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COHOMOLOGY THEOREMS FOR ASYMPTOTIC SHEAVES

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Abstract. In this paper, we study the sheaves $\mathcal{A}_E^{<0}$ and $\mathcal{A}_E^{<-\kappa}$ of strongly asymptotically developable functions with null expansion, which are subsheaves of $\mathcal{A}$ defined by Majima. Following the method developed in one variable by Sibuya, and in several variables by Majima, we compute the first cohomology group of the n-torus and the boundary of the real blow-up with coefficients in these sheaves. The same technique is used to study the multiplicative case (sheaves of non-abelian groups), in order to calculate the first cohomology set. This generalizes previous results of Majima, Haraoka and Zurro.

1. Definitions and notations. A polysector $V = V_1 \times \cdots \times V_n$ in $\boldsymbol{C}^n$ is a product of open sectors, an open sector being a set of the type

$$V_{\alpha,\beta,R} = \left\{ z \in \boldsymbol{C} \;\middle|\; \alpha < \arg z < \beta, 0 < |z| < R \right\},$$

where $R \in (0, \infty]$. The number $\beta - \alpha$ is the opening of $V_{\alpha,\beta,R}$. A subpolysector W of V ($W < V$) is $W_1 \times \cdots \times W_n$, where W_i is a closed sector of finite radius and strictly smaller opening than V_i.

$\mathcal{A}(V)$ will denote the $\boldsymbol{C}$-algebra of functions that are strongly asymptotically developable in V, introduced by Majima in [M1]. Let us recall that $f \in \mathcal{A}(V)$ if and only if there exist a family of functions

$$\mathcal{F} = \left\{ f_{\alpha_J}(z_{J^c}) \in \mathcal{O}(V_{J^c}) \;\middle|\; \emptyset \neq J \subseteq \{1, \dots, n\}, \alpha_J \in \boldsymbol{N}^J \right\}$$

such that, if $W < V$ and $N \in \boldsymbol{N}^n$, there exists $C_{W,N} > 0$ with

$$|f(z) - \mathrm{App}_N(\mathcal{F})(z)| < C_{W,N} \cdot |z^N| \text{ in } W,$$

where

$$\mathrm{App}_N(\mathcal{F})(z) = \sum_{\emptyset \neq J \subseteq \{1,\dots,n\}} \sum_{j \in J} \sum_{\alpha_J < N_J} (-1)^{\sharp J + 1} \cdot f_{\alpha_J}(z_{J^c}) \cdot z_J^{\alpha_J}$$

and $J^c = \{1, \dots, n\} \setminus J$. We have used the following notations: if $J \in \{1, \dots, n\}$, $V_J := \prod_{j \in J} V_j$, and z_J is the element of V_J obtained by projection of $z \in V$ to V_J. This family $\mathcal{F}$ (the *total family of coefficients of* f) is unique, and it will be denoted by $TA(f)$. As in [M1], for $f \in \mathcal{A}(V)$, $\emptyset \neq J \subseteq \{1, \dots, n\}$, $FA_J(f)$ will denote the series

$$FA_J(f) = \sum_{\alpha_J \in \boldsymbol{N}^J} f_{\alpha_J}(z_{J^c}) z_J^{\alpha_J} \in \mathcal{A}(V_{J^c})[[z_J]].$$

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If $J = \{1, \dots, n\}$, we shall write $FA(f)$ instead of $FA_J(f)$.

For the main properties of $\mathcal{A}(V)$ the reader can see [M1, M2, Mo].

We shall use freely multi-index notations, such us $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \dots$, as we have already done. Also, if $\alpha, \beta \in \boldsymbol{N}^n$, $\alpha < \beta$ will mean that each component of α is strictly smaller than the corresponding component of β.

If $s = (s_1, \dots, s_n) \in \boldsymbol{R}^n_{\geq 0}$, $\mathcal{A}_s(V)$ is the $\boldsymbol{C}$-subalgebra of $\mathcal{A}(V)$ of s-Gevrey type functions (see [Ha]). This means that, with previous notations, one can choose $C_{W,N} = C'_W \cdot A_W^N \cdot N!^s$, for certain $C'_W > 0$ and $A_W \in (\boldsymbol{R}_{>0})^n$, independent of N.

As usual, $\mathcal{O}$ will be the ring of germs of holomorphic functions at the origin, and $\mathcal{O}(U)$ the set of holomorphic functions defined in the open set U. This defines a sheaf, that we shall also denote by $\mathcal{O}$.

Fixing coordinates $z_1, \dots, z_n$, E will be a (germ of) normal crossing divisor or a linear subvariety. So, if H_i is the hyperplane defined by the equation $z_i = 0$ for $i = 1, \dots, k$, then $E = H_1 \cup \cdots \cup H_k$ or $E = H_1 \cap \cdots \cap H_k$. $\mathcal{A}_E^{\leq 0}(V)$ is the subset of $\mathcal{A}(V)$ whose elements vanish on E, i.e., equivalently,

1. If $f \in \mathcal{A}(V)$ and $TA(f) = \{f_{\alpha_J}(z_{J^c}) \in \mathcal{A}(V_{J^c}) \mid \alpha_J \in \boldsymbol{N}^J\}$ is the total family of coefficients of asymptotic development for f, then
   (a) if $E = H_1 \cup \cdots \cup H_k$, $f_{\alpha_J}(z_{J^c}) = 0$ whenever $\{1, \dots, k\} \cap J \neq \emptyset$,
   (b) if $E = H_1 \cap \cdots \cap H_k$, $f_{\alpha_J}(z_{J^c}) = 0$ whenever $\{1, \dots, k\} \subseteq J$.
2. If $W < V$, there exists a C^∞ extension F of $f|_W$ such that $F \equiv 0$ on E.

This equivalence is an easy consequence of the results in [Z].

Obviously, $E_1 \subseteq E_2 \Leftrightarrow \mathcal{A}_{E_2}^{\leq 0}(V) \subseteq \mathcal{A}_{E_1}^{\leq 0}(V)$. We define analogously, if $\kappa = 1/s = (1/s_1, \dots, 1/s_n)$, $\mathcal{A}_E^{\leq -\kappa}(V) = \mathcal{A}_E^{\leq 0}(V) \cap \mathcal{A}_s(V)$ (we keep the usual notations of the 1-dimensional case used by Malgrange and Ramis).

