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ISOMETRIC DEFORMATIONS OF FLAT TORI IN THE 3-SPHERE WITH NONCONSTANT MEAN CURVATURE

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Abstract. For an isometric immersion f of a flat torus into the unit 3-sphere, we show that if the mean curvature of f is not constant, then the immersion f admits a nontrivial isometric deformation preserving the total mean curvature.

1. Introduction. Let S^3 be the 3-dimensional standard unit sphere in the Euclidean space $\boldsymbol{R}^4$. For each θ satisfying $0 < \theta < \pi/2$, we consider the *Clifford torus* $M_\theta \subset S^3$ defined by

$$M_\theta = \{x \in \boldsymbol{R}^4 : x_1^2 + x_2^2 = \cos^2\theta, x_3^2 + x_4^2 = \sin^2\theta\}.$$

The Clifford torus M_θ is a flat Riemannian manifold equipped with the metric induced by the inclusion map $i_\theta : M_\theta \to S^3$. In [2] the author proved that every isometric deformation of $i_\theta : M_\theta \to S^3$ is trivial. Incidentally, it is easy to see that if M is a flat torus isometrically embedded in S^3 with constant mean curvature, then there exists a Clifford torus M_θ which is congruent to M. So we obtain

THEOREM 1.1. *If $f : M \to S^3$ is an isometric embedding of a flat torus M into S^3 with constant mean curvature, then every isometric deformation of the embedding f is trivial.*

On the other hand there are many flat tori isometrically immersed in S^3 with nonconstant mean curvature. In this paper we deal with isometric deformations of these surfaces. To state the result we recall the notion of congruence of immersions. For $i = 1, 2$, let $f_i : X_i \to Y$ be an immersion of a smooth manifold X_i into a Riemannian manifold Y. The immersions f_1 and f_2 are said to be *congruent* if there exist an isometry $A : Y \to Y$ and a diffeomorphism $\rho : X_1 \to X_2$ such that $A \circ f_1 = f_2 \circ \rho$. We shall write $f_1 \equiv f_2$ if f_1 and f_2 are congruent. The main result of this paper is the following theorem.

THEOREM 1.2. *If $f : M \to S^3$ is an isometric immersion of a flat torus M into S^3 with nonconstant mean curvature, then there exists a smooth one-parameter family of isometric immersions $f_t : M \to S^3$, $t \in \boldsymbol{R}$, such that $f_0 = f$ and*

(1) *$f_t \not\equiv f_s$ for all $s \neq t$,*

(2) *the total mean curvature of f_t does not depend on t.*

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REMARK. The total mean curvature of the immersion f_t is given by $\int_M H_t d\sigma$, where H_t denotes the mean curvature function of f_t, and $d\sigma$ denotes the volume element of the flat torus M.

The outline of this paper is as follows. In Section 2 we introduce some geometric invariants of a periodic regular curve $\gamma : \boldsymbol{R} \to S^2$. We denote by $K(\gamma)$ (resp. $L(\gamma)$) the total geodesic curvature (resp. the length) of the closed curve $\gamma|[0, l]$, where $l > 0$ is the minimum period of γ. Furthermore, using the curve $\hat{\gamma} = \dot{\gamma}/|\dot{\gamma}|$ in the unit tangent bundle of S^2, we define $I(\gamma)$ to be the homology class represented by the closed curve $\hat{\gamma}|[0, l]$.

In Section 3 we explain a method for constructing all the flat tori in S^3. A pair $\Gamma = (\gamma_1, \gamma_2)$ of periodic regular curves $\gamma_i : \boldsymbol{R} \to S^2$ is said to be a *periodic admissible pair* if the geodesic curvature of γ_1 is greater than that of γ_2 and some auxiliary conditions are satisfied (see Definition 3.1). Each periodic admissible pair Γ induces a flat torus M_Γ and an isometric immersion $f_\Gamma : M_\Gamma \to S^3$. Furthermore the immersion f_Γ is a primitive immersion (see Definition 3.2). Conversely, if $f : M \to S^3$ is a primitive isometric immersion of a flat torus M into S^3, then there exists a periodic admissible pair Γ such that $f \equiv f_\Gamma$ (Theorem 3.1).

In Section 4 we study the intrinsic structure of the flat torus M_Γ. For each periodic admissible pair $\Gamma = (\gamma_1, \gamma_2)$, we set

$$K_i(\Gamma) = K(\gamma_i)\,, \quad L_i(\Gamma) = L(\gamma_i)\,, \quad I_i(\Gamma) = I(\gamma_i)\,,$$

and define $W(\Gamma)$ to be a lattice of $\boldsymbol{R}^2$ whose generators can be written in terms of $K_i(\Gamma)$, $L_i(\Gamma)$ and $I_i(\Gamma)$. Then it is shown that the flat torus $\boldsymbol{R}^2/W(\Gamma)$ is isometric to the flat torus M_Γ (Theorem 4.1).

In Section 5 we deal with the extrinsic structure of the immersion f_Γ. For each smooth even function $\beta : \boldsymbol{R} \to \boldsymbol{R}$, we construct a functional E_β which is defined on the set of all periodic admissible pairs, and show that $E_\beta(\Gamma) = E_\beta(\bar{\Gamma})$ if $f_\Gamma \equiv f_{\bar{\Gamma}}$ (Theorem 5.1). Furthermore we show that the total mean curvature of f_Γ can be written in terms of $K_i(\Gamma)$, $L_i(\Gamma)$ and $I_i(\Gamma)$ (Theorem 5.3).

In Sections 6 and 7 we give the proof of Theorem 1.2. To establish the theorem we may assume that the immersion $f : M \to S^3$ is primitive. By Theorem 3.1 there exists a periodic admissible pair $\Gamma = (\gamma_1, \gamma_2)$ satisfying $f \equiv f_\Gamma$. Since the mean curvature of f_Γ is not constant, we see that either γ_1 or γ_2 is not a circle. Using this fact, we construct a smooth even function β and a smooth one-parameter family of periodic admissible pairs Γ_t satisfying

$$\Gamma_0 = \Gamma\,, \quad K_i(\Gamma_t) = K_i(\Gamma)\,, \quad L_i(\Gamma_t) = L_i(\Gamma)\,, \quad I_i(\Gamma_t) = I_i(\Gamma)\,,$$

and $E_\beta(\Gamma_s) \neq E_\beta(\Gamma_t)$ for all $s \neq t$. So the assertion of Theorem 1.2 follows from Theorems 4.1, 5.1 and 5.3.

REMARK. In Theorem 1.1 the word "embedding" cannot be replaced by the word "immersion". In fact, there is a flat torus M and a Riemannian covering $\pi : M \to M_\theta$ such that the composition $i_\theta \circ \pi : M \to S^3$ admits a nontrivial isometric deformation. The Riemannian coverings as above will be classified in [4].

