Tohoku Math. J.
52 (2000), 261–270

THE FIRST EIGENVALUES OF FINITE RIEMANNIAN COVERS

KATSUHIRO YOSHIJI

(Received October 5, 1998, revised January 12, 2000)

Abstract. There exists a Riemannian metric on the real projective space such that the first eigenvalue coincides with that of its Riemannian universal cover, if the dimension is bigger than 2. For the proof, we deform the canonical metric on the real projective space. A similar result is obtained for lens spaces, as well as for closed Riemannian manifolds with Riemannian double covers. As a result, on a non-orientable closed manifold other than the real projective plane, there exists a Riemannian metric such that the first eigenvalue coincides with that of its Riemannian double cover.

Introduction. Throughout this paper, we assume (M, g) to be a connected closed Riemannian manifold and fix one of its connected closed Riemannian covers, which is denoted by $(\tilde{M}, \tilde{g})$. We study the spectrum

$$\mathrm{Spec}(M, g) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots\}$$

of the Laplacian acting on the space of functions on (M, g). We investigate especially the relation between the first eigenvalues $\lambda_1(M, g)$ and $\lambda_1(\tilde{M}, \tilde{g})$. In general, we have the relation

$$\lambda_1(M, g) \geq \lambda_1(\tilde{M}, \tilde{g}),$$

and our aim is to study when the equality holds.

First, we consider the case $M = \boldsymbol{RP}^n$, an n-dimensional real projective space, and $\tilde{M} = S^n$. For the canonical metric g_0 on $\boldsymbol{RP}^n$ with constant sectional curvature 1, we have $\lambda_1(\boldsymbol{RP}^n, g_0) = 2(n+1)$ and $\lambda_1(S^n, \tilde{g}_0) = n$. We ask if there exists a Riemannian metric for which the equality holds. the answer is negative in the 2-dimensional case.

PROPOSITION ([11]). *The real projective plane $\boldsymbol{RP}^2$ does not admit any Riemannian metric g such that*

$$\lambda_1(\boldsymbol{RP}^2, g) = \lambda_1(S^2, \tilde{g}).$$

In this paper we study the higher dimensional case and obtain the following result.

PROPOSITION A. *The real projective space $\boldsymbol{RP}^n$ $(n \geq 3)$ admits a Riemannian metric g such that*

$$\lambda_1(\boldsymbol{RP}^n, g) = \lambda_1(S^n, \tilde{g}).$$

Now, let (S^{2n-1}, g_0) be the $(2n-1)$-dimensional standard sphere of constant curvature 1. We regard it as the unit sphere $\{(z_1, \ldots, z_n) \in \boldsymbol{C}^n \mid \sum_{i=1}^n |z_i|^2 = 1\}$ embedded in $\boldsymbol{C}^n = \boldsymbol{R}^{2n}$.

1991 *Mathematics Subject Classification.* Primary 58G25; Secondary 53C20.

Let γ be an element of $SO(2n)$ which acts on S^{2n-1} by

$$\gamma : (z_1, \dots, z_n) \mapsto \left(\exp\left(\frac{2\pi\sqrt{-1}}{p}\right) z_1, \dots, \exp\left(\frac{2\pi\sqrt{-1}}{p}\right) z_n\right),$$

and Γ_p the cyclic group of order p generated by γ. Then Γ_p acts without fixed points on (S^{2n-1}, g_0) and we have the lens space $L_p^{2n-1} = S^{2n-1}/\Gamma_p$, which has a homogeneous Riemannian metric of constant curvature 1 denoted also by the same symbol g (see Wolf [10]).

We deform the Riemannian metric on L_p^{2n-1} and consider the relation between the positive first eigenvalues of L_p^{2n-1} and that of its universal cover $(S^n, \tilde{g})$. Sakai [6] and Ikeda [5] computed the spectrum of the Laplacian on the lens space. For the canonical metric g_0, we have $\lambda_1(L_p^{2n-1}, g_0) = 4n$ and $\lambda_1(S^{2n-1}, \tilde{g}_0) = 2n - 1$. As to when the equality $\lambda_1(L_p^{2n-1}, g) = \lambda_1(S^{2n-1}, \tilde{g})$ holds for any Riemannian metric g on L_p^{2n-1}, we obtain the following result.

