C\dot{z}_0(s) + \varphi(s)$, with $C \in \boldsymbol{R}$—observe that $\dot{z}_0(s)$ is a bounded solution of (9). Since $\ddot{z}_0(0) \ne 0$ by H4, we can choose

$$
C = [d_l(\alpha_l, \beta_l) - \varphi(0)]/\ddot{z}_0(0)\,,
$$

so there is a bounded solution $z_1(s)$ of equation (7) that also verifies the initial condition.

Since the forcing term in (7) is of order $e^{-\zeta s}$ by hypothesis H1 and estimation (8), then we have that

$$
(10) \qquad \max\{|z_1(s)|, |\dot{z}_1(s)|\} \le k_1 e^{-\zeta s}\,, \quad s \ge 0,
$$

for some positive constant k_1. This implies that there is a bounded solution of the problem (7).

This solution is unique. To see that, we multiply (7) by $\dot{z}_0(s)$ and integrate from zero to infinity. Integrating the term $\ddot{z}_1(s)\dot{z}_0(s)$ by parts twice and using the fact the $\dot{z}_0(s)$ is a solution of equation (9) gives us the relationship

$$[\dot{z}_0\dot{z}_1 - \ddot{z}_0 z_1]_0^{+\infty} = -\int_0^{+\infty} f_x(-1, z_0(s))s\dot{z}_0(s)\mathrm{d}s\,. \tag{11}$$

Using estimations (8) and (10), the initial condition $\dot{z}_0(0) = 0$ of (6), and the fact that $f(-1, \gamma) \neq 0$ due to H4, we have that

$$z_1(0) = \frac{1}{f(-1,\gamma)}\int_0^{+\infty} f_x(-1, z_0(s))s\dot{z}_0(s)\mathrm{d}s\,.$$

For the case $\alpha_l = 1$ (that is, the Dirichlet case), the left boundary condition for Z is $Z(0, \varepsilon) = \beta_l$. This implies that $z_0(0) = \beta_l$ and $z_1(0) = 0$. The hypothesis H4 guarantees that the stable manifold of $(0, 0)$ in the plane $(z_0, \dot{z}_0)$ extends beyond β_l, giving us the existence of a bounded solution of problem (6) which verifies estimations (8). Similarly as before, we can prove the existence of a bounded solution $\varphi(s)$ of equation (7) by using the theory of exponential dichotomies. Now, we can also construct many other bounded solutions of (7) of the form $z_1(s) = C\dot{z}_0(s) + \varphi(s)$.

If $\dot{z}_0(0) \neq 0$, then we can choose $C = -\varphi(0)/\dot{z}_0(0)$, and so there is a bounded solution to (7) that verifies the initial condition $z_1(0) = 0$. Estimations (10) are also certain in this case for the same reason as before and $z_1(s)$ is again unique since we can see from (11) that

$$\dot{z}_1(0) = \frac{1}{\dot{z}_0(0)}\int_0^{+\infty} f_x(-1, z_0(s))s\dot{z}_0(s)\mathrm{d}s\,.$$

The case $\dot{z}_0(0) \neq 0$ corresponds to the boundary layer being a piece of a homoclinic orbit around zero. Let us denote this particular value of β_l by γ, as we did for the example problem (2). It turns out that for $\beta_l = \gamma$, we can show that $\varphi(s)$ itself verifies the initial condition. To see that observe that (11) now reduces to

$$\int_0^{+\infty} f_x(-1, z_0(s))s\dot{z}_0(s)\mathrm{d}s = 0\,. \tag{12}$$

This is essentially the requirement that is stated in [3] to guarantee the existence of a bounded solution of (7) for the internal layers. But for us now, it is a consequence of the existence of a bounded solution of (7). Using the fact that $\varphi(s)$ is a solution of (7), the condition (12) becomes

$$\int_0^{+\infty}\{-\ddot{\varphi}(s) - f_u(-1, z_0(s))\varphi(s)\}\dot{z}_0(s)\mathrm{d}s = 0\,.$$

This equation implies that

$$\int_0^{+\infty}\ddot{\varphi}(s)\dot{z}_0(s)\mathrm{d}s = \int_0^{+\infty}\varphi(s)\dddot{z}_0(s)\mathrm{d}s\,,$$

which is only possible if $\varphi(0) = 0$.