2. Preliminaries. Let $SU(2)$ be the group of all 2×2 unitary matrices with determinant 1. Its Lie algebra $\mathfrak{su}(2)$ consists of all 2×2 skew Hermitian matrices of trace 0. The adjoint representation of $SU(2)$ is given by

$$\mathrm{Ad}(a)x = axa^{-1},$$

where $a \in SU(2)$ and $x \in \mathfrak{su}(2)$. We set

$$\langle x, y \rangle = -\frac{1}{2}\mathrm{trace}(xy) \quad \text{for} \quad x, y \in \mathfrak{su}(2).$$

Then it follows that $\langle\, ,\, \rangle$ is a positive definite and Ad-invariant inner product on $\mathfrak{su}(2)$. Furthermore we consider the orthonormal basis of $\mathfrak{su}(2)$ given by

$$e_1 = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}.$$

Note that

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = 2e_2,$$

where $[\, ,\,]$ denotes the Lie bracket on $\mathfrak{su}(2)$. For $i = 1, 2, 3$, let E_i be the left invariant vector field on $SU(2)$ corresponding to e_i. We endow $SU(2)$ with the Riemannian metric $\langle\, ,\, \rangle$ such that $\langle E_i, E_j \rangle = \delta_{ij}$. Then $SU(2)$ is isometric to the unit 3-sphere S^3, and so we identify S^3 with $SU(2)$.

Let S^2 be the unit sphere in $\mathfrak{su}(2)$ defined by $S^2 = \{x \in \mathfrak{su}(2) : |x| = 1\}$. The unit tangent bundle of S^2, denoted by US^2, can be identified with a subset of $S^2 \times S^2$ as follows:

$$US^2 = \{(x, v) \in S^2 \times S^2 : \langle x, v \rangle = 0\},$$

where the canonical projection $p_1 : US^2 \to S^2$ is given by $p_1(x, v) = x$. Define $p_2 : S^3 \to US^2$ by

$$p_2(a) = (\mathrm{Ad}(a)e_3, \mathrm{Ad}(a)e_1). \tag{2.1}$$

The map p_2 is a double covering such that $p_2(-a) = p_2(a)$ for all $a \in S^3$. We now consider a regular curve $\gamma : \boldsymbol{R} \to S^2$, and define $\hat{\gamma} : \boldsymbol{R} \to US^2$ by

$$\hat{\gamma}(s) = (\gamma(s), \gamma'(s)/|\gamma'(s)|). \tag{2.2}$$

Then there exists a curve $c : \boldsymbol{R} \to S^3$ satisfying $p_2(c(s)) = \hat{\gamma}(s)$. By [3, Lemma 2.2] we obtain

$$c(s)^{-1}c'(s) = \frac{1}{2}|\gamma'(s)|\{e_2 + k(s)e_3\}, \tag{2.3}$$

where $k(s)$ denotes the geodesic curvature of $\gamma(s)$. Note that

$$k(s) = \langle \gamma''(s), J(\gamma'(s)) \rangle / |\gamma'(s)|^3, \tag{2.4}$$

where J denotes the almost complex structure on S^2 defined by

$$J(v) = \frac{1}{2}[x, v] \quad \text{for} \quad v \in T_xS^2. \tag{2.5}$$

We now assume that the curve $\gamma : \boldsymbol{R} \to S^2$ is periodic with the minimum period $l > 0$. The length and the total geodesic curvature of γ are given by

$$L(\gamma) = \int_0^l |\gamma'(s)| ds\,, \quad K(\gamma) = \int_0^l k(s)|\gamma'(s)| ds\,. \tag{2.6}$$

Furthermore define $I(\gamma)$ to be the element of the homology group $H_1(US^2)$ represented by the closed curve $\hat{\gamma}|[0, l]$. Note that $H_1(US^2) \cong \boldsymbol{Z}_2$. Since p_2 is a double covering and $p_2(a) = p_2(-a)$ for all $a \in S^3$, we obtain

$$c(s + l) = \begin{cases} c(s) & \text{if } I(\gamma) = 0\,, \\ -c(s) & \text{if } I(\gamma) = 1\,. \end{cases} \tag{2.7}$$

3. Construction of flat tori in S^3. In this section we explain a method for constructing all the flat tori in S^3, which was established in [1] and [3].

DEFINITION 3.1. Let $\Gamma = (\gamma_1, \gamma_2)$ be a pair of regular curves $\gamma_i : \boldsymbol{R} \to S^2$, $i = 1, 2$. The pair Γ is said to be an *admissble pair* if it satisfies the following conditions (3.1)–(3.3).

$$\hat{\gamma}_1(0) = \hat{\gamma}_2(0) = (e_3, e_1)\,, \tag{3.1}$$

$$|\gamma_i'(s)|\sqrt{1 + k_i(s)^2} = 2 \quad \text{for} \quad i = 1, 2\,, \tag{3.2}$$

$$k_1(s_1) > k_2(s_2) \quad \text{for all} \quad (s_1, s_2) \in \boldsymbol{R}^2\,, \tag{3.3}$$

where $k_i(s)$ denotes the geodesic curvature of $\gamma_i(s)$.

Let $\Gamma = (\gamma_1, \gamma_2)$ be an admissible pair. Then it follows from (3.1) that there exist curves $c_i : \boldsymbol{R} \to S^3$, $i = 1, 2$, such that

$$p_2(c_i(s)) = \hat{\gamma}_i(s)\,, \quad c_i(0) = e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.4}$$

By (2.3) and (3.2) we obtain $|c_i'(s)| = 1$. Using the group structure on S^3, we define $F_\Gamma : \boldsymbol{R}^2 \to S^3$ by

$$F_\Gamma(s_1, s_2) = c_1(s_1)c_2(s_2)^{-1}\,. \tag{3.5}$$

By [1, Lemma 3.8, Theorem 4.2] we see that the map F_Γ is a flat asymptotic Tchebychef immersion (FAT for short). For the definition of FAT, we refer the reader to [1, p. 460]. So the map F_Γ is an immersion which induces a flat Riemannian metric g_Γ on $\boldsymbol{R}^2$. Let $\alpha_i(s)$ be the function defined by

$$\cot\alpha_i(s) = k_i(s)\,, \quad 0 < \alpha_i(s) < \pi\,. \tag{3.6}$$

Then (3.3) implies $\alpha_1(s_1) < \alpha_2(s_2)$. Using (3.2), we obtain

$$\sin\alpha_i(s) = \frac{1}{2}|\gamma_i'(s)|\,, \quad \cos\alpha_i(s) = \frac{1}{2}k_i(s)|\gamma_i'(s)|\,. \tag{3.7}$$

So it follows from (2.3) that

$$c_i^{-1}(s)c_i'(s) = \sin\alpha_i(s)e_2 + \cos\alpha_i(s)e_3\,.$$

Hence the components of the Riemannian metric g_Γ for the local coordinates (s_1, s_2) satisfy

$$g_{11} = g_{22} = 1\,, \quad g_{12} = -\cos(\alpha_2(s_2) - \alpha_1(s_1))\,. \tag{3.8}$$

Furthermore the components of the second fundamental form of the immersion $F_\Gamma(s_1, s_2)$ satisfy

$$h_{11} = h_{22} = 0\,, \quad h_{12} = \sin(\alpha_2(s_2) - \alpha_1(s_1))\,, \tag{3.9}$$

where the unit normal is defined by

$$\xi = (\partial F_\Gamma/\partial s_1) \times (\partial F_\Gamma/\partial s_2)/|(\partial F_\Gamma/\partial s_1) \times (\partial F_\Gamma/\partial s_2)|\,.$$

We now consider the group

$$G(\Gamma) = \{\rho \in \mathrm{Diff}(\boldsymbol{R}^2) : F_\Gamma \circ \rho = F_\Gamma\}\,,$$

where $\mathrm{Diff}(\boldsymbol{R}^2)$ denotes the group of all diffeomorphisms of $\boldsymbol{R}^2$. Then we obtain the 2-dimensional flat Riemannian manifold $M_\Gamma = (\boldsymbol{R}^2, g_\Gamma)/G(\Gamma)$ and the isometric immersion $f_\Gamma : M_\Gamma \to S^3$ satisfying $f_\Gamma \circ \pi_\Gamma = F_\Gamma$, where π_Γ denotes the canonical projection of $\boldsymbol{R}^2$ onto M_Γ. It is easy to see that the immersion $f_\Gamma : M_\Gamma \to S^3$ is primitive in the sense of the following definition.