PROPOSITION B. *The lens space L_p^{2n-1} $(n \geq 2)$ admits a Riemannian metric g such that*

$$\lambda_1(L_p^{2n-1}, g) = \lambda_1(S^{2n-1}, \tilde{g}).$$

In Section 4, we extend the argument to the case where (M, g) is an n-dimensional closed Riemannian manifold with the connected Riemannian double cover

$$\varpi : (\tilde{M}, \tilde{g}) \to (M, g)$$

for $n \geq 3$. Then we have the following:

PROPOSITION C. *Let (M, g) be a closed Riemannian manifold whose dimension is bigger than 2 and with the connected Riemannian double cover $(\tilde{M}, \tilde{g})$. Then (M, g) admits a Riemannian metric g_0 such that*

$$\lambda_1(M, g_0) = \lambda_1(\tilde{M}, \tilde{g}_0).$$

Moreover, the Riemannian metric g_0 is obtained by the deformation of the Riemannian metric g on (M, g).

In the two-dimensional case, every non-orientable surface except $\boldsymbol{RP}^2$ admits such a Riemannian metric by virtue of the following proposition.

PROPOSITION ([11]). *If M is homeomorphic to $\#^n\boldsymbol{RP}^2$ $(n \geq 2)$, then there exists a Riemannian metric g on M such that*

$$\lambda_1(M, g) = \lambda_1(\tilde{M}, \tilde{g}),$$

where $\#^n\boldsymbol{RP}^2$ means the connected sum of n-copies of $\boldsymbol{RP}^2$.

As a conclusion, we have the following:

MAIN THEOREM. *Every non-orientable closed manifold except real projective plane $\boldsymbol{RP}^2$ admits a Riemannian metric for which the first eigenvalue coincides with that of its Riemannian double cover.*

In general, when the covering $\varpi : (\tilde{M}, \tilde{g}) \to (M, g)$ satisfies the condition

$$\pi_1(M)/\varpi_*\pi_1(\tilde{M}) \cong \mathbf{Z}_p$$

and if $\dim M \geq 3$, we have the same conclusion as in Proposition C, that is, M admits a Riemannian metric g such that

$$\lambda_1(M, g) = \lambda_1(\tilde{M}, \tilde{g}).$$

However, this equality does not hold in general in the case that the number of the generators of $\pi_1(M)/\varpi_*\pi_1(\tilde{M})$ is different from one. In Section 6 we illustrate an example for which this equality does not hold.

The author would like to express his gratitude to Professor Reiko Miyaoka and the referee for valuable advice.

1. Preliminaries. Let (M, g) be a closed Riemannian manifold and $(\tilde{M}, \tilde{g})$ be one of its connected closed Riemannian covers. Since (M, g) and $(\tilde{M}, \tilde{g})$ are both compact, the fiber of each point p in (M, g) is a finite set of points $\{\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_k\}$.

We recall the necessary properties of the eigenfunctions on $(\tilde{M}, \tilde{g})$. For each $l = 1, \dots, k$, we define an isometry $J_l : (\tilde{M}, \tilde{g}) \to (\tilde{M}, \tilde{g})$ such as

$$J_l(\tilde{p}_i) = \sigma_l(\tilde{p}_i),$$

where each σ_l is an element of $\pi_1(M)$ acting on $\tilde{M}$ as a deck transformation.

REMARK. The set $\{J_1, \dots, J_k\}$ is determined uniquely up to permutation.

Let $C^\infty(M, g)$ the set of smooth functions on (M, g). We define an inner product on $C^\infty(M, g)$ by $(f_1, f_2) = \int_M f_1 f_2 dv$, where dv is the volume element on (M, g). We define $L^2(M, g)$ to be the completion of $C^\infty(M, g)$ with respect to the inner product $(,)$. If we lift the metric g on M to $\tilde{g}$ on $\tilde{M}$, then we have an inner product on $L^2(\tilde{M}, \tilde{g})$ in the same manner as in $L^2(M, g)$. Let $E(\lambda)$ be the eigenspace of smooth functions on $\tilde{M}$ corresponding to the eigenvalue λ, and set

$$E^+(\lambda) = \{f \in E(\lambda) | f \circ J_l = f, l = 1, \dots, k\}.$$

Since all the eigenfunctions on (M, g) are lifted to those on $(\tilde{M}, \tilde{g})$ canonically, we have an inclusion $\mathrm{Spec}(M, g) \subset \mathrm{Spec}(\tilde{M}, \tilde{g})$. The eigenfunctions on $(\tilde{M}, \tilde{g})$ which come from those on (M, g) are invariant under the deck transformations $J_1, \dots, J_k$.