So in this particular case we have infinitely many bounded solutions of (7) that verify the initial condition $z_1(0) = 0$, namely $z_1(s) = C\dot{z}_0(s) + \varphi(s)$. From all of these solutions, we must pick up the one that is the limit of the $z_1(s, \beta_l)$ as β_l converges to γ. In particular, $\dot{z}_1(0, \gamma)$ should be the limit of $\dot{z}_1(0, \beta_l)$ as $\beta_l \to \gamma$. This limit turns out to be zero. This implies that $z_1(s, \gamma)$ is identically zero.

All of the other boundary layers are constructed by an analogous procedure. The internal layers are constructed similarly, although for them $s \in (-\infty, +\infty)$, and to prove their existence we have the freedom to choose their values at $s = 0$.

We will denote by Λ_i the layer that appears at the point x_i, with $i = 0, 1, \ldots, n, n+1$, and where $x_0 = 0$ and $x_{n+1} = 1$ correspond to boundary layers. So

$$\Lambda_i(s) = z_0(s) + \varepsilon z_1(s), \quad x = x_i + \varepsilon s, \tag{13}$$

and $s \in (-\infty, +\infty)$ unless $i = 0$ in which case $s \in (0, +\infty)$ or $i = n+1$ in which case $s \in (-\infty, 0)$.

2.2. Matching the layers and estimating the error. We are now going to construct the approximation $U(x, \varepsilon)$ to an equilibrium solution u. Let ρ be the minimum distance between the points x_i where there is a layer. To match our approximations for those layers, we need the following C^∞-cutoff functions defined on $[0, 1]$: for $i = 1, 2, \ldots, n$ let

$$\Phi_0(x) = \begin{cases} 0, & x \geq 2\rho/10; \\ 1, & x \leq \rho/10; \\ 0 < \Phi_0(x) < 1, & \text{otherwise}. \end{cases}$$

$$\Phi_i(x) = \begin{cases} 0, & |x - x_i| \geq 2\rho/10; \\ 1, & |x - x_i| \leq \rho/10; \\ 0 < \Phi_i(x) < 1, & \text{otherwise}. \end{cases}$$

$$\Phi_{n+1}(x) = \begin{cases} 0, & 1 - x \geq 2\rho/10; \\ 1, & 1 - x \leq \rho/10; \\ 0 < \Phi_2(x) < 1, & \text{otherwise}. \end{cases}$$

We also need a function Ψ that is going to be zero in the intervals $[x_i, x_{i+1}]$ if u is close to zero in the middle of the interval and otherwise is

$$\Psi(x) = \begin{cases} 1 - \Phi_i(x), & x - x_i \leq 2\rho/10; \\ 1, & x_i + 2\rho/10 \leq x \leq x_{i+1} - 2\rho/10; \\ 1 - \Phi_{i+1}(x), & x_{i+1} - x \leq 2\rho/10. \end{cases}$$

With the help of these functions, we define our approximate solution

$$U(x, \varepsilon) = \sum_{i=0}^{n+1} \Lambda_i(x)\Phi_i(x) + \Psi(x). \tag{14}$$

If we denote by G_U the function

$$G_U(x, \varepsilon) = \varepsilon^2 U'' + f(x, U(x, \varepsilon)), \tag{15}$$

then we can prove the following result.

LEMMA 2.1. $\sup_{x \in [0,1]} |G_U(x, \varepsilon)| = O(\varepsilon^2)$ *as* $\varepsilon \to 0$.

PROOF. Where our approximation U is constant and equal to 0 or 1, the function G_U is zero. Therefore, we only need to consider carefully the layer zones. We distinguish two cases, depending on whether Φ_i is constant or not.