DEFINITION 3.2. An immersion $f : X \to Y$ of a smooth manifold X into a smooth manifold Y is said to be *primitive* if the identity map of X is the only diffeomorphism $\phi : X \to X$ satifying $f \circ \phi = f$.

It follows from [1, Theorem 2.3] that the group $G(\Gamma)$ consists of parallel translations of $\boldsymbol{R}^2$, and so M_Γ is orientable. Furthermore it follows from [1, Theorem 5.1] that M_Γ is compact if and only if Γ is periodic, where the admissible pair $\Gamma = (\gamma_1, \gamma_2)$ is said to be *periodic* if both γ_1 and γ_2 are periodic regular curves. So we see that every periodic admissible pair Γ induces a flat torus M_Γ and a primitive isometric immersion $f_\Gamma : M_\Gamma \to S^3$. Conversely, we obtain the following theorem.

THEOREM 3.1 ([3]). *Let $f : M \to S^3$ be a primitive isometric immersion of a flat torus M. Then there exists a periodic admissible pair Γ such that $f \equiv f_\Gamma$.*

We conclude this section with the following theorem.

THEOREM 3.2. *Let $\Gamma = (\gamma_1, \gamma_2)$ be an admissible pair, and let $k_i(s)$ denote the geodesic curvature of $\gamma_i(s)$. Then the mean curvature of f_Γ is constant if and only if both $k_1(s)$ and $k_2(s)$ are constant.*

PROOF. By (3.8) and (3.9) the mean curvature H of F_Γ is given by

$$H = \cot(\alpha_2(s_2) - \alpha_1(s_1))\,. \tag{3.10}$$

So (3.6) implies the assertion of Theorem 3.2. □

4. The intrinsic structure of M_Γ. Let $\Gamma = (\gamma_1, \gamma_2)$ be a periodic admissible pair. Using the homology class $I(\gamma_i)$ defined in Section 2, we set

$$I(\Gamma) = (I(\gamma_1), I(\gamma_2))\,,$$

and define $W(\Gamma)$ to be the lattice of $\boldsymbol{R}^2$ whose generators are given by the following:

$$
\begin{cases} v_1, v_2 & \text{if } I(\Gamma) = (0,0)\,, \\ 2v_1, v_2 & \text{if } I(\Gamma) = (1,0)\,, \\ v_1, 2v_2 & \text{if } I(\Gamma) = (0,1)\,, \\ v_1 \pm v_2 & \text{if } I(\Gamma) = (1.1)\,, \end{cases} \tag{4.1}
$$

where

$$
v_1 = \frac{1}{2}(K(\gamma_1), L(\gamma_1))\,, \quad v_2 = \frac{1}{2}(-K(\gamma_2), -L(\gamma_2))\,. \tag{4.2}
$$

We now identify the lattice $W(\Gamma)$ with a group of parallel translations of $\boldsymbol{R}^2$. In this section we show that the flat torus M_Γ is isometric to the flat torus $(\boldsymbol{R}^2, g_0)/W(\Gamma)$, where g_0 denotes the canonical flat Riemannian metric on $\boldsymbol{R}^2$. Using the functions $\alpha_1(s)$ and $\alpha_2(s)$ given by (3.6), we set

$$
x_1(s_1, s_2) = \int_0^{s_1} \cos\alpha_1(s)ds - \int_0^{s_2} \cos\alpha_2(s)ds\,,
$$

$$
x_2(s_1, s_2) = \int_0^{s_1} \sin\alpha_1(s)ds - \int_0^{s_2} \sin\alpha_2(s)ds\,,
$$

and define $\Phi_\Gamma : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ by

$$
\Phi_\Gamma(s_1, s_2) = (x_1(s_1, s_2), x_2(s_1, s_2))\,. \tag{4.3}
$$

THEOREM 4.1. *Let $\Gamma = (\gamma_1, \gamma_2)$ be a periodic admissible pair, and let g_Γ be the Riemannian metric on $\boldsymbol{R}^2$ induced by the immersion $F_\Gamma : \boldsymbol{R}^2 \to S^3$. Then the map Φ_Γ is an isometry of $(\boldsymbol{R}^2, g_\Gamma)$ onto $(\boldsymbol{R}^2, g_0)$, and*

$$
W(\Gamma) = \{\Phi_\Gamma \circ \rho \circ \Phi_\Gamma^{-1} : \rho \in G(\Gamma)\}\,.
$$

In particular, the flat torus M_Γ is isometric to the flat torus $(\boldsymbol{R}^2, g_0)/W(\Gamma)$.

PROOF. By (3.8) it is easy to see that $g_\Gamma = \Phi_\Gamma^* g_0$, and so Φ_Γ is an isometry of $(\boldsymbol{R}^2, g_\Gamma)$ onto $(\boldsymbol{R}^2, g_0)$. Since the group $G(\Gamma)$ consists of parallel traslations of $\boldsymbol{R}^2$ and the quotient space $\boldsymbol{R}^2/G(\Gamma)$ is compact, the group $G(\Gamma)$ can be identified with a lattice of $\boldsymbol{R}^2$. It follows from [3, Theorem 4.1] that the lattice $G(\Gamma)$ has the following generators.

$$
\begin{cases} (l_1, 0), (0, l_2) & \text{if } I(\Gamma) = (0,0)\,, \\ (2l_1, 0), (0, l_2) & \text{if } I(\Gamma) = (1,0)\,, \\ (l_1, 0), (0, 2l_2) & \text{if } I(\Gamma) = (0,1)\,, \\ (l_1, l_2), (l_1, -l_2) & \text{if } I(\Gamma) = (1,1)\,, \end{cases} \tag{4.4}
$$

where l_i denotes the minimum period of $\gamma_i(s)$. For $m_1, m_2 \in \boldsymbol{Z}$, we consider the parallel translation $\rho : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ given by

$$
\rho(s_1, s_2) = (s_1 + m_1 l_1, s_2 + m_2 l_2)\,.
$$

Since $\alpha_i(s+l_i)=\alpha_i(s)$, it follows from (3.7) and (4.2) that

$$\begin{aligned}\Phi_\Gamma(\rho(s_1,s_2)) =&\Phi_\Gamma(s_1,s_2)+m_1\left(\int_0^{l_1}\cos\alpha_1(s)ds,\ \int_0^{l_1}\sin\alpha_1(s)ds\right)\\ &+m_2\left(-\int_0^{l_2}\cos\alpha_2(s)ds,\ -\int_0^{l_2}\sin\alpha_2(s)ds\right)\\ =&\Phi_\Gamma(s_1,s_2)+m_1v_1+m_2v_2\,.\end{aligned}$$

So we obtain

$$\Phi_\Gamma\circ\rho\circ\Phi_\Gamma^{-1}(x_1,x_2)=(x_1,x_2)+m_1v_1+m_2v_2\,.$$

Hence it follows from (4.1) and (4.4) that $\Phi_\Gamma\circ\rho\circ\Phi_\Gamma^{-1}\in W(\Gamma)$ if and only if $\rho\in G(\Gamma)$. This completes the proof of Theorem 4.1. □

5. Extrinsic invariants of f_Γ. Let $\gamma:\boldsymbol{R}\to S^2$ be a periodic regular curve with the minimum period $l>0$. For each smooth function $\beta:\boldsymbol{R}\to\boldsymbol{R}$, we define $E_\beta(\gamma)$ by

$$E_\beta(\gamma)=\frac{1}{2}\int_0^l\beta(\tau_\gamma(s))\sqrt{1+k(s)^2}|\gamma'(s)|ds\,,\tag{5.1}$$

where $k(s)$ denotes the geodesic curvature of $\gamma(s)$, and

$$\tau_\gamma(s)=2k'(s)(1+k(s)^2)^{-3/2}|\gamma'(s)|^{-1}\,.\tag{5.2}$$

Furthermore for each periodic admissible pair $\Gamma=(\gamma_1,\gamma_2)$, we set

$$E_\beta(\Gamma)=E_\beta(\gamma_1)+E_\beta(\gamma_2)\,.$$

The aim of this section is to prove the following theorem.