Conversely, every $f \in E^+(\lambda)$ is reduced to an eigenfunction on (M, g). Thus the eigenvalues of (M, g) coincide with those of $(\tilde{M}, \tilde{g})$ satisfying $E^+(\lambda) \neq \{0\}$.

PROPOSITION 1 (cf. [3, p. 143]). *The space $C^\infty(\tilde{M}, \tilde{g})$ has a complete orthonormal basis consisting of the eigenfunctions of the Laplacian on $(\tilde{M}, \tilde{g})$, and can be decomposed as follows*:

$$C^\infty(\tilde{M}, \tilde{g}) = \bigoplus_{\lambda \geq 0} E(\lambda).$$

Let $E^-(\lambda)$ be the orthogonal complement of $E^+(\lambda)$ in $E(\lambda)$. The eigenfunctions on $(\tilde{M}, \tilde{g})$ which do not come from those on (M, g) have non-zero components of $E^-(\lambda)$. Thus we concentrate our attention on the non-zero smallest eigenvalue ν such that $E^-(\nu) \neq \{0\}$. All we have to do is to compare $\lambda_1(M, g)$ with ν.

2. Proof of Proposition A. We construct a deformation of the Riemannian metric on S^n ($n \geq 3$). We regard S^n as a manifold obtained from two n-dimensional closed balls $\boldsymbol{B}_1^n$ and $\boldsymbol{B}_2^n$ pasted along their boundaries. For convenience, we take the coordinates $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ in $\boldsymbol{R}^n$ such that $\boldsymbol{B}_1^n = \{(x_1, \dots, x_n) | x_1^2 + \cdots + x_n^2 \leq 1\}$ and $\boldsymbol{B}_2^n = \{(y_1, \dots, y_n) | y_1^2 + \cdots + y_n^2 \leq 1\}$. We decompose these balls into three parts (A), (B) and (c): For $\boldsymbol{B}_1^n$ we set

$$(\mathrm{A}) = \left\{(x_1, \dots, x_n) \in \boldsymbol{B}_1^n \;\middle|\; 0 \leq x_1^2 + \cdots + x_{n-1}^2 < \frac{1}{9}\right\},$$

$$(\mathrm{B}) = \left\{(x_1, \dots, x_n) \in \boldsymbol{B}_1^n \;\middle|\; \frac{1}{9} \leq x_1^2 + \cdots + x_{n-1}^2 \leq \frac{1}{4}\right\},$$

$$(\mathrm{C}) = \left\{(x_1, \dots, x_n) \in \boldsymbol{B}_1^n \;\middle|\; \frac{1}{4} < x_1^2 + \cdots + x_{n-1}^2 \leq 1\right\}.$$

We decompose $\boldsymbol{B}_2^n$ in the same way.

We paste these two balls by the equivalence relation $\sim$ defined on $\boldsymbol{B}_1^n \cup \boldsymbol{B}_2^n$ as follows. For two points $x \in \boldsymbol{B}_1^n$ and $y \in \boldsymbol{B}_2^n$, $x = (x_1, \dots, x_n) \sim y = (y_1, \dots, y_n)$ means that $(x_1, \dots, x_n) \in \partial\boldsymbol{B}_1^n$, $(y_1, \dots, y_n) \in \partial\boldsymbol{B}_2^n$ and $(x_1, \dots, x_n) = (-y_1, y_2, \dots, y_n)$. Then we regard S^n as the quotient space $(\boldsymbol{B}_1^n \cup \boldsymbol{B}_2^n)/\sim$ so that S^n is decomposed into three parts (A), (B) and (C).

Next, we regard $\boldsymbol{RP}^n$ as the quotient space of S^n with respect to the antipodal involution of S^n. In other words, we obtain $\boldsymbol{RP}^n$ from $\boldsymbol{B}_1^n$ by identifying two points $(x_1, \dots, x_n)$ and $(-x_1, \dots, -x_n)$ on the boundary of $\boldsymbol{B}_1^n$. Then we also decompose $\boldsymbol{RP}^n$ into three parts (A), (B) and (C). As for the topology of these parts (A), (B) and (C), we have the following:

LEMMA 1. *Let $\boldsymbol{I} = [0, 1]$ be the closed interval. Then the following hold.*

(1) *The parts* (A), (B) *and* (C) *of $\boldsymbol{B}^n$ are homeomorphic to $\boldsymbol{B}^{n-1} \times \boldsymbol{I}$, $S^{n-2} \times \boldsymbol{I} \times \boldsymbol{I}$ and $S^{n-2} \times \boldsymbol{I} \times \boldsymbol{I}$, respectively.*

