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SURFACES OF CONSTANT MEAN CURVATURE ONE IN THE HYPERBOLIC THREE-SPACE WITH IRREGULAR ENDS

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Abstract. We investigate surfaces of constant mean curvature one in the hyperbolic three-space with irregular ends, and prove that their irregular ends must self-intersect, which answers affirmatively a conjecture of Umehara and Yamada. Moreover we also obtain an explicit representation of a constant mean curvature one surface and a new minimal surface in the Euclidean three-space.

1. Introduction. In the hyperbolic 3-space, surfaces of constant mean curvature one (abbreviated to CMC-1) share many properties with minimal surfaces in the Euclidean 3-space. In 1987, Bryant established a hyperbolic analogue of the Weierstrass representation of minimal surfaces in the Euclidean 3-space for CMC-1 surfaces in the hyperbolic space, and also showed that even if it has finite total curvature, the hyperbolic Gauss map of a complete CMC-1 surface can not meromorphically extend across its ends. This is an important difference between CMC-1 surfaces and minimal surfaces. Subsequently, Umehara and Yamada made many profound studies on CMC-1 surfaces. In particular, they found a necessary and sufficient condition of embeddedness of a regular end and constructed many CMC-1 surfaces subject to certain topological conditions.

If a complete CMC-1 surface is of finite total curvature, then it is conformally equivalent to a compact surface with finite points removed, and these points correspond to the ends of this surface. When the hyperbolic Gauss map could meromorphically extend up to an end, then we call the end regular, and call it irregular otherwise. Umehara and Yamada have constructed many surfaces with irregular ends, and observed through a lot of numerical computation that the irregular ends of these surfaces all self-intersect. Based on this observation in [6] they conjectured that *no irregular ends of CMC-*1 *surfaces are embedded.* In this paper we utilize an asymptotic analysis of ordinary linear differential equations with irregular singular points to study this question, and affirmatively answer to their conjecture:

THEOREM. *No irregular ends of CMC-*1 *surfaces are embedded.*

In addition to proving this, we get an explicit representation of a CMC-1 surface and also a new minimal surface in the Euclidean 3-space.

In their recent work [2] Collin, Hausworth and Rosenberg also answer this conjecture in the general case of surfaces properly embedded in the hyperbolic 3-space.

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2. Asymptotic analysis. Let $\Delta^* = \{z \in \boldsymbol{C} \mid 0 < |z| < 1\}$ be a punctured disc, and $f : \Delta^* \mapsto H^3(-1)$ a conformal CMC-1 immersion of finite total curvature, which is complete at the origin $z = 0$. Suppose that its hyperbolic Gauss map does not meromorphically extend across the end $z = 0$. Then, from a result of Bryant [1] or Umehara-Yamada [5], one can write the Weierstrass data as follows:

$$g = z^{\mu}, \quad \mu > 0, \tag{1}$$

$$\omega = z^{-\nu}\omega_0(1 + \omega_1 z + \cdots)dz, \quad \nu > 0, \tag{2}$$

for which the Hopf differential is given by

$$Q = \omega dg = \mu\omega_0 z^{\mu-\nu-1}(1 + \omega_1 z + \cdots)dz^2, \tag{3}$$

where $\mu - \nu$ is an integer. Since the end is irregular, by using a proposition in [1], we know that $\mu - \nu - 1 \leq -3$, namely $\nu - \mu \geq 2$.

Now, let $F : \widetilde{\Delta^*} \mapsto SL(2, \boldsymbol{C})$ denote Bryant's representation of the surface $f : \Delta^* \mapsto H^3(-1)$, where $\widetilde{\Delta^*}$ is the universal cover of Δ^*, and write F as follows :

$$F = \begin{pmatrix} F_1 & F_3 \\ F_2 & F_4 \end{pmatrix}, \tag{4}$$

which satisfies the equation

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega. \tag{5}$$

In their paper [5], Umehara and Yamada have showed that F_1, F_3 and F_2, F_4 satisfy the following equations (E1) and (E2), respectively,

$$X'' - \frac{\omega'}{\omega}X' - g'\omega X = 0, \tag{E1}$$

$$Y'' - \frac{(g^2\omega)'}{g^2\omega}Y' - g'\omega Y = 0, \tag{E2}$$

and the coefficients of (E1) and (E2) are given by

$$P = \frac{\omega'}{\omega} = \sum_{i=-1}^{\infty} p_i z^i, \quad P' = \frac{(g^2\omega)'}{g^2\omega} = \sum_{i=-1}^{\infty} p'_i z^i,$$

and

$$Q = Q' = g'\omega = \sum_{i=-m}^{\infty} q_i z^i,$$

where

$$p_{-1} = -\nu, \quad p'_{-1} = 2\mu - \nu, \quad p_0 = p'_0, \quad p_1 = p'_1, \ldots,$$

and

$$m = \nu - \mu + 1, \quad q_{-m} = \mu\omega_0, \quad q_{-m+1} = \mu\omega_{0\omega_1}, \ldots.$$