Let $\pi : \tilde{\boldsymbol{C}} \to \boldsymbol{C}$ be the real blow-up of $\boldsymbol{C}$ at 0, with $\pi^{-1}(0) = S^1$, the points of S^1 representing real directions from 0 in $\boldsymbol{C}$. $\tilde{\boldsymbol{C}}^n$ will be the componentwise real blow-up (we blow up each component separately, see [M2, p. 37]). It is a real manifold with boundary $\partial\tilde{\boldsymbol{C}}^n$, where $\partial\tilde{\boldsymbol{C}}^n$ is the inverse image by $\pi_n : \tilde{\boldsymbol{C}}^n \to \boldsymbol{C}^n$ of the coordinate hyperplanes $H_1 \cap \cdots \cap H_n$. Let us put $X_i = \pi_n^{-1}(H_i)$. So, $\partial\tilde{\boldsymbol{C}}^n = X_1 \cup \cdots \cup X_n$. Over $\tilde{\boldsymbol{C}}^n$ we define sheaves $\mathcal{A}$, $\mathcal{A}_s$, $\mathcal{A}_E^{\leq 0}$, $\mathcal{A}_E^{\leq -\kappa}$ from the above definitions. If $z \notin \partial\tilde{\boldsymbol{C}}^n$, the stalk $\mathcal{F}_z$ ($\mathcal{F}$ being one of the above sheaves) coincides with $\mathcal{O}$. So we are mainly interested in the study of these sheaves over $\partial\tilde{\boldsymbol{C}}^n$. In particular, we shall compute the first cohomology group of $\mathcal{A}_E^{\leq 0}$, $\mathcal{A}_E^{\leq -\kappa}$ following the method developed by Sibuya. We shall need to compute H^1 also over $\boldsymbol{T}^n = X_1 \cap \cdots \cap X_n$. In this case, some of the results had already been obtained by Majima [M1], Haraoka [Ha] and Zurro [Z] (the last author uses a different method, following Malgrange in [Ml]). We will reprove them in order to obtain a global vision.

Given a Fréchet space L, $\mathcal{A}(V; L)$, $\mathcal{A}_s(V; L), \dots$ will denote the sets of strongly asymptotically developable functions with values in L (so, $\mathcal{A}(V) = \mathcal{A}(V; \boldsymbol{C})$), with obvious modifications of the definitions. The set $\mathcal{A}(V)$ is itself a Fréchet space, where a family of seminorms is given by

$$p_{W,N}(f) = \sup\{|D^N f(z)| / z \in W\},$$

and in fact we have canonical isomorphisms $\mathcal{A}(V_1; \mathcal{A}(V_2)) \cong \mathcal{A}(V_1 \times V_2)$. Precise details of this approach can be read in [He]. $L\{z\}$ will denote the $\boldsymbol{C}$-algebra of convergent series with coefficients in L, i.e., if $\sum a_\alpha z^\alpha \in L[[z]]$ and p is a continuous seminorm on L, $\sum p(a_\alpha) z^\alpha \in \boldsymbol{C}\{z\}$.

2. First cohomology group of the asymptotic sheaves.

2.1. Main lemma. The following result generalizes Cartan's decomposition lemma of complex analysis.

LEMMA 2.1. *Let* $f \in \mathcal{A}^{<0}_{(z_1=0)}$. *If* $V_1 = \{z_1 \in \boldsymbol{C} \,|\, a < \arg z_1 < b, |z_1| < R\}$, *denote* $\tilde{V}_1 = \{z \in \boldsymbol{C} \,|\, a < \arg z < b + 2\pi, |z| < R\}$. *Then there exist* $F \in \mathcal{A}(\tilde{V}_1 \times V_2 \times \cdots \times V_n)$ *such that*

$$F(z_1 e^{2\pi i}, z_2, \ldots, z_n) - F(z_1, \ldots, z_n) = f(z).$$

If $1 \notin J$, *then*

$$F_{\alpha_J}(z_1 e^{2\pi i}, z_{J^c\setminus\{1\}}) - F_{\alpha_J}(z_{J^c}) = f_{\alpha_J}(z_{J^c}).$$

If in addition, $f \in \mathcal{A}_s(V)$, $F \in \mathcal{A}_s(\tilde{V}_1 \times V_2 \times \cdots \times V_n)$.

SKETCH OF PROOF. As in the case of one variable (see [S]), we define $F(z)$ as the Cauchy-Heine transform (with parameters) of f in the first variable. This F verifies the required properties. □

2.2. Cohomology over $\boldsymbol{T}^n$. A good covering of S^1 will be a covering by (at least three) open intervals, each three of them having empty intersection. Every interval in S^1 represents a sector at the origin in $\boldsymbol{C}$, forgetting the radius. A good covering of $\boldsymbol{T}^n$ is a product of good open coverings of S^1. As every open covering of $\boldsymbol{T}^n$ can be refined by a good covering $\mathcal{U}$, we only need to compute $H^1(\mathcal{U}; *)$. In the sequel, we shall identify sectors with intervals of S^1.

PROPOSITION 2.2. *Let* $\mathcal{V} = \{V_1, \ldots, V_r\}$ *be a good covering of* S^1, *and* V' *a poly-sector in* $\boldsymbol{C}^{n-1}$. *If* $V_{ij} = V_i \cap V_j$, *let* $f_{i,i+1} \in \mathcal{A}(V_{i,i+1} \times V')$ *such that*

$$\sum_i FA_{\{1\}}(f_{i,i+1}) = 0 \in \mathcal{A}(V')[[z_1]].$$

Then, there exist $f_i \in \mathcal{A}(V_i \times V')$ *such that* $f_{i,i+1} = f_{i+1}|_{V_{i,i+1}\times V'} - f_i|_{V_{i,i+1}\times V'}$.

If $f_{i,i+1} \in \mathcal{A}_s(V_{i,i+1} \times V')$, *then* $f_i \in \mathcal{A}_s(V_i \times V')$ *(provided that the openings of* V_i *and* V' *are sufficiently small).*

REMARK 2.3. One can observe that this is a reformulation of [M2, Th. 3.2. (iii)].

PROOF OF PROPOSITION 2.2. By a theorem of Borel-Ritt, take $g_i \in \mathcal{A}(V_i \times V')$ such that

$$FA_{\{1\}}(g_i) = \sum_{k=1}^{i-1} FA_{\{1\}}(f_{k,k+1})$$

$(i = 2, \dots, r+1 = 1)$. If $h_{i,i+1} = f_{i,i+1} - g_{i+1} + g_i$, $FA_{\{1\}}(h_{i,i+1}) = 0$. Let us suppose first that $h_{i,i+1} = 0$ if $i \neq 1$. Then, Lemma 2.1 gives us a function H_{12}, and we define $h_i = H_{12}\big|_{V_i \times V'}$. It follows that $h_{i+1} - h_i = h_{i,i+1}$.

In general, let $h_j^{i,i+1} \in \mathcal{A}(V_j \times V')$ such that

$$h_{j+1}^{i,i+1} - h_j^{i,i+1} = \delta_{ij} h_{i,i+1} .$$

The functions

$$f_i = g_i + \sum_{i=1}^{r} h_j^{i,i+1}$$

give the result. The Gevrey case is similar. □

Proposition 2.2 is essentially a result in one variable, but it can be used in order to prove the corresponding assertion in several variables:

PROPOSITION 2.4. *Let $\mathcal{V} = \{V_{i_1} \times \cdots \times V_{i_n}\}$ be a good covering of $\boldsymbol{T}^n$. Then, the map*

$$H^1(\mathcal{V}; \mathcal{A}) \to H^1(\mathcal{V}; \mathcal{A}/\mathcal{A}_E^{\leq 0})$$

is injective. Namely, for each $i = (i_1, \dots, i_n)$, $i' = (i'_1, \dots, i'_n)$, let $f_{i,i'} \in \mathcal{A}(V_i \cap V_{i'})$ satisfying the cocycle condition, such that

$$[(f_{i,i'})_{i,i'}] = 0 \text{ as an element of } H^1(\mathcal{V}; \mathcal{A}/\mathcal{A}_E^{\leq 0}) ,$$

$[\cdots]$ meaning the equivalence class. Then $[(f_{i,i'})_{i,i'}] = 0$ as an element of $H^1(\mathcal{V}; \mathcal{A})$.