1. In the case of an interval where Φ_i is constant, we know that U is given by $U(x,\varepsilon) = \Lambda_i(x)$. So

$$\begin{aligned}G_U(x) &= z_0''((x-x_i)/\varepsilon) + \varepsilon z_1''((x-x_i)/\varepsilon) + f(x,\Lambda_i(x))\\ &= f(x,\Lambda_i(x)) - f(x_i,z_0) + \varepsilon[f_u(x_i,z_0)z_1 + f_x(x_i,z_0)(x-x_i)/\varepsilon]\\ &= (1/2)f_{uu}(\tilde{x},\tilde{z})[\varepsilon z_1]^2 + f_{ux}(\tilde{x},\tilde{z})(x-x_i)\varepsilon z_1 + (1/2)f_{xx}(\tilde{x},\tilde{z})(x-x_i)^2\\ &= \varepsilon^2[(1/2)f_{uu}(\tilde{x},\tilde{z})z_1^2 + f_{ux}(\tilde{x},\tilde{z})sz_1 + (1/2)f_{xx}(\tilde{x},\tilde{z})s^2],\end{aligned}$$

where $\tilde{z} = z_0 + \theta\varepsilon z_1$ and $\tilde{x} = x_i + \theta(x-x_i)$ for some $0 \le \theta(x,\varepsilon) \le 1$ by the Mean Value Theorem. Since we have the estimations (8), (10) and $f_{xx}(x,0) = 0 = f_{xx}(x,1)$ the function between braces is bounded in $(0,+\infty)$ as a function of $s = (x-x_i)/\varepsilon$ (remember that $f \in C^\infty$ and $u \in [0,1]$). So our result is verified for this type of interval.

2. If Φ_i is not constant, then our approximation U has the form

$$U(x,\varepsilon) = \Lambda_i(x)\Phi_i(x) + \Psi(x)$$

and

$$\begin{aligned}|G_U(x)| \le &|\Lambda_i''(x)\Phi_i(x)|_0 + 2\varepsilon|\Lambda_i'(x)\Phi_i'(x)|_0\\ &+ \varepsilon^2|\Lambda_i(x)\Phi_i''(x) + \Psi''(x)|_0 + |f(x,U(x,\varepsilon))|_0,\end{aligned}$$

where $|\ |_0$ stands for the supremum norm in the interval. Because of the estimations (8) and (10), we only have to prove that the last term on the right hand side is of order ε^2. We have two possibilities. If Λ_i is a layer that is going down to zero as s goes to $\pm\infty$ (depending of which side of x_i we are), then $\Psi(x) = 0$ and we have that

$$f(x,U(x,\varepsilon)) = f(x,0) + f_u(x,0)\Lambda_i(x)\Phi_i(x) + o(\Lambda_i(x)\Phi_i(x)).$$

Since $f(x,0) = 0$, estimations (8) and (10) can be applied again to obtain the desired result.

Finally, we consider the case in which Λ_i is a layer going up to 1, and then $\Psi(x) = 1 - \Phi_i(x)$ so

$$f(x,U(x,\varepsilon)) = f(x,1) + f_u(x,1)\Phi_i(x)(\Lambda_i(x)-1) + o(\Phi_i(x)[\Lambda_i(x)-1]).$$

The fact that $f(x,1) = 0$ and the corresponding estimations (8) and (10) enable us to conclude the proof for this case.

As a last note, observe from (7) that the general Robin boundary conditions are verified up to order ε^2. Boundary conditions of Dirichlet or Neumann types are verified exactly.

q.e.d.

3. The linear operator. We wish now to discuss the spectral properties of the linear operator $\mathcal{L}$ around the approximate solution U:

$$\mathcal{L}\phi = \varepsilon^2\phi'' + f_u(x,U(x,\varepsilon))\phi, \tag{16}$$

with the boundary conditions

$$\begin{cases} \alpha_l\phi(-1) - (1-\alpha_l)\phi'(-1) = 0\,, \\ \alpha_r\phi(1) + (1-\alpha_r)\phi'(1) = 0\,. \end{cases} \tag{17}$$

In order to do so, we define

$$X = \left\{ u \in C^2([-1,1]) : \quad \begin{matrix} \alpha_l u(-1) - (1-\alpha_l)u'(-1) = 0 \\ \alpha_r u(1) + (1-\alpha_r)u'(1) = 0 \end{matrix} \right\},$$
$$Y = C^0([-1,1])\,,$$

where Y is a Banach space with the supremum norm represent by $|u|_0$, and X is a Banach space with the norm

$$|u|_{2,\varepsilon} = |u|_0 + \varepsilon|u'|_0 + \varepsilon^2|u''|_0 \quad \text{for } u \in X\,.$$

The operator $\mathcal{L} : X \to Y$ is continuous.