THEOREM 5.1. *Let Γ and $\bar\Gamma$ be periodic admissible pairs such that $f_\Gamma\equiv f_{\bar\Gamma}$. Then $E_\beta(\Gamma)=E_\beta(\bar\Gamma)$ for any smooth even function β.*

It is easy to see that $f_\Gamma\equiv f_{\bar\Gamma}$ implies $F_\Gamma\equiv F_{\bar\Gamma}$. So Theorem 5.1 follows from the following lemma.

LEMMA 5.2. *Let $\Gamma=(\gamma_1,\gamma_2)$ and $\bar\Gamma=(\bar\gamma_1,\bar\gamma_2)$ be periodic admissible pairs. If $F_\Gamma\equiv F_{\bar\Gamma}$, then $E_\beta(\Gamma)=E_\beta(\bar\Gamma)$ for any smooth even function β.*

PROOF. Let $c_i(s)$ and $\bar c_i(s)$ be the curves in S^3 defined by (3.4). Then

$$F_\Gamma(s_1,s_2)=c_1(s_1)c_2(s_2)^{-1}\,,\quad F_{\bar\Gamma}(s_1,s_2)=\bar c_1(s_1)\bar c_2(s_2)^{-1}\,.\tag{5.3}$$

By (3.2) and (5.1) we obtain

$$E_\beta(\gamma_i)=\int_0^{l_i}\beta(\tau_{\gamma_i}(s))ds\,,\quad E_\beta(\bar\gamma_i)=\int_0^{\bar l_i}\beta(\tau_{\bar\gamma_i}(s))ds\,,\tag{5.4}$$

where l_i (resp. $\bar l_i$) denotes the minimum period of γ_i (resp. $\bar\gamma_i$). Let $\kappa_i(s)$ be the curvature of the curve $c_i(s)$. Since $|c_i'|=1$, it follows from [1, Lemmas 3.7 and 3.8] that

$$\kappa_i=|D_{c_i'}c_i'|=|\alpha_i'|\,,$$

where D denotes the Riemannian connection on S^3, and α_i is the function defined by (3.6). Differentiating (3.6), we have $\alpha_i'(s) = -k_i'(s)\sin^2\alpha_i(s)$, where $k_i(s)$ denotes the geodesic curvature of $\gamma_i(s)$. So it follows from (3.7) and (5.2) that $\alpha_i'(s) = -\tau_{\gamma_i}(s)$. Hence

$$\kappa_i(s) = |\tau_{\gamma_i}(s)|\,. \tag{5.5}$$

Similarly we obtain

$$\bar{\kappa}_i(s) = |\tau_{\bar{\gamma}_i}(s)|\,, \tag{5.6}$$

where $\bar{\kappa}_i(s)$ denotes the curvature of the curve $\bar{c}_i(s)$.

Let g_{ij} (resp. $\bar{g}_{ij}$) and h_{ij} (resp. $\bar{h}_{ij}$) denote the first and second fundamental forms of the immersion $F_\Gamma(s_1, s_2)$ (resp. $F_{\bar{\Gamma}}(s_1, s_2)$). Since $F_\Gamma \equiv F_{\bar{\Gamma}}$, there exist an isometry A of S^3 and a diffeomorphism ρ of $\boldsymbol{R}^2$ such that $A \circ F_\Gamma = F_{\bar{\Gamma}} \circ \rho$. Then we obtain

$$g_{ij} = \sum_{kl} \bar{g}_{kl}(\rho)\frac{\partial \rho_k}{\partial s_i}\frac{\partial \rho_l}{\partial s_j}\,, \quad h_{ij} = \pm\sum_{kl} \bar{h}_{kl}(\rho)\frac{\partial \rho_k}{\partial s_i}\frac{\partial \rho_l}{\partial s_j}\,,$$

where $\rho(s_1, s_2) = (\rho_1(s_1, s_2), \rho_2(s_1, s_2))$. So it follows from (3.8) and (3.9) that the Jacobi matrix of the diffeomorphism $\rho : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ satisfies the following relation.

$$\frac{\partial(\rho_1, \rho_2)}{\partial(s_1, s_2)} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \quad \text{or} \quad \frac{\partial(\rho_1, \rho_2)}{\partial(s_1, s_2)} = \begin{bmatrix} 0 & a_2 \\ a_1 & 0 \end{bmatrix}, \tag{5.7}$$

where $|a_1| = |a_2| = 1$.

We now consider the first case of (5.7). Then we obtain

$$\rho(s_1, s_2) = (a_1 s_1 + b_1, a_2 s_2 + b_2)\,.$$

Since $A \circ F_\Gamma = F_{\bar{\Gamma}} \circ \rho$, it follows from (5.3) that

$$A(c_1(s_1)c_2(s_2)^{-1}) = \bar{c}_1(a_1 s_1 + b_1)\bar{c}_2(a_2 s_2 + b_2)^{-1}\,.$$

Since $c_1(0) = c_2(0) = e$, the relation above implies

$$(R \circ A)c_1(s) = \bar{c}_1(a_1 s + b_1)\,, \quad (L \circ A)c_2(s)^{-1} = \bar{c}_2(a_2 s + b_2)^{-1}\,,$$

where R denotes the right translation by $\bar{c}_2(b_2)$, and L denotes the left translation by $\bar{c}_1(b_1)^{-1}$. So there exist isometries A_1 and A_2 of S^3 such that

$$c_i(s) = A_i \bar{c}_i(a_i s + b_i)\,. \tag{5.8}$$

This shows that $\kappa_i(s) = \bar{\kappa}_i(a_i s + b_i)$. Since $\beta : \boldsymbol{R} \to \boldsymbol{R}$ is an even function, it follows from (5.5) and (5.6) that

$$\beta(\tau_{\gamma_i}(s)) = \beta(\tau_{\bar{\gamma}_i}(a_i s + b_i))\,. \tag{5.9}$$

By (2.7) and (5.8) we obtain

$$c_i(s + \bar{l}_i) = A_i \bar{c}_i(a_i s + b_i + a_i \bar{l}_i) = \pm A_i \bar{c}_i(a_i s + b_i) = \pm c_i(s)\,.$$

Since $p_2(\pm c_i(s)) = \hat{\gamma}_i(s)$, we obtain $\hat{\gamma}_i(s+\bar{l}_i) = \hat{\gamma}_i(s)$, and so $\gamma_i(s+\bar{l}_i) = \gamma_i(s)$. Hence $\bar{l}_i/l_i$ must be an integer. Similarly we see that $l_i/\bar{l}_i$ is an integer, and so we have $l_i = \bar{l}_i$. Therefore

$$\int_0^{l_i} \beta(\tau_{\gamma_i}(s))ds = \int_0^{\bar{l}_i} \beta(\tau_{\bar{\gamma}_i}(a_i s + b_i))ds = \int_0^{\bar{l}_i} \beta(\tau_{\bar{\gamma}_i}(s))ds \,,$$

where the first equality follows from (5.9), and the second equality follows from the fact that $\tau_{\bar{\gamma}_i}(s)$ is $\bar{l}_i$-periodic. Hence (5.4) implies

$$E_\beta(\gamma_1) = E_\beta(\bar{\gamma}_1)\,, \quad E_\beta(\gamma_2) = E_\beta(\bar{\gamma}_2)\,.$$

For the second case of (5.7), in the same way as above, we obtain

$$E_\beta(\gamma_1) = E_\beta(\bar{\gamma}_2)\,, \quad E_\beta(\gamma_2) = E_\beta(\bar{\gamma}_1)\,.$$

This completes the proof of Lemma 5.2. □

We conclude this section with the following theorem.