If $f_{i,i'}(z) \in \mathcal{A}_s(V_i \cap V_{i'})$ and $[(f_{i,i'})_{i,i'}] = 0$ in $H^1(\mathcal{V}; \mathcal{A}_s/\mathcal{A}_E^{\leq -\kappa})$, then $[(f_{i,i'})_{i,i'}] = 0$ in $H^1(\mathcal{V}; \mathcal{A}_s)$.

PROOF. Suppose first that $E = (0)$. The exactness of the sequence

$$0 \to \mathcal{A}_E^{\leq 0} \to \mathcal{A} \to \boldsymbol{C}[[z]] \to 0$$

shows that $\mathcal{A}/\mathcal{A}_E^{\leq 0} \cong \boldsymbol{C}[[z]]$ (Taylor map).

If $\tilde{\imath}$ and $\tilde{\imath}'$ are $(n-1)$-tuples, we denote

$$A_{\tilde{\imath},\tilde{\imath}'} = \sum_k FA_{\{1\}}\big(f_{(k,\tilde{\imath}),(k+1,\tilde{\imath}')}\big) \in \mathcal{A}(V_{\tilde{\imath}} \cap V_{\tilde{\imath}'})[[z_1]] ,$$

$$B_{\tilde{\imath},\tilde{\imath}'} = \sum_k FA_{\{1\}}\big(f_{(k,\tilde{\imath}),(k,\tilde{\imath}')}\big) \in \mathcal{A}(V_{\tilde{\imath}} \cap V_{\tilde{\imath}'})[[z_1]] .$$

The cocycle condition implies that

$$f_{(k,\tilde{\imath}),(k,\tilde{\imath}')} + f_{(k,\tilde{\imath}'),(k+1,\tilde{\imath}')} = f_{(k,\tilde{\imath}),(k+1,\tilde{\imath})} + f_{(k+1,\tilde{\imath}),(k+1,\tilde{\imath}')}$$

and so

$$B_{\tilde{\imath},\tilde{\imath}'} + A_{\tilde{\imath}',\tilde{\imath}'} = A_{\tilde{\imath},\tilde{\imath}} + B_{\tilde{\imath},\tilde{\imath}'} .$$

In the intersection $V_i \cap V_{i'}$, $A_{\tilde{\imath},\tilde{\imath}} = A_{\tilde{\imath}',\tilde{\imath}'}$ and so, they glue together in

$$A \in \mathcal{O}(D_2 \times \cdots \times D_n)[[z_1]] .$$

The hypothesis shows that the Taylor series of A at the origin is 0, so $A = 0 = A_{\bar{\imath},\bar{\imath}}$. Now we use a kind of argument that has been developed in [M1]. By the previous proposition, we can find $f_{(i_1,\bar{\imath})} \in \mathcal{A}(V_{i_1} \times V_{\bar{\imath}})$ such that

$$f_{(i_1,\bar{\imath}),(i'_1,\bar{\imath})} = f_{(i'_1,\bar{\imath})} - f_{(i_1,\bar{\imath})}$$

and

$$\sum_{i_1} f_{(i_1,\bar{\imath}),\alpha_J}(z_{J^c}) = 0 \quad \text{if } 1 \notin J\,.$$

Again by the cocycle condition, the equality

$$f_{(i_1,\bar{\imath}),(i'_1,\bar{\imath}')} - f_{(i'_1,\bar{\imath}')} + f_{(i_1,\bar{\imath})} = f_{(i''_1,\bar{\imath}),(i'''_1,\bar{\imath}')} - f_{(i'''_1,\bar{\imath}')} + f_{(i''_1,\bar{\imath})}$$

holds in the intersection of the domains (for indices i_1, i'_1, i''_1, i'''_1 and $(n-1)$-tuples $\bar{\imath}$, $\bar{\imath}'$). For, add $f_{(i_1,\bar{\imath}),(i'_1,\bar{\imath}')} + f_{(i''_1,\bar{\imath})} = f_{(i_1,\bar{\imath}),(i'_1,\bar{\imath}')} + f_{(i_1,\bar{\imath}),(i''_1,\bar{\imath})} + f_{(i_1,\bar{\imath})}$ to each side of the equality $f_{(i'_1,\bar{\imath}')} + f_{(i'_1,\bar{\imath}'),(i'''_1,\bar{\imath}')} = f_{(i'''_1,\bar{\imath}')}$. By glueing together the preceding functions, we obtain $f_{(\bar{\imath},\bar{\imath}')} \in \mathcal{A}(D_1 \times (V_{\bar{\imath}} \cap V_{\bar{\imath}'}))$.

We keep the same conditions as in the beginning of the proof (identify $\mathcal{A}(D \times V) \cong \mathcal{A}(V; \mathcal{O}(D))$ and consider asymptotic developments with values in the Fréchet space $\mathcal{O}(D)$). So, for each $(n-2)$-tuple $\bar{\imath}$, construct functions

$$f_{(i_2,\bar{\imath})} \in \mathcal{A}(D_1 \times V_{i_2} \times V_{\bar{\imath}})$$

such that

$$f_{(i'_2,\bar{\imath})} - f_{(i_2,\bar{\imath})} = f_{(i_2,\bar{\imath}),(i'_2,\bar{\imath})}$$

and we iterate the process, obtaining for $(n-k)$-tuples $j=(j_{k+1}, \dots, j_n)$, $j'=(j'_{k+1}, \dots, j'_n)$, and indices j_k, j'_k, functions

$$f_j \in \mathcal{A}(D_1 \times \cdots \times D_k \times V_{j_{k+1}} \times \cdots \times V_{j_n})\,,$$

$$f_{(j_k,j),(j'_k,j')} \in \mathcal{A}(D_1 \times \cdots \times D_{k-1} \times (V_{j_k} \cap V_{j'_k}) \times (V_j \cap V_{j'}))$$

such that

$$f_{j,j'} = f_{(j_k,j),(j'_k,j')} - f_{(j_k,j')} + f_{(j_k,j)}\,.$$

Defining

$$F_{(j_1,\dots,j_n)} = f_{(j_1,\dots,j_n)} + f_{(j_2,\dots,j_n)} + \cdots + f_{j_n}\,,$$

a straightforward computation shows the required property, i.e.,

$$F_{j'} - F_j = f_{j,j'}\,.$$

Consider now a general E. We have an exact diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}_E^{<0} & \to & \mathcal{A} & \to & \mathcal{A}/\mathcal{A}_E^{<0} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{A}_{(0)}^{<0} & \to & \mathcal{A} & \to & \mathbf{C}[[z]] & \to & 0\,, \end{array}$$

from which we deduce an exact cohomology sequence (taking global sections)

$$\begin{array}{ccccccccc} 0 \to & \mathbf{C}\{z\} & \to & \Gamma(\mathbf{T}^n; \mathcal{A}/\mathcal{A}_E^{<0}) & \to & H^1(\mathcal{V}; \mathcal{A}_E^{<0}) & \to & H^1(\mathcal{V}; \mathcal{A}) & \xrightarrow{\delta_1} & H^1(\mathcal{V}; \mathcal{A}/\mathcal{A}_E^{<0}) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow^{\alpha} \\ 0 \to & \mathbf{C}\{z\} & \to & \mathbf{C}[[z]] & \to & H^1(\mathcal{V}; \mathcal{A}_E^{<0}) & \to & H^1(\mathcal{V}; \mathcal{A}) & \xrightarrow{\delta_2} & H^1(\mathcal{V}, \mathbf{C}[[z]]), \end{array}$$