(2) *The parts* (A), (B) *and* (C) *of S^n are homeomorphic to $\boldsymbol{B}^{n-1} \times S^1$, $S^{n-2} \times \boldsymbol{I} \times S^1$ and $S^{n-2} \times \boldsymbol{I} \times \boldsymbol{I}$, respectively.*

(3) *The parts* (A), (B) *and* (C) *of $\boldsymbol{RP}^n$ are homeomorphic to $(\boldsymbol{B}^{n-1} \times S^1)/\boldsymbol{Z}_2$, $(S^{n-2} \times \boldsymbol{I} \times S^1)/\boldsymbol{Z}_2$ and $(S^{n-2} \times \boldsymbol{I} \times \boldsymbol{I})/\boldsymbol{Z}_2$, respectively.*

REMARK. The parts (A), (B) and (C) of $\boldsymbol{B}^n$, S^n and $\boldsymbol{RP}^n$, respectively, are all connected n-dimensional manifolds in the cases (1), (2) and (3) Lemma 1.

Now we construct a deformation g_ε of a Riemannian metric on $\boldsymbol{RP}^n$ and the corresponding $\tilde{g}_\varepsilon$ on S^n. In [4] Cheeger constructed a deformation g_ε of a Riemannian metric on S^2 such that $\lambda_1(S^2, g_\varepsilon)$ converges to 0 as $\varepsilon \to 0$. We apply Cheeger's method to our situation as follows.

Step 1. We take g_ε on the part (B) of $\boldsymbol{RP}^n$ as

$$g_\varepsilon = dS^2 + dt^2 + \varepsilon^2 du^2 ,$$

where dS^2 is the canonical metric on S^{n-2}, dt^2 is the canonical metric on $\boldsymbol{I}$ such that the length of $\boldsymbol{I}$ is 1 and du^2 is the canonical metric on S^1 such that the length of S^1 is 1.

Step 2. We take a suitable Riemannian metric g on the parts (A) and (C) of $\boldsymbol{RP}^n$ such that the volumes of (A) and (C) are equal.

Step 3. Then we take the deformation g_ε on (A) and (C) with a slight modification on the connected parts with (B), keeping invariant on the remaining part.

We denote it by $(\boldsymbol{RP}^n, g_\varepsilon)$. If ε goes to 0, then the part (B) of $(\boldsymbol{RP}^n, g_\varepsilon)$ collapses onto the $(n-1)$-dimensional manifold, that is, $(S^{n-2} \times \boldsymbol{I})/\boldsymbol{Z}_2$.

On the other hand, we lift the above deformation g_ε of a metric on $\boldsymbol{RP}^n$ to metric $\tilde{g}_\varepsilon$ on S^n. If ε goes to 0, then the part (B) of $(S^n, \tilde{g}_\varepsilon)$ collapses onto the $(n-1)$-dimensional manifold $S^{n-2} \times \boldsymbol{I}$.

LEMMA 2.

$$\lim_{\varepsilon \to 0} \lambda_1(\boldsymbol{RP}^n, g_\varepsilon) = 0 .$$

PROOF. The first eigenvalue λ_1 is characterized by the Rayleigh quotient, that is,

$$\lambda_1(M, g) = \inf \frac{\int_M |\nabla f|^2 dv}{\int_M f^2 dv} ,$$

where f runs over all non-vanishing functions orthogonal to the constant functions in $L^2(M, g)$.

We consider a test function f_ε on $(\boldsymbol{RP}^n, g_\varepsilon)$, which is equal to 1 on (A) and -1 on (C), and decreases from 1 to -1 linearly across (B). We may take f_ε to be orthogonal to the constant functions in $L^2(\boldsymbol{RP}^n, g_\varepsilon)$ and $|\nabla f_\varepsilon| \le d$, where d is a constant depending only on n. If ε goes to 0, then the volume of the part (B) decreases to 0. This means that $\int_M |\nabla f_\varepsilon|^2 dv$ converges to 0 as $\varepsilon \to 0$. It follows from the Rayleigh quotient that $lim_{\varepsilon \to 0} \lambda_1(\boldsymbol{RP}^n, g_\varepsilon) = 0$. □

Next we consider the double cover $(S^n, \tilde{g}_\varepsilon)$ of $(\boldsymbol{RP}^n, g_\varepsilon)$. Then $\lambda_1(\boldsymbol{RP}^n, g_\varepsilon)$ belongs to $\mathrm{Spec}(S^n, \tilde{g}_\varepsilon)$. We denote it simply by $\lambda(\varepsilon)$. We frequently use the following result due to Anné. In the following proposition, $S^1(\varepsilon)$ means the circle of radius ε.