The operator $\mathcal{L}$ has been extensively studied. We summarize in the following theorem some of the results found in [3] for the case of homogeneous Neumann conditions.

THEOREM 3.1. *Let us assume that the approximate solution $U = U(x,\varepsilon)$ exhibits n internal layers at the points $\{x_i : i = 1, \dots, n\}$. Let $\mathcal{L}^n$ be the linear operator around U. Then we have that*

(1) *All the eigenvalues of $\mathcal{L}^n$ are simple.*

(2) *The first n eigenvalues of $\mathcal{L}^n$, $\lambda_1(\varepsilon) > \lambda_2(\varepsilon) > \cdots > \lambda_n(\varepsilon)$, approach zero as* $\varepsilon \downarrow 0$.

(3) *If $\phi_j(x,\varepsilon)$, $j \in \{1, \dots, n\}$, is an eigenfunction corresponding to $\lambda_j(\varepsilon)$, then there exist positive constants k, β and d such that*

$$|\phi_j(x_i + \varepsilon s, \varepsilon)| \le k|\phi_j(x_i,\varepsilon)|e^{-\beta|s|} \quad \textit{for } |s| \le d/\varepsilon\,.$$

(4) *If $Z^i(s,\varepsilon) = z_0^i(s) + \varepsilon z_1^i(s)$ represents the asymptotic approximation to the internal layer of U at x_i then*

$$\phi_j(x_i + \varepsilon s, \varepsilon)/\phi_j(x_i,\varepsilon) \to \dot{z}_0^i(s)/\dot{z}_0^i(0) \quad \textit{as } \varepsilon \downarrow 0\,.$$

(5) *The remaining eigenvalues of $\mathcal{L}^n$ are bounded away from zero; namely there is a positive constant μ_0 such that*

$$\lambda_{n+1}(\varepsilon) \le -\mu_0 \quad \textit{for } 0 < \varepsilon \le \varepsilon_0\,.$$

To extend some of these results to more general boundary conditions we have to study the phase plane associated with the eigenvalue equation $\mathcal{L}\phi = \lambda\phi$, $\lambda \in \boldsymbol{C}$; i.e., the equation

$$\varepsilon^2\phi'' + f_u(x, U(x,\varepsilon))\phi = \lambda\phi \tag{18}$$

with the corresponding boundary conditions. In order to study the phase plane of this equation, we let $\rho = \varepsilon\phi'$. Then we have

$$\begin{cases} \varepsilon\phi' = \rho\,, \\ \varepsilon\rho' = B(x,\varepsilon,\lambda)\phi\,, \end{cases}$$

where $B = B(x, \varepsilon, \lambda) = \lambda - f_u(U(x, \varepsilon), x)$. Changing to the polar coordinates (r, θ) such that $\phi = r\cos\theta$ and $\rho = -r\sin\theta$, we obtain the equations

$$\begin{cases} \varepsilon r' = r(1+B)\sin\theta\cos\theta\,, \\ \varepsilon\theta' = \sin^2\theta + B\cos^2\theta\,. \end{cases}$$

Observe that

$$\lim_{\lambda\to+\infty} \theta(x, \varepsilon, \lambda) = +\infty\,,$$

$$\lim_{\lambda\to-\infty} \theta(x, \varepsilon, \lambda) = -\pi/2 + (n+1)\pi\,, \quad \text{where } n = E\left[\frac{\theta(x, \varepsilon, \lambda) - \pi/2}{\pi}\right]$$

and E is the Integer Part function. Since the equation for θ does not depend on r and we are interested only in the functions $\theta(x, \varepsilon, \lambda)$ such that

$$\theta(0, \varepsilon, \lambda) = q_0 \ (\operatorname{mod}\pi)\,, \quad \theta(1, \varepsilon, \lambda) = q_1 \ (\operatorname{mod}\pi)\,;$$

and

$$\tan q_0 = -\alpha_l/(1-\alpha_l)\,, \quad \text{and} \quad \tan q_1 = \alpha_r/(1-\alpha_r)\,,$$