We have shown that $\delta_2 = \alpha \circ \delta_1$ is injective, so δ_1 is also injective. The same argument applies to the Gevrey case. □

REMARK 2.5. The spaces $\Gamma(\mathbf{T}^n; \mathcal{A}/\mathcal{A}_E^{<0})$ may be computed explicitly. Moreover, if we write

$$(*) \qquad \hat{f}(z) = \sum_{\alpha} \hat{f}_\alpha(z_1, \dots, \hat{z}_i, \dots, z_n) \cdot z_i^\alpha$$

and $E = H_1 \cup \cdots \cup H_k$, it can be canonically identified with the set of series $\hat{f} \in \mathbf{C}[[z]]$ such that if $1 \le i \le k$, then there exists a disk D around the origin in $\mathbf{C}^{n-1}$ such that $\hat{f}_\alpha \in \mathcal{O}(D)$. This set is precisely the stalk at the origin of the formal completion of the sheaf $\mathcal{O}$ along the divisor E (see [Gr]), i.e.,

$$\left(\varprojlim_k \mathcal{O}/\mathcal{I}_E^k \right)_0 .$$

If E is a linear subvariety $H_1 \cap \cdots \cap H_k$, the above stalk is precisely

$$\varinjlim_{D \to 0} \mathcal{O}(D_{k+1} \times \cdots \times D_n)[[z_1, \dots, z_k]],$$

i.e., the set of formal power series

$$(**) \qquad \sum f_{\alpha_1, \dots, \alpha_k}(z_{k+1}, \dots, z_n) \cdot z_1^{\alpha_1} \cdots z_k^{\alpha_k}$$

such that $f_\alpha(z_{k+1}, \dots, z_n) \in \mathcal{O}(D)$, D being a disk in $\mathbf{C}^{n-k}$.

Analogously, $\Gamma(\mathbf{T}^n; \mathcal{A}_s/\mathcal{A}_E^{\le -\kappa})$ is identified with the "formal Gevrey completion", i.e., if E is a normal crossing divisor $(z_1 \cdots z_k = 0)$, the set of series $(*)$ such that, if K is a compact set in D,

$$\sum \| \hat{f}_\alpha(z_1, \dots, \hat{z}_i, \dots, z_n) \|_K \cdot z_i^\alpha \in \mathbf{C}[[z_i]]_{s_i}$$

and when E is the linear subvariety $H_1 \cap \cdots \cap H_k$ we need in $(**)$ that

$$\sum \| f_{\alpha_1, \dots, \alpha_k}(z_{k+1}, \dots, z_n) \|_K \cdot z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \mathbf{C}[[z_1, \dots, z_k]]_{(s_1, \dots, s_k)}$$

for every compact set K in D.

THEOREM 2.6. *There are canonical isomorphisms*

$$H^1(\mathbf{T}^n; \mathcal{A}_E^{<0}) \cong \Gamma(\mathbf{T}^n; \mathcal{A}/\mathcal{A}_E^{<0})/\mathbf{C}\{z\},$$

$$H^1(\mathbf{T}^n; \mathcal{A}_E^{\le -\kappa}) \cong \Gamma(\mathbf{T}^n; \mathcal{A}_s/\mathcal{A}_E^{\le -\kappa})/\mathbf{C}\{z\}.$$

PROOF. Take a good covering $\mathcal{V}$ of $\mathbf{T}^n$, and the long exact sequences

$$0 \to \mathbf{C}\{z\} \to \Gamma(\mathbf{T}^n; \mathcal{A}/\mathcal{A}_E^{<0}) \to H^1(\mathbf{T}^n; \mathcal{A}_E^{<0}) \to H^1(\mathbf{T}^n; \mathcal{A}) \xrightarrow{\delta} H^1(\mathbf{T}^n; \mathcal{A}/\mathcal{A}_E^{<0}),$$

$$0\to \boldsymbol{C}\{z\}\to \Gamma(\boldsymbol{T}^n;\mathcal{A}_s/\mathcal{A}_E^{\leq -\kappa})\to H^1(\boldsymbol{T}^n;\mathcal{A}_E^{\leq -\kappa})\to H^1(\boldsymbol{T}^n;\mathcal{A}_s)\xrightarrow{\delta} H^1(\boldsymbol{T}^n;\mathcal{A}_s/\mathcal{A}_E^{\leq -\kappa}).$$

The assertion of the theorem is equivalent to the injectivity of δ, that is assured by the previous proposition. □

REMARK 2.7. When $E = H_1 \cup \cdots \cup H_n$, this result can be seen in [M1] and [Ha]. When $E = H_1 \cap \cdots \cap H_n$, it is proved in [Z] using a different method, only in the general case (not Gevrey).

2.3. Cohomology over $\partial\tilde{\boldsymbol{C}}^n$. In order to compute $H^1(\partial\tilde{\boldsymbol{C}}^n; *)$, where $*$ is one of the sheaves $\mathcal{A}_E^{<0}$, $\mathcal{A}_E^{\leq -\kappa}$, we write $\partial\tilde{\boldsymbol{C}}^n = X_1 \cup \cdots \cup X_n$ as in the beginning of the paper, and we shall apply a Mayer-Vietoris argument. At the end, we will remark the local case.

We need a previous technical lemma:

LEMMA 2.8. *If $h \in \bigcap_{k=2}^n \mathcal{O}(\boldsymbol{C}^{n-2})\{z_1, z_k\}$, there exist $h_1 \in \bigcap_{k=2}^n \mathcal{O}(\boldsymbol{C}^{n-1})\{z_k\}$, $h_2 \in \mathcal{O}(\boldsymbol{C}^{n-1})\{z_1\}$ with $h_1 - h_2 = h$.*

PROOF. h may be considered as a holomorphic function defined in a neighbourhood U of $\bigcup_{k=2}^n H_1 \cap H_k$ in $\boldsymbol{C}^n$. Let us suppose that U is a logarithmically convex Reinhardt domain (this is always possible because they form a fundamental system of neighbourhoods of the origin). We can find open subsets U_1 and U_2 of $\boldsymbol{C}^n$ such that U_1 is a neighbourhood of $\bigcup_{k=2}^n H_k$ and U_2 is a neighbourhood of H_1.