PROPOSITION 2 (Anné [1]). *Let (M_1, g_1) and (M_2, g_2) be two connected, closed Riemannian manifolds of the same dimension n. Let D_1 and D_2 be submanifolds of (M_1, g_1) and (M_2, g_2), which are both diffeomorphic to $S^{n-2} \times S^1(\varepsilon)$, and remove D_1 and D_2 from (M_1, g_1) and (M_2, g_2), respectively. Then we attach $S^{n-2} \times \boldsymbol{I} \times S^1(\varepsilon)$ to (M_1, g_1) and (M_2, g_2) along the boundaries, and make it smooth at the connected part and denote it by $(M_\varepsilon, g_\varepsilon)$.*

We express $\mathrm{Spec}(M_\varepsilon, g_\varepsilon)$ *as* $\mathrm{Spec}(M_\varepsilon, g_\varepsilon) = \{\lambda_0(\varepsilon) < \lambda_1(\varepsilon) \le \lambda_2(\varepsilon) \le \cdots\}$, *and denote by*

$$\{\mu_0 \le \mu_1 \le \mu_2 \le \mu_3 \le \cdots\}$$

the union of $\mathrm{Spec}(M_1, g_1)$, $\mathrm{Spec}(M_2, g_2)$ *and* $\mathrm{Spec}_D(S^{n-2} \times \boldsymbol{I}, g)$ *(the subscript D means the Dirichlet condition) counted with multiplicity. Then for any n we have*

$$\lim_{\varepsilon \to 0} \lambda_n(\varepsilon) = \mu_n .$$

REMARK 1. The Dirichlet spectrum $\mathrm{Spec}_D(S^{n-2} \times \boldsymbol{I}, g)$ has no zero-eigenvalue. So $\mu_0 = \mu_1 = 0$ and $\mu_2 > 0$.

REMARK 2. Anné [1] treats the case of one body with one handle, but the connectedness of the body is not necessary in the proof. As for the case of many bodies with many tubes, see [2].

Now we are in a position to prove Proposition A. We apply Proposition 2 to $(S^n, \tilde{g}_\varepsilon)$. By Lemma 2, we have $\lim_{\varepsilon \to 0} \lambda(\varepsilon) = 0$ and $\nu(\varepsilon)$ converges to some positive value which is greater than or equal to μ_2. This positive value depends on the original metric on $\boldsymbol{RP}^n$ before the deformation. By the continuous dependence of the eigenvalue on the parameter ε, there exists ε_0 such that

$$\lambda(\varepsilon) < \nu(\varepsilon)$$

for $0 < \varepsilon < \varepsilon_0$. Therefore we have

$$\lambda_1(\boldsymbol{RP}^n, g_\varepsilon) = \lambda(\varepsilon) = \lambda_1(S^n, \tilde{g}_\varepsilon)$$

for all ε with $0 < \varepsilon < \varepsilon_0$. □

3. Proof of Proposition B. We take a $(2n-1)$-dimensional ball

$$\boldsymbol{B}^{2n-1} = \{(x_1, \dots, x_{2n-1}) | x_1^2 + \cdots + x_{2n-1}{}^2 \leq 1\} .$$

We define an isometry $R(p) : \boldsymbol{B}^{2n-1} \to \boldsymbol{B}^{2n-1}$ by

$$\begin{aligned} &R(p)(x_1, x_2, \dots, x_{2n-3}, x_{2n-2}, x_{2n-1}) \\ &\quad = (\cos\theta x_1, \sin\theta x_2, \dots, \cos\theta x_{2n-3}, \sin\theta x_{2n-2}, -x_{2n-1}) , \end{aligned}$$

where $\theta = 2\pi/p$. We also define an equivalence relation $\sim$ on $\boldsymbol{B}^{2n-1}$ such that $x = (x_1, \dots, x_{2n-1}) \sim y = (y_1, \dots, y_{2n-1})$ means that $x \in \partial \boldsymbol{B}^{2n-1}$, $y \in \partial \boldsymbol{B}^{2n-1}$ and $R(p)x = y$ or $R(p)y = x$. Then the lens space L_p^{2n-1} can be regarded topologically as $\boldsymbol{B}^{2n-1}/\sim$.