THEOREM 5.3. *Let $\Gamma = (\gamma_1, \gamma_2)$ be a periodic admissible pair, and let H be the mean curvature of the isometric immersion $f_\Gamma : M_\Gamma \to S^3$. Then*

$$\int_{M_\Gamma} H d\sigma = \frac{c}{4}\{K(\gamma_1)K(\gamma_2) + L(\gamma_1)L(\gamma_2)\}\,, \quad c = \begin{cases} 1 & \text{if } I(\Gamma) = (0,0)\,, \\ 2 & \text{if } I(\Gamma) \neq (0,0)\,, \end{cases}$$

where $d\sigma$ denotes the volume element of the flat torus M_Γ.

PROOF. Let $l_i > 0$ be the minimum period of γ_i, and let ξ_1 and ξ_2 denote the generators of the lattice $G(\Gamma)$ given by (4.4). We consider the domain

$$D = \{x\xi_1 + y\xi_2 : 0 \le x \le 1, 0 \le y \le 1\} \subset \boldsymbol{R}^2\,.$$

Since D is a fundamental domain of $G(\Gamma)$, it follows from (3.8) and (3.10) that

$$\begin{aligned}\int_{M_\Gamma} H d\sigma &= \int_D \cos(\alpha_2(s_2) - \alpha_1(s_1))ds_1 ds_2 \\ &= c\int_0^{l_2} ds_2 \int_0^{l_1} \cos(\alpha_2(s_2) - \alpha_1(s_1))ds_1\,,\end{aligned}$$

where the second equality follows from the fact that the function $\alpha_i(s)$ is l_i-periodic. On the other hand (3.7) implies

$$\cos(\alpha_2(s_2) - \alpha_1(s_1)) = \frac{1}{4}(k_1(s_1)k_2(s_2) + 1)|\gamma_1'(s_1)||\gamma_2'(s_2)|\,.$$

This completes the proof. □

6. Proof of Theorem 1.2.

LEMMA 6.1. *Let $f : M \to S^3$ be a primitive isometric immersion of a flat torus M, and let $\pi : \bar{M} \to M$ be a Riemannian covering. If $\rho : \bar{M} \to \bar{M}$ is a diffeomorphism satisfying the relation $f \circ \pi \circ \rho = f \circ \pi$, then $\pi \circ \rho = \pi$.*

PROOF. Since $\bar{M}$ is a complete connected flat surface and $f \circ \pi : \bar{M} \to S^3$ is an isometric immersion, it follows from [5] that there exists a covering $T : \boldsymbol{R}^2 \to \bar{M}$ such that

$$\bar{g}\left(\frac{\partial T}{\partial s_i}, \frac{\partial T}{\partial s_i}\right) = 1\,, \quad \bar{h}\left(\frac{\partial T}{\partial s_i}, \frac{\partial T}{\partial s_i}\right) = 0 \quad \text{for } i = 1, 2\,,$$

where $\bar{g}$ denotes the Riemannian metric on $\bar{M}$, and $\bar{h}$ denotes the second fundamental form of the immersion $f \circ \pi : \bar{M} \to S^3$. Note that the immersion $F = f \circ \pi \circ T$ is a FAT.

Since T is a universal covering, there exist $\bar{\rho} \in \mathrm{Diff}(\boldsymbol{R}^2)$ such that $T \circ \bar{\rho} = \rho \circ T$. Using the relation $f \circ \pi \circ \rho = f \circ \pi$, we obtain $F \circ \bar{\rho} = F$, and so it follows from [1, Theorem 2.3] that $\bar{\rho}$ is a parallel translation of $\boldsymbol{R}^2$. Let ϕ be a covering transformation of π. We take $\bar{\phi} \in \mathrm{Diff}(\boldsymbol{R}^2)$ such that $T \circ \bar{\phi} = \phi \circ T$. Since $\pi \circ \phi = \pi$, in the same way as above, we see that $\bar{\phi}$ is a parallel translation of $\boldsymbol{R}^2$. Hence $\bar{\rho} \circ \bar{\phi} = \bar{\phi} \circ \bar{\rho}$, and so we obtain

$$\rho \circ \phi = \phi \circ \rho\,. \tag{6.1}$$

Since the covering π is regular, it follows from (6.1) that there exists a diffeomorphism ρ' : $M \to M$ such that $\pi \circ \rho = \rho' \circ \pi$. Then

$$f \circ \rho' \circ \pi = f \circ \pi \circ \rho = f \circ \pi\,.$$

Hence $f \circ \rho' = f$. Since the immersion f is primitive, we see that $\rho' = 1$, and so $\pi \circ \rho = \pi$. □

LEMMA 6.2. *Let f_1 and f_2 be primitive isometric immersions of a flat torus M into S^3, and let $\pi : \bar{M} \to M$ be a Riemannian covering. If $f_1 \circ \pi \equiv f_2 \circ \pi$, then $f_1 \equiv f_2$.*

PROOF. Since $f_1 \circ \pi \equiv f_2 \circ \pi$, there exist an isometry A of S^3 and a diffeomorphism ρ of $\bar{M}$ such that $A \circ f_1 \circ \pi = f_2 \circ \pi \circ \rho$. We now denote by $G(\pi)$ the covering transformation group of π. Then, for each $\phi \in G(\pi)$, we obtain

$$f_2 \circ \pi \circ \rho \circ \phi \circ \rho^{-1} = A \circ f_1 \circ \pi \circ \phi \circ \rho^{-1} = A \circ f_1 \circ \pi \circ \rho^{-1} = f_2 \circ \pi\,.$$

So it follows from Lemma 6.1 that $\pi \circ \rho \circ \phi \circ \rho^{-1} = \pi$. Hence

$$\rho \circ \phi \circ \rho^{-1} \in G(\pi) \quad \text{for all} \quad \phi \in G(\pi)\,. \tag{6.2}$$

Since the covering π is regular, it follows from (6.2) that there exists a diffeomorphism ρ' : $M \to M$ satisfying the relation $\pi \circ \rho = \rho' \circ \pi$. Then

$$A \circ f_1 \circ \pi = f_2 \circ \pi \circ \rho = f_2 \circ \rho' \circ \pi\,.$$

Hence $A \circ f_1 = f_2 \circ \rho'$, and so $f_1 \equiv f_2$. □

By Lemma 6.2 it is easy to see that Theorem 1.2 follows from the following theorem.