If we rewrite the equation (E1) as $X'' = PX' + QX$, and set

$$U = \begin{pmatrix} X \\ X' \end{pmatrix},$$

then we easily get

$$z^m \frac{dU}{dz} = \begin{pmatrix} 0 & z^m \\ Qz^m & Pz^m \end{pmatrix} U. \tag{6}$$

This is an ordinary linear homogeneous differential system. Since the end is irregular, the number $m \geq 3$, and hence $z = 0$ is an irregular singular point of (6). This is a special case studied in Turrittin's paper [4], which is successfully treated there. The fundamental solutions of (6) have the representation of convergent generalized factorial series in a sector domain. Since the functions F_1, F_3 and their derivatives satisfy the system (6), Bryant's representation of the end will have an explicit form in a sector domain.

In what follows, following the idea of Turrittin we first compute the canonical form of the above equation (6), and discuss the question separately in two cases, according to the number $m = 2n$, $n \geq 2$, or $m = 2n + 1$, $n \geq 1$. We only treat the even number case ($m = 2n$) in detail, and the other case can be dealt with similarly.

Taking the shearing transformation

$$U = (\delta_{ij} z^{n(2n-i)}) V$$

to the system (6), where i, $j = 1, 2$, and multiplying the system (6) by z^{-n}, we get

$$z^n \frac{dV}{dz} = \begin{pmatrix} n(1-2n)z^{n-1} & 1 \\ Qz^{2n} & n(2-2n)z^{n-1} + Pz^n \end{pmatrix} V. \tag{7}$$

Applying the normalizing transformation

$$V = \begin{pmatrix} 1 & -1 \\ \sqrt{q_{-2n}} & \sqrt{q_{-2n}} \end{pmatrix} W$$

to the above equation (7), the leading coefficient matrix becomes in diagonal form. We have

$$z^n \frac{dW}{dz} = (A + B)\, W, \tag{8}$$

where

$$A = \sigma_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sum_{i=1}^{n-1} \frac{q_{-2n+i}}{2\sqrt{q_{-2n}}} z^i \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} \sigma_n & d \\ d & \sigma_n \end{pmatrix} z^{n-1},$$

$$\sigma_0 = \sqrt{q_{-2n}}, \quad d = \frac{1}{2}(n - \nu), \quad \sigma_n = \frac{1}{2}(3n - 4n^2 + p_{-1}),$$

and from now on, B will always denote the sum of terms of z^i $(i \geq n)$. The zero-inducing transformations

$$W = (I + zQ_1)R_1, \quad R_1 = (I + z^2 Q_2) R_2, \ldots,$$

will change the coefficients matrix of z^k into diagonal form, and can not affect that of z^i $(i = 1, 2, \cdots, k-1)$. In fact, substituting

$$W = (I + zQ_1)R_1$$

into (8), we get

$$z^n \frac{dR_1}{dz} = MR_1\,, \tag{9}$$

where

$$M = (I + zQ_1)^{-1}((A + B)(I + zQ_1) - z^n Q_1)\,. \tag{10}$$

By using

$$(I + zQ_1)^{-1} = I - zQ_1 + z^2 Q_1^2 - \cdots\,,$$

and

$$Q_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

into (9) and (10), the coefficient matrix of z can be computed as

$$\sigma_0 \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix} + \frac{q_{-2n+1}}{2\sqrt{q_{-2n}}} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

In order to let it become in diagonal form, we may choose the entries of the matrix Q_1 as

$$a = d = 0\,, \quad b = c = \frac{q_{-2n+1}}{4q_{-2n}}\,,$$

which is equivalent to that

$$Q_1 = \frac{q_{-2n+1}}{4q_{-2n}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By a straightforward compution the equation (9) takes the following form in which the first two terms become in diagonal form

$$z^n \frac{dR_1}{dz} = \left((\sigma_0 + \sigma_1 z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + M_1 + B \right) R_1\,, \tag{11}$$

where

$$M_1 = \sum_{i=2}^{n-1} \begin{pmatrix} A_i & -B_i \\ B_i & -A_i \end{pmatrix} z^i + \begin{pmatrix} \sigma_n & d \\ d & \sigma_n \end{pmatrix} z^{n-1}\,, \quad \sigma_1 = \frac{q_{-2n+1}}{2\sqrt{q_{-2n}}}\,,$$