We decompose $\boldsymbol{B}^{2n-1}$ into three parts (A), (B) and (c) in the same way as in Section 2. Hence the lens space L_p^{2n-1} is decomposed into three parts (A), (B), and (C). It is easily checked that the parts (A), (B) and (C) of L_p^{2n-1} are homeomorphic to

$$(\boldsymbol{B}^{2n-3} \times S^1)/\boldsymbol{Z}_p , \quad (S^{2n-3} \times \boldsymbol{I} \times S^1)/\boldsymbol{Z}_p \quad \text{and} \quad (S^{2n-3} \times \boldsymbol{I} \times \boldsymbol{I})/\boldsymbol{Z}_p ,$$

respectively. Note that (A) and (C) are connected n-dimensional manifolds.

We define the deformation g_ε of a Riemannian metric on L_p^{2n-1} in the same manner as in Section 2. Then we have

$$\lim_{\varepsilon \to 0} \lambda_1(L_p^{2n-1}, g_\varepsilon) = 0 .$$

the union of $\mathrm{Spec}(M_1, g_1)$, $\mathrm{Spec}(M_2, g_2)$ *and* $\mathrm{Spec}_D(S^{n-2} \times \boldsymbol{I}, g)$ (*the subscript D means the Dirichlet condition*) *counted with multiplicity. Then for any n we have*

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$$\lambda(\varepsilon) < \nu(\varepsilon)$$

for $0 < \varepsilon < \varepsilon_0$. Therefore we have

$$\lambda_1(\boldsymbol{RP}^n, g_\varepsilon) = \lambda(\varepsilon) = \lambda_1(S^n, \tilde{g}_\varepsilon)$$

for all ε with $0 < \varepsilon < \varepsilon_0$. □

3. Proof of Proposition B. We take a $(2n-1)$-dimensional ball

$$\boldsymbol{B}^{2n-1} = \{(x_1, \dots, x_{2n-1}) | x_1^2 + \cdots + x_{2n-1}{}^2 \le 1\} .$$

We define an isometry $R(p) : \boldsymbol{B}^{2n-1} \to \boldsymbol{B}^{2n-1}$ by

$$\begin{aligned} &R(p)(x_1, x_2, \dots, x_{2n-3}, x_{2n-2}, x_{2n-1}) \\ &\qquad = (\cos\theta x_1, \sin\theta x_2, \dots, \cos\theta x_{2n-3}, \sin\theta x_{2n-2}, -x_{2n-1}) , \end{aligned}$$

where $\theta = 2\pi/p$. We also define an equivalence relation $\sim$ on $\boldsymbol{B}^{2n-1}$ such that $x = (x_1, \dots, x_{2n-1}) \sim y = (y_1, \dots, y_{2n-1})$ means that $x \in \partial \boldsymbol{B}^{2n-1}$, $y \in \partial \boldsymbol{B}^{2n-1}$ and $R(p)x = y$ or $R(p)y = x$. Then the lens space L_p^{2n-1} can be regarded topologically as $\boldsymbol{B}^{2n-1}/\sim$.

We decompose $\boldsymbol{B}^{2n-1}$ into three parts (A), (B) and (c) in the same way as in Section 2. Hence the lens space L_p^{2n-1} is decomposed into three parts (A), (B), and (C). It is easily checked that the parts (A), (B) and (C) of L_p^{2n-1} are homeomorphic to

$$(\boldsymbol{B}^{2n-3} \times S^1)/\boldsymbol{Z}_p , \quad (S^{2n-3} \times \boldsymbol{I} \times S^1)/\boldsymbol{Z}_p \quad \text{and} \quad (S^{2n-3} \times \boldsymbol{I} \times \boldsymbol{I})/\boldsymbol{Z}_p ,$$

respectively. Note that (A) and (C) are connected n-dimensional manifolds.

We define the deformation g_ε of a Riemannian metric on L_p^{2n-1} in the same manner as in Section 2. Then we have

$$\lim_{\varepsilon \to 0} \lambda_1(L_p^{2n-1}, g_\varepsilon) = 0 .$$

Step 4. We follow [9] and [8] to define g_t as

$$g_t = t^{-1}g + (t^2 - t^{-1})\eta \otimes \eta, \quad 0 < t < \infty.$$

The spectrum $\mathrm{Spec}(S^3, g_t)$ was computed by Urakawa [9] and Tanno [8] as follows.