THEOREM 6.3. *If $f : M \to S^3$ is a primitive isometric immersion of a flat torus M into S^3 with nonconstant mean curvature, then there exists a smooth one-parameter family of primitive isometric immersions $f_t : M \to S^3$, $t \in \boldsymbol{R}$, such that $f_0 = f$ and $f_t \not\equiv f_s$ for all $s \neq t$. Furthermore the total mean curvature of the immersion f_t is equal to that of f_0 for all $t \in \boldsymbol{R}$.*

PROOF. By Theorem 3.1 there exists a periodic admissible pair $\Gamma = (\gamma_1, \gamma_2)$ such that $f \equiv f_\Gamma$. So we may assume that $f = f_\Gamma$ and $M = M_\Gamma$. Since the mean curvature of f_Γ is not constant, it follows from Theorem 3.2 that either $k_1(s)$ or $k_2(s)$ is not constant, where $k_i(s)$ denotes the geodesic curvature of $\gamma_i(s)$. Without loss of generality, we may assume that $k_1(s)$ is not constant.

We now use the following theorem which will be proved in Section 7.

THEOREM 6.4. *Let $\gamma : \boldsymbol{R} \to S^2$ be a periodic regular curve whose geodesic curvature $k(s)$ satisfies $|\gamma'(s)|\sqrt{1+k(s)^2} = 2$. If $k(s)$ is not constant, then there exist a smooth even function $\beta : \boldsymbol{R} \to \boldsymbol{R}$ and a smooth one-parameter family of periodic regular curves $\gamma_t : \boldsymbol{R} \to S^2$, $-\varepsilon < t < \varepsilon$, such that $\gamma_0(s) = \gamma(s)$ and*

(1) $|\gamma_t'(s)|\sqrt{1+k_t(s)^2} = 2$,

(2) $K(\gamma_t) = K(\gamma)$, $L(\gamma_t) = L(\gamma)$, $E_\beta(\gamma_t) = E_\beta(\gamma) + t$,

(3) $I(\gamma_t) = I(\gamma)$,

where $k_t(s)$ denotes the geodesic curvature of $\gamma_t(s)$.

So there exist a smooth even function β and a smooth one-parameter family of periodic regular curves $\gamma_1^t : \boldsymbol{R} \to S^2$, $t \in \boldsymbol{R}$, such that $\gamma_1^0(s) = \gamma_1(s)$ and

$$\Gamma_t = (\gamma_1^t, \gamma_2) \text{ is a periodic admissible pair}, \tag{6.3}$$

$$K(\gamma_1^t) = K(\gamma_1), \quad L(\gamma_1^t) = L(\gamma_1), \quad I(\gamma_1^t) = I(\gamma_1), \tag{6.4}$$

$$E_\beta(\gamma_1^s) \neq E_\beta(\gamma_1^t) \quad \text{for all} \quad s \neq t. \tag{6.5}$$

By (6.3) we obtain the flat torus M_{Γ_t} and the primitive isometric immersion $f_{\Gamma_t} : M_{\Gamma_t} \to S^3$. For each Γ_t, define $\Phi_{\Gamma_t} : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ in the same way as (4.3). Then it follows from Theorem 4.1 that the map Φ_{Γ_t} induces the isometry $\phi_t : M_{\Gamma_t} \to (\boldsymbol{R}^2, g_0)/W(\Gamma_t)$. On the other hand, (6.4) implies that $W(\Gamma_t) = W(\Gamma)$. So we obtain the primitive isometric immersion $f_t : M \to S^3$ defined by

$$f_t = f_{\Gamma_t} \circ \phi_t^{-1} \circ \phi_0 .$$

We now show that the family f_t, $t \in \boldsymbol{R}$, satisfies the properties required in Theorem 6.3. Since $\Gamma_0 = \Gamma$, we obtain $f_0 = f_\Gamma = f$. By (6.5) it follows from Theorem 5.1 that $f_{\Gamma_s} \not\equiv f_{\Gamma_t}$ for all $s \neq t$, and so

$$f_s \not\equiv f_t \quad \text{for all} \quad s \neq t .$$

Let H_t denote the mean curvature of the immersion f_t. Since $f_t \equiv f_{\Gamma_t}$, it follows from (6.4) and Theorem 5.3 that

$$\int_M H_t d\sigma = \int_M H_0 d\sigma \quad \text{for all} \quad t \in \boldsymbol{R},$$

where $d\sigma$ denotes the volume element of the flat torus M. To establish the property that the map $(t, x) \mapsto f_t(x)$ is smooth, we consider the maps $Q_1 : \boldsymbol{R} \times \boldsymbol{R}^2/W(\Gamma) \to S^3$ and $Q_2 : \boldsymbol{R} \times \boldsymbol{R}^2 \to \boldsymbol{R} \times \boldsymbol{R}^2/W(\Gamma)$ defined by

$$Q_1(t, p) = f_{\Gamma_t}(\phi_t^{-1}(p)), \quad Q_2(t, x_1, x_2) = (t, \pi(x_1, x_2)),$$

where π denotes the canonical projection of $\boldsymbol{R}^2$ onto $\boldsymbol{R}^2/W(\Gamma)$. Note that the map Q_2 is a local diffeomorphism. Furthermore we define the diffeomorphism $Q_3 : \boldsymbol{R} \times \boldsymbol{R}^2 \to \boldsymbol{R} \times \boldsymbol{R}^2$ by

$$Q_3(t, s_1, s_2) = (t, \Phi_{\Gamma_t}(s_1, s_2)).$$

Then it follows that

$$Q_1(Q_2(Q_3(t, s_1, s_2))) = F_{\Gamma_t}(s_1, s_2),$$

and so the map $Q_1 \circ Q_2 \circ Q_3 : \boldsymbol{R} \times \boldsymbol{R}^2 \to S^3$ is smooth. Since the map $Q_2 \circ Q_3$ is a local diffeomorphism, we see that the map Q_1 is smooth. Hence the map $(t, x) \mapsto f_t(x)$ is smooth. This completes the proof of Theorem 6.3. □

7. Deformations of periodic regular curves in S^2. The aim of this section is to prove Theorem 6.4. We first prove the following lemma.

LEMMA 7.1. *Let U be an open subset of $\boldsymbol{R}^n$ which contains the origin $o \in \boldsymbol{R}^n$. Let $f : U \times \boldsymbol{R} \to \boldsymbol{R}^m$ be a continuous map such that $f_x : \boldsymbol{R} \to \boldsymbol{R}^m$ is nonconstant and periodic for all $x \in U$, where $f_x(s) = f(x, s)$. Suppose that there exists a continuous positive function $l : U \to \boldsymbol{R}^+$ satisfying*

(1) *$f_x(s + l(x)) = f_x(s)$ for all $(x, s) \in U \times \boldsymbol{R}$,*

(2) *$l(o)$ is the minimum period of $f_o(s)$.*

Then there exists an open neighborhood U' of the origin o in U such that the minimum period of $f_x(s)$ is equal to $l(x)$ for all $x \in U'$.

PROOF. For each $x \in U$, let $\bar{l}(x) > 0$ be the minimum period of $f_x(s)$, and let $q(x) = l(x)/\bar{l}(x)$. Note that $q(x)$ is a positive integer. Now assume that the assertion of the lemma is not true. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in U such that $q(x_n) \geq 2$ and $\lim_{n\to\infty} x_n = o$. We first consider the case where the sequence $q(x_n)$ is bounded. Then we may assume that there exists an integer $p \geq 2$ such that $q(x_n) = p$ for all n. Hence

$$f_{x_n}(s) = f_{x_n}(s + \bar{l}(x_n)) = f_{x_n}(s + l(x_n)/p).$$

Letting n tend to infinity, we have $f_o(s) = f_o(s + l(o)/p)$. So the minimum period of $f_o(s)$ is smaller than $l(o)$. This is a contradiction.