(or $q_0 = -\pi/2$, $q_1 = \pi/2$ if $\alpha_l = 1$ or $\alpha_r = 1$, resp.), we can restrict ourselves to the study of the equation for θ.

Each one of the $\lambda_i(\varepsilon)$ is the first eigenvalue associated with the internal layer at x_i in the sense that the angle θ corresponding to the eigenfunction ϕ_i, experiences a π rotation when $c(x)$ crosses 1/2 at the point x_i. We are interested in studying what happens to the first eigenvalue associated with each of the boundary layers. Our results are summarized in the following lemma.

LEMMA 3.2. *Let us assume that $U = U(x, \varepsilon)$ is the approximate solution given by* (14). *Let $\mathcal{L}$ be the linear operator around U given by* (16) *with the boundary conditions* (17). *Then we have that*

(1) *There exists a positive number $\Lambda > 0$ such that if we denote by $\lambda_l(\varepsilon)$* (*resp. $\lambda_r(\varepsilon)$*) *the first eigenvalue of $\mathcal{L}$ associated to the left* (*resp. right*) *boundary layer, then $\lambda_l(\varepsilon) \in (-\Lambda, \Lambda)$* (*resp. $\lambda_r(\varepsilon) \in (-\Lambda, \Lambda)$*).

(2) *If $\phi_l(x, \varepsilon)$* (*resp. $\phi_r(x, \varepsilon)$*) *is an eigenfunction corresponding to $\lambda_l(\varepsilon)$* (*resp. $\lambda_r(\varepsilon)$*), *then there exist positive constants k, β and d such that*

$$|\phi_l(-1+\varepsilon s, \varepsilon)| \le ke^{-\beta s} \quad \text{for } 0 \le s \le d/\varepsilon$$

(*resp.* $|\phi_r(1+\varepsilon s, \varepsilon)| \le ke^{-\beta s}$ *for* $-d/\varepsilon \le s \le 0$).

(3) *Moreover we have that $\phi_l(-1+\varepsilon s, \varepsilon) \to \psi(s)$ as $\varepsilon \to 0$, where $\psi(s)$ verifies*

$$\ddot{\psi} + f_u(-1, z_0(s))\psi = \lambda(0)\psi\,, \quad 0 < s < +\infty\,, \tag{19}$$

(*resp.* $-\infty < s < 0$) *with the boundary condition $\psi(0) = 0$ if $\alpha_l = 1$* (*resp. $\alpha_r = 1$*) *and $\psi'(0) = 0$ otherwise.*

PROOF. We are going to consider only the left boundary layer because the proof for the right one is very similar.

First, observe that the eigenvalue $\lambda_l(\varepsilon)$ must belong to some interval $[-\Lambda, \Lambda]$ for a certain $\zeta > 0$, because the angle θ corresponding to the eigenfunction $\phi_l(x, \varepsilon)$ changes by an amount approximately π in the boundary layer zone. In other words, $\lambda_l(\varepsilon)$ can not be too positive because otherwise it would make θ jump by a bigger amount, and it can not be too negative because otherwise θ would not jump at all.

Consequently, $B(x, \varepsilon, \lambda)$ is bounded, and that gives us the bound for the eigenfunction shown in the statement of the lemma. Then, since the eigenfunction verifies

$$\ddot{\phi}_l + f_u(-1 + \varepsilon s, U(-1 + \varepsilon s, \varepsilon))\phi_l = \lambda(\varepsilon)\phi_l\,, \tag{20}$$

we can conclude that $\ddot{\phi}_l$ is also bounded. And so is $\dot{\phi}_l$ because of the interpolation inequality

$$|u'|_0 \leq \nu|u|_0 + (2/\nu)|u''|_0\,, \quad \text{for } \nu > 0\,.$$

We only need now to apply the Arzelá-Ascoli theorem to show that the eigenfunction $\phi_l(-1+\varepsilon s, \varepsilon)$ converges to the solution of (19). q.e.d.