PROPOSITION ([9] and [8]). *The positive first eigenvalue of* (S^3, g_t) *is given by*

$$\lambda_1(g_t) = \begin{cases} 2t + t^{-2} & t \geq 1/\sqrt[3]{6}, \\ 8t & t \leq 1/\sqrt[3]{6}. \end{cases}$$

It is easy to see that

$$V_1 = E(2t + t^{-2}) = \mathrm{span}\{x, y, z, w\}$$

and

$$V_2 = E(8t) = \mathrm{span}\{x^2 + y^2 - z^2 - w^2, xz + yw, xw - yz\}.$$

Each eigenfunction $\varphi \in V_1$ satisfies $\varphi(-x, -y, -z, -w) = -\varphi(x, y, z, w)$ and $\varphi \in V_2$ satisfies $\varphi(-x, -y, -z, -w) = \varphi(x, y, z, w)$.

Thus $\varphi \in V_2$ is reduced to an eigenfunction on $(\boldsymbol{RP}^3, g_t)$ with the eigenvalue $8t$, while $\varphi \in V_1$ is not reduced to one on $(\boldsymbol{RP}^3, g_t)$. Therefore we have

$$\lambda_1(\boldsymbol{RP}^3, g_t) > \lambda_1(S^3, g_t) \quad \text{for } t > \frac{1}{\sqrt[3]{6}},$$
$$\lambda_1(\boldsymbol{RP}^3, g_t) = \lambda_1(S^3, g_t) \quad \text{for } t \leq \frac{1}{\sqrt[3]{6}}.$$

The above deformation g_t is a kind of collapsing which we used for the part (B). Actually, the deformation makes S^3 into collapsing along the direction of each fiber.

REMARK. For $t \leq 1/\sqrt[3]{6}$, the multiplicity of $\lambda_1(\boldsymbol{RP}^3, g_t) = 8t$ is 3. So the deformation g_t is not the one used in the proof of Proposition A.

6. Counterexample. Finally, we give an example which shows that Proposition C cannot be generalized to a non-double finite Riemannian cover.

Let T be a 2-dimensional torus. We regard it as the quotient space of $\boldsymbol{R}^2$ by identifying (x, y) with $(x + a, y)$ and (x, y) with $(x, y + b)$. We take the finite covering

$$\varpi : \tilde{\tilde{\mathrm{T}}} \to \mathrm{T},$$

where $\tilde{\tilde{\mathrm{T}}}$ is the quotient space of $\boldsymbol{R}^2$ by identifying (x, y) with $(x + 2a, y)$ and (x, y) with $(x, y + 2b)$. The covering transformation group is

$$\pi_1(\mathrm{T})/\varpi_*\pi_1(\tilde{\tilde{\mathrm{T}}}) \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2.$$

The projection ϖ splits as

$$\tilde{\tilde{\mathrm{T}}} \xrightarrow{\varpi_2} \tilde{\mathrm{T}} \xrightarrow{\varpi_1} \mathrm{T},$$

where $\tilde{\mathrm{T}}$ is the quotient space of $\boldsymbol{R}^2$ by identifying (x, y) with $(x + 2a, y)$ and (x, y) with $(x, y + b)$.

For the covering $\tilde{\tilde{\mathrm{T}}} \xrightarrow{\varpi_2} \tilde{\mathrm{T}} \xrightarrow{\varpi_1} \mathrm{T}$ and any Riemannian metric g on T, let $\tilde{g}$ and $\tilde{\tilde{g}}$ be the induced Riemannian metrics on $\tilde{\mathrm{T}}$ and $\tilde{\tilde{\mathrm{T}}}$, respectively.

PROPOSITION D. *For the four-fold cover* $\tilde{\tilde{\mathrm{T}}}$ *onto* T *and any Riemannian metric* g *on* T, *we have*

$$\lambda_1(\mathrm{T}, g) > \lambda_1(\tilde{\tilde{\mathrm{T}}}, \tilde{\tilde{g}}).$$

PROOF. Suppose that the first eigenfunction φ on (T, g) lifts to the first eigenfunction $\tilde{\tilde{\varphi}}$ on $(\tilde{\tilde{\mathrm{T}}}, \tilde{\tilde{g}})$. Then we see that (1) the number of the nodal domain of $\tilde{\varphi}$ is two, and (2) that of $\tilde{\tilde{\varphi}}$ is two. We fix the sign of φ, and denote its positive nodal domain by D_+ and its negative nodal domain by D_-.