If $\mathcal{U} = \{U_1, U_2\}$ is an open covering of $U_1 \cup U_2$, since

$$H^1(\mathcal{U};\mathcal{O}) \hookrightarrow H^1(U_1 \cup U_2;\mathcal{O}) = 0,$$

we can find $h_i \in \mathcal{O}(U_i)$ such that $h_1 - h_2 = h$. These are the required functions. □

THEOREM 2.9. *There are isomorphisms*

$$H^1(\partial\tilde{\boldsymbol{C}}^n;\mathcal{A}_E^{<0}) \cong \Gamma(\partial\tilde{\boldsymbol{C}}^n;\mathcal{A}/\mathcal{A}_E^{<0}) \Big/ \bigcap_{i=1}^n \mathcal{O}(\boldsymbol{C}^{n-1})\{z_i\}$$

$$H^1(\partial\tilde{\boldsymbol{C}}^n;\mathcal{A}_E^{\leq -\kappa}) \cong \Gamma(\partial\tilde{\boldsymbol{C}}^n;\mathcal{A}_s/\mathcal{A}_E^{\leq -\kappa}) \Big/ \bigcap_{i=1}^n \mathcal{O}(\boldsymbol{C}^{n-1})\{z_i\}.$$

PROOF. First of all, we will compute $H^1(X_i; *)$ with $*$ one of the above sheaves. Suppose $i = 1$. Recall that the real blow-up $\tilde{\boldsymbol{C}}^n$ of $\boldsymbol{C}^n$ is constructed as a product of n times the real blow-up of $\boldsymbol{C}$. This latter is a product $S^1 \times \boldsymbol{R}_{\geq 0}$ (polar coordinates). So, a good covering of X_1 is composed by open sets such as

$$U = I_i \times (I_{j_2}^2 \times A_{k_2}^2) \times \cdots \times (I_{j_n}^n \times A_{k_n}^n),$$

where the I's are open intervals in S^1 and the A's in $\boldsymbol{R}_{\geq 0}$ (it is a product of good coverings of $S^1, \boldsymbol{R}_{\geq 0}$). We reorder the indices in such a way that $0 \in A_{k_l}^l \Leftrightarrow k_l = 0$.

As before, we are going to prove that the natural maps

$$H^1(X_1;\mathcal{A}) \xrightarrow{\delta} H^1(X_1;\mathcal{A}/\mathcal{A}_E^{<0}),$$

$$H^1(X_1;\mathcal{A}_s) \xrightarrow{\delta} H^1(X_1;\mathcal{A}_s/\mathcal{A}_E^{\leq -\kappa})$$

are injective. Suppose $E = 0$. In this case, if $z = (z_1, \dots, z_n) \in X_1$ and

$$J = \left\{ j \in \{2, \dots, n\} \mid z_j \in S^1 \times \{0\} \subseteq \tilde{C} \right\},$$

then the sheaves of coefficients of the right hand side are

$$(\mathcal{A}/\mathcal{A}_E^{<0})_z = \mathcal{O}_{\{2,\dots,n\}\setminus J}[[z_1, z_J]],$$
$$(\mathcal{A}_s/\mathcal{A}_E^{\leq -\kappa})_z = \mathcal{O}_{\{2,\dots,n\}\setminus J}[[z_1, z_J]]_s.$$

Take

$$\left[f_{i,(j,k)}^{i',(j',k')} \in \Gamma\left((I_i \cap I_{i'}) \times \left(\prod_{l=2}^{n} (I_{j_l}^l \times A_{k_l}^l) \cap (I_{j'_l}^l \times A_{k'_l}^l) \right), * \right) \right] \in \ker \delta .$$

Fixing i,

$$\left[\{ f_{i,(j,k)}^{i,(j',k')} \}_{(j,k),(j',k')} \right]$$

can be seen as a 1-cocycle over $\tilde{C}^{n-1}$, with parameters. We need here the following result, that we shall prove later. Remark that the first assertion is a particular case of [M2, Chapter 1, Theorem 3.3].

PROPOSITION 2.10.

$$H^1(\tilde{C}^n; \mathcal{A}_E^{<0}) \cong \Gamma(\partial\tilde{C}^n; \mathcal{A}/\mathcal{A}_E^{<0})/\mathcal{O}(C^n),$$
$$H^1(\tilde{C}^n; \mathcal{A}_E^{\leq -\kappa}) \cong \Gamma(\partial\tilde{C}^n; \mathcal{A}_s/\mathcal{A}_E^{\leq -\kappa})/\mathcal{O}(C^n).$$

Then, there are $f_{i,(j,k)}$ on $I \times \prod_{l=2}^{n}(I_{j_l^l} \times A_{k_l^l})$ such that

$$f_{i,(j,k)}^{i,(j',k')} = f_{i,(j',k')} - f_{i,(j,k)}.$$

As in Proposition 2.4, the equality

$$f_{i,(j,k)}^{i',(j',k')} - f_{i',(j',k')} + f_{i,(j,k)} = f_{i,(j'',k'')}^{i',(j''',k''')} - f_{i',(j''',k''')} + f_{i,(j'',k'')}$$

defines $F_{i,i'} \in \mathcal{A}_E^{<0}(I_i \cap I_{i'}; \mathcal{O}(C^{n-1}))$ and hence $F_{i,i'} = F_{i'} - F_i$ with $F_i \in \mathcal{A}(I_i; \mathcal{O}(C^{n-1}))$. The family of functions

$$g_{i,(j,k)} := f_{i,(j,k)} + F_i$$

has as coboundary the desired cocycle. The Gevrey case is analogous, and so,

$$H^1(X_1; \mathcal{A}_E^{<0}) \cong C[[z]]/\mathcal{O}(C^{n-1})\{z_1\},$$
$$H^1(X_1; \mathcal{A}_E^{\leq -\kappa}) \cong C[[z]]_s/\mathcal{O}(C^{n-1})\{z_1\}.$$

Now we apply induction on n. Suppose that

$$H^1(X_2 \cup \dots \cup X_n; \mathcal{A}_E^{<0}) \cong C[[z]] \Big/ \bigcap_{k=2}^{n} \mathcal{O}(C^{n-1})\{z_k\},$$

$$H^1(X_1 \cap (X_2 \cup \dots \cup X_n); \mathcal{A}_E^{<0}) \cong C[[z]] \Big/ \bigcap_{k=2}^{n} \mathcal{O}(C^{n-2})\{z_1, z_k\}.$$

The exact Mayer-Vietoris sequence [I]

$$0 \to H^1(\partial\tilde{C}^n; \mathcal{A}_E^{<0}) \to H^1(X_1; \mathcal{A}_E^{<0}) \oplus H^1(X_2 \cup \cdots \cup X_n; \mathcal{A}_E^{<0}) \\ \to H^1(X_1 \cap (X_2 \cup \cdots \cup X_n); \mathcal{A}_E^{<0})$$

is in this case

$$0 \to H^1(\partial\tilde{C}^n; \mathcal{A}_E^{<0}) \xrightarrow{\varepsilon} C[[z]]/\mathcal{O}(C^{n-1})\{z_1\} \oplus C[[z]] \Big/ \bigcap_{k=2}^{n} \mathcal{O}(C^{n-1})\{z_k\} \\ \xrightarrow{\alpha} C[[z]] \Big/ \bigcap_{k=2}^{n} \mathcal{O}(C^{n-2})\{z_1, z_k\}.$$

Let

$$\beta : C[[z]] \to C[[z]]/\mathcal{O}(C^{n-1})\{z_1\} \oplus C[[z]] \Big/ \bigcap_{k=2}^{n} \mathcal{O}(C^{n-1})\{z_k\}$$

be the natural quotient map. It is clear that $\alpha \circ \beta = 0$. Moreover, if

$$f = \left(f_1 + \mathcal{O}(C^{n-1})\{z_1\}, f_2 + \bigcap_{k=2}^{n} \mathcal{O}(C^{n-1})\{z_k\} \right) \in \ker \alpha$$

then $f_1 - f_2 \in \bigcap_{k=2}^{n} \mathcal{O}(C^{n-2})\{z_1, z_k\}$ and by Lemma 2.8, $f_1 - f_2 = h_1 - h_2$ with

$$h_1 \in \mathcal{O}(C^{n-1})\{z_1\}; \quad h_2 \in \bigcap_{k=2}^{n} \mathcal{O}(C^{n-1})\{z_k\}$$

and $f = \beta(f_1 - h_1 = f_2 - h_2)$. Then, $\operatorname{im} \beta = \ker \alpha = \operatorname{im} \varepsilon$, and there is an injective map

$$\gamma : C[[z]] \to H^1(\partial\tilde{C}^n; \mathcal{A}_E^{<0})$$

such that $\varepsilon \circ \gamma = \beta$. So, $\ker \gamma = \ker \beta = \bigcap_{k=1}^{n} \mathcal{O}(C^{n-1})\{z_k\}$ and this ends the result for $E = (0)$ (the Gevrey case is analogous).