A_i, B_i being constant depending only on q_i. Similarly, applying the zero-inducing transformations

$$R_{i-1} = (I + z^i Q_i)R_i\,, \quad i = 2, 3, \ldots, n-2\,,$$

to the equation (11), taking the constant matrix Q_i always in the form

$$Q_i = c_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the coefficient matrix of z^i $(i = 2, \dots, n-2)$ becomes in the diagonal form, and (11) becomes

$$z^n \frac{dR_{n-2}}{dz} = \left(\sum_{i=0}^{n-2} \sigma_i z^i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + M_2 + B \right) R_{n-2}, \tag{12}$$

where

$$M_2 = \begin{pmatrix} A'_{n-1} & -B'_{n-1} \\ B'_{n-1} & -A'_{n-1} \end{pmatrix} z^{n-1} + \begin{pmatrix} \sigma_n & d \\ d & \sigma_n \end{pmatrix} z^{n-1},$$

and A'_{n-1}, B'_{n-1}, σ_i $(i = 0, \dots, n-2)$ being constant depending only on q_i. Finally, we use the transformation

$$R_{n-2} = (I + z^{n-1} Q_{n-1}) R,$$

to the equation (12), where

$$Q_{n-1} = \begin{pmatrix} 0 & (B'_{n-1} - d)/2\sigma_0 \\ (B'_{n-1} + d)/2\sigma_0 & 0 \end{pmatrix}.$$

By setting $\sigma_{n-1} = A'_{n-1}$, the equation (12) becomes

$$z^n \frac{dR}{dz} = \left(\sum_{i=0}^{n-1} \sigma_i z^i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sigma_n z^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B \right) R. \tag{13}$$

Summing up these, we get the following lemma:

LEMMA 1. *Corresponding to the system* (7), *there exists a transformation*

$$V = S(z) R,$$

where

$$S(z) = \begin{pmatrix} \frac{1}{\sqrt{q-2n}} & \frac{-1}{\sqrt{q-2n}} \end{pmatrix} (I + zQ_1) \cdots (I + z^{n-1} Q_{n-1}),$$

which reduces the equation (7) *to the canonical form* (13).

To proceed further several new symbols are needed. Set $B = (B_{ij})$, $B_{ij} = \sum_{k=n}^{\infty} B_{ijk} z^k$, and

$$\rho_j(z) = (-1)^{j+1} \sum_{i=0}^{n-1} \sigma_i z^i + \sigma_n z^{n-1}, \quad j = 1, 2.$$

If $i \neq j$, let

$$\rho_i - \rho_j = r_{ij,0} + r_{ij,1} z + \cdots + r_{ij,n-1} z^{n-1},$$

where $r_{ij,0} \neq 0$, and define the matrices Γ_{ij} by the equations

$$r_{ij,0} \Gamma_{ij} + B_{ijn} = 0.$$

Keeping this notation in mind, we apply the results of Turrittin to the equation (13) to obtain

THEOREM 2. *The fundamental solutions of the system* (13) *have the following forms*:

$$R_1(z) = \begin{pmatrix} 1 + U_{11}(z) \\ z^n(\Gamma_{21} + U_{21}(z)) \end{pmatrix} \exp(f_1(z)),$$

$$R_2(z) = \begin{pmatrix} z^n(\Gamma_{12} + U_{12}(z)) \\ 1 + U_{22}(z) \end{pmatrix} \exp(f_2(z)),$$

where

$$f_1(z) = (\sigma_n + \sigma_{n-1}) \ln z - \frac{\sigma_{n-2}}{z} - \cdots - \frac{\sigma_0}{(n-1)z^{n-1}},$$

$$f_2(z) = (\sigma_n - \sigma_{n-1}) \ln z + \frac{\sigma_{n-2}}{z} + \cdots + \frac{\sigma_0}{(n-1)z^{n-1}},$$