We regard the torus T as the quotient space of $\boldsymbol{R}^2$ by identifying (x, y) with $(x + a, y)$ and (x, y) with $(x, y + b)$. We have the double cover $\tilde{\mathrm{T}}$ by identifying (x, y) with $(x + 2a, y)$ and (x, y) with $(x, y + b)$, and the four-fold cover $\tilde{\tilde{\mathrm{T}}}$ by identifying (x, y) with $(x + 2a, y)$ and (x, y) with $(x, y + 2b)$. Let c_a and c_b be the fundamental cycles of $H_1(\mathrm{T}; \boldsymbol{Z}) \cong \boldsymbol{Z}^2$ such that c_a and c_b are represented by the closed curves on T satisfying $c_a(t) = (t, 0)$ $(0 \le t \le a)$ and $c_b(t) = (0, t)$ $(0 \le t \le b)$, respectively. We notice that the cycle $[c_a]$ does not come from $(\varpi_1)_*(H_1(\tilde{\mathrm{T}}; \boldsymbol{Z}))$.

Since the number of the connected components of $\varphi^{-1}(D_+)$ is one, D_+ contains a closed curve which is represented by the homology cycle $pc_a + qc_b$, where p and q are given by one of the following three cases:

(1) $p = 0$ and $q = 1$,
(2) $p = 1$ and $q = 0$ and
(3) $p \neq 0, q \neq 0$ and p, q are relatively prime integers.

We denote the homology cycle $pc_a + qc_b$ simply by c.

Let $\tilde{c}_a$ and $\tilde{c}_b$ be the fundamental cycles of $H_1(\tilde{\tilde{\mathrm{T}}}; \boldsymbol{Z}) \cong \boldsymbol{Z}^2$ such that $(\varpi_*)(\tilde{c}_a) = c_a$ and $(\varpi_*)(\tilde{c}_b) = c_b$, respectively. They are the homology bases on $(\tilde{\tilde{\mathrm{T}}}, \tilde{\tilde{g}})$. We consider the cycle $\tilde{c} = p\tilde{c}_a + q\tilde{c}_b$. We regard these homology cycles of $H_1(\mathrm{T}; \boldsymbol{Z})$ and $H_1(\tilde{\tilde{\mathrm{T}}}; \boldsymbol{Z})$ as the closed curves on T and $\tilde{\tilde{\mathrm{T}}}$, respectively. Also we denote the length of the closed curve c by $l(c)$. Regarding the construction of the covering $\varpi : \tilde{\tilde{\mathrm{T}}} \to \mathrm{T}$, we have:

$$l(\tilde{c}_a) : l(c_a) = 2 : 1 \quad \text{and} \quad l(\tilde{c}_b) : l(c_b) = 2 : 1 .$$

Then we have $l(\tilde{c}) : l(c) = 2 : 1$.

D_- also contains a closed curve which is represented by the same homology cycle c. Thus D_+ and D_- make two stripes on (T, g). We consider the nodal lines of φ on (T, g) which are the closed curves on (T, g) dividing it into D_+ and D_-. There are two nodal lines and they are both represented by the homology cycle c.

To see the connectedness of $\varpi^{-1}(D_+)$ and $\varpi^{-1}(D_-)$, we only used to count the number of connected components of the inverse image of the closed curve represented by the cycle. We only have to compare $l(c)$ with $l(\varpi^{-1}(c))$.

Regarding the construction of the covering, we have $l(\varpi^{-1}(c)) = 4l(c)$. But $\tilde{c}$ is a connected component of $\varpi^{-1}(c)$ and $l(\tilde{c}) = 2l(c)$. Every deck transformation $\sigma \in \pi_1(\mathrm{T})/\varpi_*\pi_1(\tilde{\tilde{\mathrm{T}}}) \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ is an isometry on $(\tilde{\tilde{\mathrm{T}}}, \tilde{\tilde{g}})$ without fixed points. Therefore the connected components of $\varpi^{-1}(c)$ is two and both of them are represented by the cycle $\tilde{c}$.

Consequently, the number of the nodal lines on $\tilde{\tilde{\varphi}}$ is twice of that of φ. Hence there are four nodal lines of $\tilde{\tilde{\varphi}}$. But this contradicts our assumption, completing the proof of Proposition D. □

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DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
2–12–1 OH-OKAYAMA
MEGURO-KU, TOKYO 152–8551
JAPAN