For E general, we apply a cohomological argument as in Proposition 2.4.

The only remaining thing is the proof of Proposition 2.10. The sheaf $\mathcal{A}/\mathcal{A}_E^{<0}$ over $\tilde{C}^n$ ($E = (0)$) has stalks

$$(\mathcal{A}/\mathcal{A}_E^{<0})_z = \mathcal{O}_{\{1,\dots,n\}\setminus J}[[z_J]]$$

or

$$(\mathcal{A}_s/\mathcal{A}_E^{\le -\kappa})_z = \mathcal{O}_{\{1,\dots,n\}\setminus J}[[z_J]]$$

$\big(J = \{j \in \{1,\dots,n\} \mid z_j \in S^1 \times \{0\} \subseteq \tilde{C}\}\big)$.

We argue as before. For a 1-cocycle $f_{(j,k)}^{(j',k')}$ of the sheaf $\mathcal{A}$, that is cohomologically trivial over $\mathcal{A}/\mathcal{A}_E^{<0}$, by the previous results (again with parameters) we can find, for $J \subsetneq \{1,\dots,n\}$, $f_{(j,k)_J,(j',0)_{J^c}}$ such that

$$f_{(j,k)_J,(j',0)_{J^c}} - f_{(j,k)_J,(j,0)_{J^c}} = f_{(j,k)_J,(j,0)_{J^c}}^{(j,k)_J,(j',0)_{J^c}}$$

($(j,k)_J$ fixed, $k_l \neq 0$ if $l \in J$).

For each $J \subseteq \{1,\dots,n\}$, $k_l \neq 0$ if $l \notin J$, we have open sets $D_J \times U_{(j,k)_{J^c}}$, where D_J is a polydisk obtained by glueing the polysectors corresponding to $\prod_{i\in J}(I_{j_i}^i \times A_0^i)$ and

$U_{(j,k)_{J^c}}$ is the open set in $\boldsymbol{C}^{J^c}$ obtained from $\prod_{i\in J}(I^l_{j_i}\times A^l_{k_l})$. If $\mathcal{U}$ is the open covering of $\boldsymbol{C}^n$ consisting in the just defined sets, it is well-known that $H^1(\mathcal{U};\mathcal{O})=0$. We define a 1-cocycle on $\mathcal{U}$ with values in the sheaf $\mathcal{O}$ as follows:

i. If $J=\emptyset$, then $g^{(j',k')}_{(j,k)} := f^{(j',k')}_{(j,k)}$.

ii. If $J\neq\emptyset$, then on $(D_{J_1}\times U_{(j,k)_{J_1^c}})\cap(D_{J_2}\times U_{(j',k')_{J_2^c}})$,

$$g^{(j',k')_{J_2^c}}_{(j,k)_{J_1^c}} := f^{(j',0)_{J_2},(j',k')_{J_2^c}}_{(j,0)_{J_1},(j,k)_{J_1^c}} - f_{(j',0)_{J_2},(j',k')_{J_2^c}} + f_{(j,0)_{J_1},(j,k)_{J_1^c}}$$

(as before, independent of $(j,0)_{J_1}$, $(j',0)_{J_2}$).

So, we have

$$g^{(j',k')_{J_2^c}}_{(j,k)_{J_1^c}} = g_{(j',k')_{J_2^c}} - g_{(j,k)_{J_1^c}}\,.$$

As before, the cochain defined by

$$F_{(j,0)_J,(j,k)_{J^c}} := f_{(j,0)_J,(j,k)_{J^c}} + g_{(j,k)_{J^c}}$$

has as coboundary the desired cocycle. □

REMARK 2.11. Again, the spaces $\Gamma(\partial\tilde{\boldsymbol{C}}^n;\mathcal{A}/\mathcal{A}_E^{\leq 0})$ and $\Gamma(\partial\tilde{\boldsymbol{C}}^n;\mathcal{A}_s/\mathcal{A}_E^{\leq -\kappa})$ may be computed explicitly, and they agree with the space of sections over $\boldsymbol{T}^n$.

REMARK 2.12. If, instead of $\partial\tilde{\boldsymbol{C}}^n$, we restrict ourselves to a neighbourhood of the origin, say D, we obtain the same with $\boldsymbol{C}^n$ replaced by D, i.e.,

$$H^1(\partial\tilde{D};\mathcal{A}_E^{\leq 0}) \cong \Gamma(\partial\tilde{\boldsymbol{C}}^n;\mathcal{A}/\mathcal{A}_E^{\leq 0})\Big/\bigcap_{i=1}^n \mathcal{O}(D_1\times\cdots\times\hat{D}_i\times\cdots\times D_n)\{z_i\}\,,$$

$$H^1(\partial\tilde{D};\mathcal{A}_E^{\leq -\kappa}) \cong \Gamma(\partial\tilde{\boldsymbol{C}}^n;\mathcal{A}_s/\mathcal{A}_E^{\leq -\kappa})\Big/\bigcap_{i=1}^n \mathcal{O}(D_1\times\cdots\times\hat{D}_i\times\cdots\times D_n)\{z_i\}\,.$$

If we take the inductive limit $(\partial\tilde{\boldsymbol{C}}^n,0)=\varinjlim_r \partial\tilde{D}_r$, we obtain

$$H^1((\partial\tilde{\boldsymbol{C}}^n,0);\mathcal{A}_E^{\leq 0}) \cong \varinjlim H^1(\partial\tilde{D}_r;\mathcal{A}_E^{\leq 0}) = \Gamma(\partial\tilde{\boldsymbol{C}}^n;\mathcal{A}/\mathcal{A}_E^{\leq 0})/\boldsymbol{C}\{z\}\,,$$

$$H^1((\partial\tilde{\boldsymbol{C}}^n,0);\mathcal{A}_E^{\leq -\kappa}) \cong \varinjlim H^1(\partial\tilde{D}_r;\mathcal{A}_E^{\leq -\kappa}) = \Gamma(\partial\tilde{\boldsymbol{C}}^n;\mathcal{A}_s/\mathcal{A}_E^{\leq -\kappa})/\boldsymbol{C}\{z\}\,,$$

as H^1 commutes with $\varinjlim$.

3. First cohomology set in the non-abelian case. Now, we shall consider sheaves of invertible matrices whose coefficients have asymptotic development. These are sheaves of non-abelian groups, for which only the 0th and the 1st order Čech cohomology are well defined (several non equivalent definitions of H^2 have been given). Moreover, H^1 is not a group, but a pointed set (set with a distinguished element). The main definitions and results of non-abelian sheaf cohomology can be seen in [F, Gi, Hz].