Now consider the other case. Then we may assume that $\lim_{n\to\infty} q(x_n) = \infty$. For each $s \in \boldsymbol{R}$, let s_n be the real number such that $(s_n - s)/\bar{l}(x_n)$ is an integer and

$$0 \leq s_n < \bar{l}(x_n).$$

Then $f_{x_n}(s) = f_{x_n}(s_n)$ and $0 \leq s_n < l(x_n)/q(x_n)$. Letting n tend to infinity, we have $f_o(s) = f_o(0)$, which shows that $f_o(s)$ is constant. This is a contradiction. □

LEMMA 7.2. *Let $\gamma : \boldsymbol{R} \to S^2$ be a periodic regular curve parametrized by arclength, and let $l > 0$ be the minimum period of $\gamma(s)$. If the geodesic curvature of $\gamma(s)$ is not constant, then there exist a smooth even function $\beta : \boldsymbol{R} \to \boldsymbol{R}$ and a smooth one-parameter family of periodic regular curves $\gamma_t : \boldsymbol{R} \to S^2$, $-\varepsilon < t < \varepsilon$, such that*

(1) *$\gamma_0(s) = \gamma(s)$,*

(2) *the minimum period of $\gamma_t(s)$ is equal to l,*

(3) $K(\gamma_t) = K(\gamma)$, $L(\gamma_t) = L(\gamma)$, $E_\beta(\gamma_t) = E_\beta(\gamma) + t$.

PROOF. Let $k(s)$ be the geodesic curvature of $\gamma(s)$, and let $\tau(s) = \tau_\gamma(s)$. Since $|\gamma'(s)| = 1$, it follows from (5.2) that

$$\tau(s) = 2k'(s)(1 + k(s)^2)^{-3/2}. \tag{7.1}$$

Since $\tau = 2(k/\sqrt{1+k^2})'$ and $k(l) = k(0)$, we obtain

$$\int_0^l \tau(s)ds = 0.$$

If $\tau(s)$ is constant, then $\tau(s) = 0$, and so $k'(s) = 0$. This contradicts the assumption that the geodesic curvature of $\gamma(s)$ is not constant. Hence $\tau(s)$ is not constant. So there exists a real number s_0 such that

$$\tau(s_0) \neq 0, \quad \tau'(s_0) \neq 0. \tag{7.2}$$

We now choose a smooth even function $\beta : \boldsymbol{R} \to \boldsymbol{R}$ such that

$$\beta^{(i)}(\tau(s_0)) = \begin{cases} 0 & \text{if } 0 \leq i \leq 5, \\ \text{nonzero} & \text{if } i = 6, \end{cases} \tag{7.3}$$

where $\beta^{(i)}$ denotes the i-th derivative of the function β.

Let $f_1(s)$, $f_2(s)$ and $f_3(s)$ be l-periodic smooth functions which will be specified later. For each $x = (x_1, x_2, x_3) \in \boldsymbol{R}^3$, define $p_x : \boldsymbol{R} \to S^2$ by

$$p_x(s) = \cos\left(\sum_{i=1}^3 f_i(s)x_i\right)\gamma(s) + \sin\left(\sum_{i=1}^3 f_i(s)x_i\right)\nu(s), \quad \nu(s) = J(\gamma'(s)),$$

where J denotes the almost complex structure given by (2.5). Let $B_\delta(o)$ denote the δ-neighborhood of the origin $o \in \boldsymbol{R}^3$. Since $p_o(s) = \gamma(s)$ and $p_x(s+l) = p_x(s)$, there exists a positive number δ such that for each $x \in B_\delta(o)$ the map $p_x : \boldsymbol{R} \to S^2$ is a periodic regular curve. By Lemma 7.1 we may assume that the minimum period of $p_x(s)$ is equal to l for all $x \in B_\delta(o)$. So we obtain

$$L(p_x) = \int_0^l |p_x'(s)|ds, \quad K(p_x) = \int_0^l k_x(s)|p_x'(s)|ds \quad \text{for} \quad x \in B_\delta(o),$$

where $k_x(s)$ denotes the geodesic curvature of $p_x(s)$. Furthermore

$$E_\beta(p_x) = \frac{1}{2}\int_0^l \beta(\tau_x(s))\sqrt{1 + k_x(s)^2}|p_x'(s)|ds \quad \text{for} \quad x \in B_\delta(o),$$

where $\tau_x(s) = 2k_x'(s)(1 + k_x(s)^2)^{-3/2}|p_x'(s)|^{-1}$. Therefore we obtain the smooth map $F : B_\delta(o) \to \boldsymbol{R}^3$ defined by

$$F(x) = (K(p_x), L(p_x), E_\beta(p_x)).$$

We now show that for a suitable choice of the functions $f_j(s)$, the Jacobi matrix of F is non-singular at the origin o. By a straightforward calculation we obtain

$$\left.\frac{\partial}{\partial x_j}L(p_x)\right|_{x=o} = -\int_0^l k(s)f_j(s)ds, \tag{7.4}$$

(7.5) $$\frac{\partial}{\partial x_j} K(p_x)\bigg|_{x=o} = \int_0^l f_j(s)ds\,,$$

(7.6) $$\frac{\partial}{\partial x_j} E_\beta(p_x)\bigg|_{x=o} = \int_0^l \sum_{i=1}^{3} a_i(s) f_j^{(i)}(s)ds\,,$$

where $f_j^{(i)}(s)$ denote the i-th derivatives of the functions $f_j(s)$, and the functions $a_i(s)$ are given by

(7.7) $$a_1 = \beta'(\tau)\,,\quad a_2 = \frac{k}{2\sqrt{1+k^2}}\beta(\tau) - \frac{3kk'}{(1+k^2)^2}\beta'(\tau)\,,\quad a_3 = \frac{1}{1+k^2}\beta'(\tau)\,.$$

Since the functions $a_i(s)$ and $f_j(s)$ are l-periodic, it follows from integration by parts that

(7.8) $$\frac{\partial}{\partial x_j} E_\beta(p_x)\bigg|_{x=o} = \int_0^l u_\beta(s) f_j(s)ds\,,$$

where $u_\beta(s) = \sum_{i=1}^3 (-1)^i a_i^{(i)}(s)$.