In order to determine the stability properties of the boundary layers we need to study the limit of $\lambda_l(\varepsilon)$ (resp. $\lambda_r(\varepsilon)$) as $\varepsilon \to 0$. We will first discuss the Dirichlet case in the following theorem.

THEOREM 3.3 (Dirichlet case). *If $\lambda_l(\varepsilon)$ (resp. $\lambda_r(\varepsilon)$) is the first eigenvalue of $\mathcal{L}$ associated to the left (resp. right) boundary layer with $\alpha_l = 1$ (resp. $\alpha_r = 1$) then, for ε small enough, $\lambda_l(\varepsilon)$ (resp. $\lambda_r(\varepsilon)$) has the same sign as $\dot{z}_0(0)$ if the boundary layer goes down to zero and opposite if it goes up to one. In particular, $\lambda_l(0)$ (resp. $\lambda_r(0)$) is zero if and only if $\dot{z}_0(0)$ is zero.*

PROOF. Let us consider here only the left boundary layer since the analysis of the right one is very similar. First observe that our boundary condition is now $\psi(0) = 0$ so $\dot{\psi}(0)$ must always be different from zero. Also notice that since $\dot{z}_0(s)$ is a solution of $\ddot{\psi} + f_u(-1, z_0(s))\psi = 0$, we can write $\psi(s) = \xi\dot{z}_0(s) + \eta(s)$, where $\eta(s)$ is a function orthogonal to $\dot{z}_0(s)$.

If we multiply (20) by $\dot{z}_0(s)$ and integrate from 0 to $+\infty$, after a couple of integrations by parts, we obtain

$$\begin{aligned} R + \int_0^{+\infty} \{\dddot{z}_0(s) + f_u(-1 + \varepsilon s, Z(s, \varepsilon))\dot{z}_0(s)\}\phi_l(-1 + \varepsilon s, \varepsilon)\mathrm{d}s \\ = \lambda_l(\varepsilon) \int_0^{+\infty} \phi_l(-1 + \varepsilon s, \varepsilon)\dot{z}_0(s)\mathrm{d}s\,, \end{aligned} \tag{21}$$

where

$$R = [\dot{\phi}_l\dot{z}_0 - \phi_l\ddot{z}_0]_0^{+\infty} = \phi_l(-1, \varepsilon)\ddot{z}_0(0) - \dot{\phi}_l(-1, \varepsilon)\dot{z}_0(0)\,.$$

Taking the limit as ε goes to zero and applying the Dirichlet boundary condition we conclude that

$$-\dot{\psi}(0)\dot{z}_0(0) = \lambda_l(0)\xi \int_0^{+\infty} \dot{z}_0^2(s)\mathrm{d}s\,. \tag{22}$$

Relation (22) implies that the eigenvalue $\lambda_l(0)$ (resp. $\lambda_r(0)$) is different from zero if $\dot{z}_0(0)$ is different from zero. To determine the sign of $\lambda_l(0)$ (resp. $\lambda_r(0)$) we can make β_l (resp. β_r) go to zero or one, depending on whether the boundary layer goes down to zero or up to one. Then $\lambda_l(0)$ (resp. $\lambda_r(0)$) will converge to the value of the first eigenvalue for the non-layer situation ($z_0(s)$ identically zero or one), which can be seen to be smaller than $-\zeta^2$, by hypothesis H2. So the sign of $\lambda_l(0)$ (resp. $\lambda_r(0)$) is negative if $\dot{z}_0(0)$ is negative and the layer goes down to zero or if $\dot{z}_0(0)$ is positive and the layer goes up to one.