The sheaves we shall mainly use are:

i. $GL(m, \mathcal{A})$, $GL(m, \mathcal{A}_s)$ of invertible $m \times m$ matrices with coefficients in $\mathcal{A}$, $\mathcal{A}_s$, respectively.

ii. $GL(m, \mathcal{A})_{I_m,E}$, $GL(m, \mathcal{A}_s)_{I_m,E}$ of invertible matrices that have the identity matrix I_m as asymptotic development in the divisor or subvariety E. These are the invertible matrices M such that $M - I_m$ have coefficients in $\mathcal{A}_E^{<0}$, $\mathcal{A}_E^{\leq -\kappa}$, respectively.

3.1. Main non-abelian lemma. The main lemma of Subsection 2.1 may be rephrased as follows.

LEMMA 3.1. *Let $f \in GL(m, \mathcal{A})_{I_m,(z_1=0)}(V)$ and V_{11}, V_{12} be sectors in $\mathbf{C}$ with $V_{11} \cap V_{12} = V_1$. If $W' < V$, $W'_{11} < V_{11}$, $W'_{12} < V_{12}$ and $W'_{11} \cap W'_{12} = W'_1$, then there exist $f_i(z) \in GL(m, \mathcal{A})_{I_m,(z_1=0)}(W)$ (perhaps reducing the radius of W) such that $f = f_1^{-1} \cdot f_2$.*

PROOF. We follow here the idea of the proof of Sibuya [S] for the case of one variable. We will omit the details that are similar to those encountered there.

Take a sequence of subsectors $W_1 > W_2 > \cdots$ such that $\bigcap_{k=1}^{\infty} W_k = W'$. If $g_1(z) = f(z) - I_m$, write (by Lemma 2.1) $g_1(z) = g_{12}(z) - g_{11}(z)$ on $W_1 \times V_2 \times \cdots \times V_n$. Define $g_2(z) = g_{11}(z) \cdot g_1(z) \cdot (I_m + g_{12}(z))^{-1}$ and write $g_2(z) = g_{22}(z) - g_{21}(z)$ on $W_2 \times V_2 \times \cdots \times V_n$.

Iterating this process, you have a sequence $\{g_k(z) = g_{k2}(z) - g_{k1}(z)\}_{k=1}^{\infty}$. Define now

$$f_1(z) = \lim_{k\to\infty} (I_m + g_{ki}(z)) \cdot (I_m + g_{k-1,i}(z)) \cdots (I_m + g_{k1}(z)).$$

By the equality

$$(I_m + g_{k+1}(z)) \cdot (I_m + g_{k2}(z)) = (I_m + g_{k1}(z)) \cdot (I_m + g_k(z)),$$

it can be seen that $f_1(z) \cdot f(z) = f_2(z)$. Choosing $\{W_k\}_k$ carefully, the convergence of the expression defining $f_i(z)$ is assured, from the expression of the Cauchy-Heine transform and reducing the radius conveniently.

In order to show taht $f_i(z)$ has an asymptotic development, we use [M1, prop. 3] (for a proof, see [Mo]). A bound $C^k_{W'',N}$ for the derivative $D^N g_{ki}(z)$ on $W'' < W'_{i_1} \times W'_2 \times \cdots \times W'_n$ can be obtained by a universal polynomial P_N with natural coefficients

$$C^k_{W'',N} = P_N(\{C^{k-1}_{W'',N'}/N' \leq N\}, R'')$$

(R'' the radius), such that R'' divides P_N, and so, reducing R'', the series

$$\sum_{k=1}^{\infty} C^k_{W'',N}$$

converges, which completes the proof. □

4. Computation of the cohomology over T^n in the non-abelian case. The multiplicative analogue of Proposition 2.2 can be stated as

PROPOSITION 4.1. *Let $\mathcal{V} = \{V_1, \ldots, V_r\}$ be a good covering of S^1, and V' a polysector in $\mathbf{C}^{n-1}$. If $F_{i,i+1}$ are $m \times m$ matrices of functions with asymptotic development on $V_{i,i+1} \times V'$, and*

$$FA_1(F_{12}) \cdot FA_1(F_{23}) \cdots FA_1(F_{r1}) = I_m,$$

then there are F_i on $V_i \times V'$ such that $F_i \cdot F_{i,i+1} = F_{i+1}$ on $V_{i,i+1} \times V'$.

PROOF. As in the function case, if G_i on $V_i \times V'$ is such that

$$FA_1(G_i) = FA_1(F_{12}) \cdot FA_1(F_{23}) \cdots FA_1(F_{i-1,i})$$

and $f_{i,i+1} = G_i \cdot F_{i,i+1} \cdot G_{i+1}^{-1}$ ($G_1 = I_m$), then we can suppose $FA_1(F_{i,i+1}) = I_m$. As in that case, consider first the situation where only perhaps $F_{12} \neq I_m$. Take $f_1' \in GL(m, \mathcal{A})_{I_m,(z_1=0)}(V_1 \times V')$, $f_2' \in GL(m, \mathcal{A})_{I_m,(z_1=0)}((V_2 \cup \cdots \cup V_r) \times V')$ (Lemma 3.1). If, in the Riemann surface of $\log z$, we write $f_2' = f_r^{-1} \cdot f_r'$, with f_r on $V_2 \cup \cdots \cup V_r \cup V_1$, f_r' on $V_1 \cup \cdots \cup V_r$, the equality $f_r \cdot f_2' = f_r'$ allows us to define a function f on a big sector, and its restrictions to V_1 are the functions F_i.

In the general case, we argue as in the abelian case, by "multiplicative linearity". This argument was also used by Sibuya [S] and Majima [M1]. □

THEOREM 4.2. *In the category of pointed sets, there are isomorphisms*

$$H^1(\boldsymbol{T}^n; GL(m, \mathcal{A})_{I_m,E}) \cong \Gamma(\boldsymbol{T}^n; GL(m, \mathcal{A})/GL(m, \mathcal{A})_{I_m,E})/GL(m, \boldsymbol{C}\{z\}).$$

PROOF. We shall suppose $E = (0)$. In this case, we have the equivalence

$$\Gamma(\boldsymbol{T}^n; GL(m, \mathcal{A})/\mathcal{A}_E^{\leq 0}) \cong GL(m, \boldsymbol{C}[[z]]).$$

By a theorem of Borel-Ritt, the natural map (fixing a good covering $\mathcal{V}$ of $\boldsymbol{T}^n$)

$$GL(m, \boldsymbol{C}[[z]]) \xrightarrow{\alpha} H^1(\boldsymbol{T}^n; GL(m, \mathcal{A})_{I_m,E})$$

can be constructed in such a way that $\alpha(F) = \alpha(FG)$ if $G \in GL(m, \boldsymbol{C}\{z\})$. So, it defines a map (again denoted by α)

$$GL(m, \boldsymbol{C}[[z]])/GL(m, \boldsymbol{C}\{z\}) \xrightarrow{\alpha} H^1(\boldsymbol{T}^n; GL(m, \mathcal{A})_{I_m,E}).$$

It is easily checked that α is injective. Let us prove its surjectivity. If $[(F_{i,i'})_{i,i'}] \in H^1(\mathcal{V}; GL(m, \mathcal{A})_{I_m,E})$, we shall show, by induction on n, that:

1. There are $F_i \in \Gamma(V_i; GL(m, \mathcal{A}))$ such that $F_{ii'} = F_i^{-1} \cdot F_{i'}$.
2. Fixing $\bar{\imath} \in \boldsymbol{N}^{n-1}$, then

$$FA_1(F_{(1,\bar{\imath}),(2,\bar{\imath})}) \cdot FA_1(F_{(2,\bar{\imath}),(3,\bar{\imath})}) \cdots FA_1(F_{(r,\bar{\imath}),(1,\bar{\imath})}) = I_m$$

as a matrix of elements of $\mathcal{A}(V_{\bar{\imath}})[[z_1]]$.