We now specify the functions $f_j(s)$ as follows:

$$f_1(s) = 1\,,\quad f_2(s) = -k(s)\,,\quad f_3(s) = u_\beta(s)\,.$$

Then it follows from (7.4), (7.5) and (7.8) that the Jacobi matrix of F at the origin o is given by

$$F'(o) = [c_{ij}]\,,\quad c_{ij} = \int_0^l f_i(s) f_j(s)ds\,.$$

By using (7.7), the function $u_\beta(s)$ can be written as $u_\beta(s) = \sum_{i=0}^4 b_i(s)\beta^{(i)}(\tau(s))$. Since $b_4(s) = \tau'(s)^3/(1+k(s)^2)$, it follows from (7.2) and (7.3) that

$$u'_\beta(s_0) = 0\,,\quad u''_\beta(s_0) \neq 0\,.$$

On the other hand, (7.1) and (7.2) imply $k'(s_0) \neq 0$. Hence

(7.9) $$\det\begin{bmatrix} f_1(s_0) & f_2(s_0) & f_3(s_0) \\ f'_1(s_0) & f'_2(s_0) & f'_3(s_0) \\ f''_1(s_0) & f''_2(s_0) & f''_3(s_0) \end{bmatrix} = -\det\begin{bmatrix} 1 & k(s_0) & u_\beta(s_0) \\ 0 & k'(s_0) & u'_\beta(s_0) \\ 0 & k''(s_0) & u''_\beta(s_0) \end{bmatrix} \neq 0\,.$$

Let ξ_1, ξ_2, ξ_3 be real numbers satisfying the following relation.

$$\sum_{j=1}^3 c_{ij}\xi_j = 0 \quad \text{for} \quad i = 1, 2, 3\,.$$

Since $\sum_{i,j=1}^3 c_{ij}\xi_i\xi_j = 0$, we obtain

$$\int_0^l \left|\sum_{i=1}^3 \xi_i f_i(s)\right|^2 ds = 0\,.$$

Hence $\sum_{i=1}^{3} \xi_i f_i(s) = 0$ for all $s \in \boldsymbol{R}$, and so

$$\sum_{i=1}^{3} \xi_i f_i(s_0) = \sum_{i=1}^{3} \xi_i f_i'(s_0) = \sum_{i=1}^{3} \xi_i f_i''(s_0) = 0\,.$$

Therefore it follows from (7.9) that $\xi_1 = \xi_2 = \xi_3 = 0$. This implies that the matrix $F'(o)$ is non-singular.

Using the inverse function theorem, we see that there exists a positive number ε such that the map $F : B_\delta(o) \to \boldsymbol{R}^3$ carries a neighborhood of the origin o diffeomorphically onto the ε-neighborhood of $F(o) \in \boldsymbol{R}^3$. Since $F(o) = (K(\gamma), L(\gamma), E_\beta(\gamma))$, we obtain a smooth curve $x : (-\varepsilon, \varepsilon) \to B_\delta(o)$ such that

$$F(x(t)) = (K(\gamma), L(\gamma), E_\beta(\gamma) + t)\,, \quad x(0) = o\,.$$

Then the smooth one-parameter family of the periodic regular curves $\gamma_t(s) = p_{x(t)}(s)$ satisfies the required properties (1)–(3). □

PROOF OF THEOREM 6.4. Let $\theta : \boldsymbol{R} \to \boldsymbol{R}$ be the diffeomorphism given by

$$\theta(s) = \int_0^s |\gamma'(x)|dx\,,$$

and let $\bar{\gamma} : \boldsymbol{R} \to S^2$ be the curve defined by $\bar{\gamma}(\theta(s)) = \gamma(s)$. Then $\bar{\gamma}$ is a regular curve parametrized by arclength, and its geodesic curvature $\bar{k}$ satisfies $\bar{k}(\theta(s)) = k(s)$. Since $|\gamma'(s)|\sqrt{1 + k(s)^2} = 2$, we obtain

$$s = \frac{1}{2}\int_0^{\theta(s)} \sqrt{1 + \bar{k}(x)^2}dx\,.$$

So it follows that γ is m-periodic if and only if $\bar{\gamma}$ is $\theta(m)$-periodic. Hence the minimum period of $\bar{\gamma}$ is equal to $\theta(l)$, where l denotes the minimum period of γ. Since $\bar{k}$ is not constant, Lemma 7.2 implies that there exist a smooth even function β and a smooth one-parameter family of periodic regular curves $\bar{\gamma}_t : \boldsymbol{R} \to S^2$, $-\varepsilon < t < \varepsilon$, such that

$$\text{(7.10)} \qquad \bar{\gamma}_0 = \bar{\gamma}\,, \quad K(\bar{\gamma}_t) = K(\bar{\gamma})\,, \quad L(\bar{\gamma}_t) = L(\bar{\gamma})\,, \quad E_\beta(\bar{\gamma}_t) = E_\beta(\bar{\gamma}) + t\,,$$

and the minimum period of $\bar{\gamma}_t$ is equal to $\theta(l)$.

We now consider the smooth one-parameter family of the diffeomorphisms $\theta_t : \boldsymbol{R} \to \boldsymbol{R}$, $-\varepsilon < t < \varepsilon$, defined by the following relation:

$$\text{(7.11)} \qquad s = \frac{1}{2}\int_0^{\theta_t(s)} |\bar{\gamma}_t'(x)|\sqrt{1 + \bar{k}_t(x)^2}dx\,,$$

where $\bar{k}_t$ denotes the geodesic curvature of $\bar{\gamma}_t$. Furthermore we consider the smooth one-parameter family of regular curves $\gamma_t : \boldsymbol{R} \to S^2$, $-\varepsilon < t < \varepsilon$ given by

$$\gamma_t(s) = \bar{\gamma}_t(\theta_t(s))\,.$$

Since $\bar{\gamma}_0 = \bar{\gamma}$ and $|\bar{\gamma}'| = 1$, we obtain $\theta_0(s) = \theta(s)$ and so $\gamma_0(s) = \gamma(s)$. We set

$$l_t = \frac{1}{2}\int_0^{\theta(l)} |\bar{\gamma}_t'(x)|\sqrt{1 + \bar{k}_t(x)^2}dx\,.$$

Then it follows that $\theta_t(l_t) = \theta(l)$ and $l_0 = l$. Since $\bar{\gamma}_t$ is $\theta(l)$-periodic, we obtain $\theta_t(s + l_t) = \theta_t(s) + \theta(l)$. Hence

$$\gamma_t(s + l_t) = \gamma_t(s)\,. \tag{7.12}$$

We now show that the family γ_t, $-\varepsilon < t < \varepsilon$, satisfies the properties (1)–(3) required in Theorem 6.4. Let $k_t(s)$ denote the geodesic curvature of $\gamma_t(s)$. Then it follows that $k_t(s) = \bar{k}_t(\theta_t(s))$, and so (7.11) implies

$$|\gamma_t'(s)|\sqrt{1 + k_t(s)^2} = 2\,.$$

Since $l_0 = l$ and $\gamma_0(s) = \gamma(s)$, the minimum period of $\gamma_0(s)$ is equal to l_0. Hence, using (7.12) and Lemma 7.1, we may assume that the minimum period of $\gamma_t(s)$ is equal to l_t for $-\varepsilon < t < \varepsilon$. So we obtain

$$L(\gamma_t) = \int_0^{l_t} |\gamma_t'(s)|ds = \int_0^{l_t} |\bar{\gamma}_t'(\theta_t(s))|\theta_t'(s)ds = \int_0^{\theta(l)} |\bar{\gamma}_t'(x)|dx = L(\bar{\gamma}_t)\,,$$

where the third equality follows from the relation $\theta_t(l_t) = \theta(l)$. Similarly we obtain $K(\gamma_t) = K(\bar{\gamma}_t)$ and $E_\beta(\gamma_t) = E_\beta(\bar{\gamma}_t)$. Hence (7.10) implies that

$$K(\gamma_t) = K(\gamma_0)\,, \quad L(\gamma_t) = L(\gamma_0)\,, \quad E_\beta(\gamma_t) = E_\beta(\gamma_0) + t\,.$$

Since l_t is continuous in t, the closed curves $\hat{\gamma}_0|[0, l_0]$ and $\hat{\gamma}_t|[0, l_t]$ represent the same homology class in $H_1(US^2)$. Hence

$$I(\gamma_t) = I(\gamma_0)\,.$$

This completes the proof of Theorem 6.4. □

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