If there is a value γ of β_l (resp. β_r) such that the corresponding value of $\dot{z}_0(0)$ is zero, then $z_0(s)$ is half of a homoclinic orbit, as it may be the case for $c \neq 1/2$ in the example (2) which is depicted in Figure 1. In this case $\lambda_l(0)$ (resp. $\lambda_r(0)$) is zero and if we continue to move along the homoclinic orbit so $\dot{z}_0(0)$ changes sign, $\lambda_l(0)$ (resp. $\lambda_r(0)$) will also change sign.

To see this we need to normalize the eigenfunction $\psi(s)$. Since $\psi'(0)$ is different from zero, we can assign to it the value $\ddot{z}_0(0)$, which, in a neighborhood of the turning point of the homoclinic orbit, is always different from zero and makes $\xi = 1$. From (22) it follows that the eigenvalue would change sign in this case and the eigenvalue corresponding to the turning point would be zero. This normalization of the eigenfunction is valid in a neighborhood of the turning point of the homoclinic orbit because the eigenfunction corresponding to the eigenvalue at the turning point is precisely $\dot{z}_0(s)$. q.e.d.

From the previous theorem it is obvious that the value of the Dirichlet condition corresponding to the turning point of a homoclinic orbit is a bifurcation point, where a stable and an unstable branch of solutions start up or die out. In any case, as can be seen in Figure 1, there would always be another stable layer that connects all the points in the interval (0, 1) to the other fixed point (the one that is not part of the homoclinic orbit).

For the rest of the Robin cases, the presence of a boundary layer automatically makes the solution unstable. This was shown to be the case for example (2) and $c(x)$ a step function in [6] for the case of homogeneous Neumann conditions and in [4] for all the Robin cases. The next theorem extends this result to our more general setting.

THEOREM 3.4 (The other cases). *If $\lambda_l(\varepsilon)$ (resp. $\lambda_r(\varepsilon)$) is the first eigenvalue of $\mathcal{L}$ associated to the left (resp. right) boundary layer with $\alpha_l \neq 1$ (resp. $\alpha_r \neq 1$), then, for ε small enough, $\lambda_l(\varepsilon)$ (resp. $\lambda_r(\varepsilon)$) is positive. Its value is*

$$\lambda_l(0) = \lambda_r(0) = \gamma \Big/ \int_0^{+\infty} \dot{z}_0^2(s)\mathrm{d}s\,.$$

PROOF. If $\alpha_l \neq 1$ (resp. $\alpha_r \neq 1$), as ε goes to zero, we obtain an equivalent relationship to (22) of the form

$$\psi(0)\ddot{z}(0) = \lambda_l(0)\xi \int_0^{+\infty} \dot{z}_0^2(s)\mathrm{d}s\,, \tag{23}$$

(similarly with $\lambda_r(0)$). This eigenvalue should be positive when γ gets very close to one (the non-layer situation $z_0(s)$ identically one is always stable). But this implies that its sign is always positive, since from (23) it is clear that it can not ever be zero.

In order to determine the value of the eigenvalue we have to normalize the eigenfunction $\psi(s)$. Since for $\varepsilon > 0$,

$$\frac{\dot{\phi}(-1,\varepsilon)}{\phi(-1,\varepsilon)} = \frac{\varepsilon\alpha_l}{1-\alpha_l} = \frac{\dot{z}_0(0)}{z_0(0)} \quad \text{and} \quad \frac{\dot{\phi}(1,\varepsilon)}{\phi(1,\varepsilon)} = \frac{-\varepsilon\alpha_r}{1-\alpha_r} = \frac{\dot{z}_0(0)}{z_0(0)},$$

a natural choice for $\psi(0)$ is γ. To determine the value of ξ we observe ξ should go to zero as γ goes to one and ξ should have the same sign as $\ddot{z}_0(0)$ in order to make the eigenvalue positive. It is clear that the election $\xi = \ddot{z}_0(0)$ verifies all the requirements and give us the value of the eigenvalue stated in the theorem. q.e.d.

The thesis of this last theorem is not surprising, since the other two possible situations that verify the boundary conditions in this case are both stable: they are non-layer situations identically equal to zero and one. Also, this last result is consistent with Theorem 3.1 proven in [3] in the sense that only the first eigenvalues corresponding to the internal layers get arbitrary close to zero as ε goes to zero.

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