If $n = 1$, the result is trivial. For bigger n, let

$$A_{\bar{\imath}} = FA_1(F_{(1,\bar{\imath}),(2,\bar{\imath})}) \cdot FA_1(F_{(2,\bar{\imath}),(3,\bar{\imath})}) \cdots FA_1(F_{(r,\bar{\imath})(1,\bar{\imath})}) \in \mathcal{A}(V_{\bar{\imath}})[[z_1]].$$

From the equality

$$F_{(k,\bar{\imath}),(k,\bar{\imath}')} \cdot F_{(k,\bar{\imath}'),(k+1,\bar{\imath}')} = F_{(k,\bar{\imath}),(k+1,\bar{\imath})} \cdot F_{(k+1,\bar{\imath}),(k+1,\bar{\imath}')},$$

we obtain

$$A_{\bar{\imath}} \cdot FA_1(F_{(1,\bar{\imath}),(1,\bar{\imath}')}) = FA_1(F_{(1,\bar{\imath}),(1,\bar{\imath}')}) \cdot A_{\bar{\imath}'}.$$

The family

$$(FA_1(F_{(1,\bar{\imath}),(1,\bar{\imath}')}))_{\bar{\imath},\bar{\imath}'}$$

is a cocycle of the sheaf $GL(m, \boldsymbol{C}[[z]])$. If L is a Fréchet space, $L[[z]]$ has also a natural Fréchet structure, and the space $\mathcal{A}(V)[[z]]$ can be identified with $\mathcal{A}(V; \boldsymbol{C}[[z]])$. Using the induction hypothesis (for asymptotic developments with coefficients in a Fréchet space), we obtain $F_{\tilde{\imath}} \in GL(m, \mathcal{A}(*, \boldsymbol{C}[[z]]))(V_{\tilde{\imath}})$ such that

$$FA_1(F_{(1,\tilde{\imath}),(1,\tilde{\imath}')}) = F_{\tilde{\imath}}^{-1} \cdot F_{\tilde{\imath}'} ,$$

and so the equality

$$F_{\tilde{\imath}} \cdot A_{\tilde{\imath}} \cdot F_{\tilde{\imath}}^{-1} = F_{\tilde{\imath}'} \cdot A_{\tilde{\imath}'} \cdot F_{\tilde{\imath}'}^{-1}$$

defines a matrix with elements in $\boldsymbol{C}\{z_2, \ldots , z_n\}[[z_1]]$. As $A_{\tilde{\imath}}$ has the identity matrix as an asymptotic development at the origin, $F_{\tilde{\imath}} \cdot A_{\tilde{\imath}} F_{\tilde{\imath}}^{-1} = I_m$ and hence $A_{\tilde{\imath}} = I_m$.

The rest of the reasoning is exactly as in the abelian case. We obtain $F_{(k,\tilde{\imath})}$ with

$$F_{(k,\tilde{\imath})} \cdot F_{(k,\tilde{\imath})(k+1,\tilde{\imath})}$$

and the identity

$$F_{(i_1,\tilde{\imath})} \cdot F_{(i_1,\tilde{\imath}),(i_1',\tilde{\imath}')} \cdot F^{-1}_{(i_1',\tilde{\imath}')} = F_{(i_1'',\tilde{\imath})} \cdot F_{(i_1'',\tilde{\imath}),(i_1''',\tilde{\imath}')} \cdot F^{-1}_{(i_1''',\tilde{\imath}')}$$

allows us to reason by recurrence. At last, we construct $F_i \in GL(m, \mathcal{A})(V_i)$ with $F_{i,i'} = F_i^{-1} F_{i'}$. If $\hat{F} \in \boldsymbol{C}[[z]]$ is the series of asymptotic development at the origin of F_i, $\alpha(\hat{F}) = (F_{i,i'})_{i,i'}$ and so α is surjective.

For E general, it is possible to use a cohomological argument as in the function case (there is a "long" exact sequence in the category of pointed sets, see [F]), or to repeat the preceding argument for each E. □

REMARK 4.3. As in the abelian case, if $E = H_1 \cup \cdots \cup H_n$, the result can be read in [M1] and [Ha] (general and Gevrey cases, respectively).

5. General and Gevrey cases. The only essential difference with the function case is in the use of Mayer-Vietoris principle. Such a result, for the non-abelian sheaf cohomology, seems to be known by the specialists, but we have not found a precise reference. With this result in hand, we could show:

THEOREM 5.1. *There is an isomorphism of pointed sets*

$$H^1(\partial\tilde{\boldsymbol{C}}^n; GL(m, \mathcal{A})_{I_m,E}) \cong \Gamma(\partial\tilde{\boldsymbol{C}}^n; GL(m, \mathcal{A})/\mathcal{A}_E^{<0}) \Big/ GL\left(m, \bigcap_{i=1}^{n} \mathcal{O}(\boldsymbol{C}^{n-1})\{z_i\}\right) .$$

The Gevrey case is deduced from the general asymptotic case by the following observation: If $(f_{i,i'})_{i,i'}$ is a Gevrey 1-cocycle, we can write $(f_{i,i'})_{i,i'} = f_i^{-1} \cdot f_{i'}$ where f_i has asymptotic expansion (not necessarily Gevrey). If $(f_{i,i'})_{i,i'} = I_m + g_{i,i'}$, we have $f_i \cdot g_{i,i'} = f_{i'} - f_i$ with null asymptotic development s-Gevrey over the corresponding divisor or subvariety E. So, $f_{i'} - f_i = g_{i'} - g_i$, g_i s-Gevrey (by the function case). Then, $f_i = u + g_i$, with u convergent (u is formed by glueing $f_i - g_i = f_{i'} - g_{i'}$) and g_i is s-Gevrey, so f_i is s-Gevrey.

THEOREM 5.2. *There are isomorphisms of pointed sets*

$$H^1(\boldsymbol{T}^n; GL(m, \mathcal{A}_s)_{I_m,E}) \cong \Gamma(\boldsymbol{T}^n; GL(m, \mathcal{A}_s)/GL(m, \mathcal{A}_s)_{I_m,E})/GL(m, \boldsymbol{C}\{z\}) ,$$

$$H^1(\partial\tilde{C}^n; GL(m,\mathcal{A}_s)_{I_m,E}) \cong \Gamma(T^n; GL(m,\mathcal{A}_s)/GL(m,\mathcal{A}_s)_{I_m,E}) \Big/ GL\left(m, \bigcap_{i=1}^{n} \mathcal{O}(C^{n-1})\{z_i\}\right).$$

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