and

$$U_{ij}(z) = \sum_{k=1}^{n-1} \{z^k U_{ijk} + z^k L_{ijk}(z)\},$$

$$L_{ijk}(z) = \sum_{\nu=0}^{\infty} \frac{K_{ijk\nu}(\Phi, \gamma)}{z^{1-n}(z^{1-n} + \gamma e^{-i\Phi}) \cdots (z^{1-n} + \nu\gamma e^{-i\Phi})},$$

are convergent factorial series provided

(i) *constant number* $\Phi \neq \arg(\pm\sigma_0)$,

(ii) *positive constant* γ *is sufficiently large*,

(iii) $|z|$ *is sufficiently small and* z *is located in one of the sectors*

$$(2\Phi - \pi + 2\varepsilon + 4\pi k)/2(n-1) \leq \arg z \leq (2\Phi + \pi - 2\varepsilon + 4\pi k)/2(n-1),$$

where $\varepsilon > 0$ *is arbitrary and* $k = 0, 1, 2, \ldots, n-2$. *Here* U_{ijk} *and* $K_{ijk\nu}$ *are appropriate known constant matrices.*

REMARK. The constant number Φ can be taken arbitrary except $\arg(\pm\sigma_0)$, and the above factorial series are convergent only in a sector domain. When z tends to zero in the sector domain, $U_{ij}(z)$ also tends to zero.

In the following, we consider problems only in a sector domain which depends on Φ. Now we return to the beginning equation (E1). Then its fundamental solutions X_1, X_2 take the following forms:

$$X_1(z) = z^{a+\sigma}(1 + A(z)) \exp(-\zeta), \tag{14}$$

$$X_2(z) = z^{a-\sigma}(1 + B(z)) \exp \zeta, \tag{15}$$

where

$$a = n(2n-1) + \frac{1}{2}(3n - 4n^2 + p_{-1}), \quad \sigma = \sigma_{n-1},$$

$$\zeta = \frac{\sigma_{n-2}}{z} + \cdots + \frac{\sigma_0}{(n-1)z^{n-1}},$$

and A, B are analytic functions, which tend to zero as z tends to zero. On the other hand, since (E1) and (E2) have the same coefficients except p_{-1}, the fundamental solutions Y_1, Y_2 of (E2) should have the forms:

$$Y_1(z) = z^{a'+\sigma}(1 + C(z))\exp(-\zeta), \tag{16}$$

$$Y_2(z) = z^{a'-\sigma}(1 + D(z))\exp\zeta, \tag{17}$$

where

$$a' = n(2n-1) + \frac{1}{2}(3n - 4n^2 + p'_{-1}),$$

and $C(z)$, $D(z)$ have the same properties as $A(z)$, $B(z)$. Recall that $p_{-1} = -\nu$, $p'_{-1} = 2\mu - \nu$ and $\mu - \nu - 1 = -2n$, so that

$$a = \frac{1}{2}(-n - \mu + 1) < 0, \quad a' - a = \mu, \quad a + a' = 1 - n. \tag{18}$$

3. Main results. Next we establish a lemma which plays an important rule in the rest of our argument.

LEMMA 3. *Let $\alpha = \alpha_1 + \sqrt{-1}\alpha_2$ be a constant complex number, and*

$$p(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_{m-1} z,$$

a polynomial in z, and $a_0 \neq 0$. In the domain $\{z \in \boldsymbol{C} \mid 0 < |z| < 1, -\pi < \arg z \leq \pi\}$, for the function

$$z^\alpha \exp p\left(\frac{1}{z}\right),$$

there exists an essential singular direction $\theta_0 \in (-\pi, \pi]$, that is, there exists a sector

$$S = \{z \in \boldsymbol{C} \mid 0 < |z| < 1, \ \theta_0 - \varepsilon < \arg z < \theta_0 + \varepsilon\}$$

such that for any constant number $b \neq 0, \infty$, there exists a sequence $z_1, z_2, \cdots \in S$ satisfying $\lim z_i = 0$ and

$$z_i^\alpha \exp p\left(\frac{1}{z_i}\right) = b.$$

PROOF. Set

$$z^\alpha \exp p\left(\frac{1}{z}\right) = b,$$

and take logarithm of the equation. Then we have

$$\alpha \ln z + \frac{a_0}{z^m} + \frac{a_{m-1}}{z^{m-1}} + \cdots + \frac{a_1}{z} = \ln b.$$

For convenience, let $a_0 = 1$ and write $|z| = r$, $\arg b = -(\beta + 2k\pi)$. Taking the real and imaginary part the above equation can be written as

$$\alpha_1 \ln r - \alpha_2\theta - \ln|b| + \frac{1}{r^m}(\cos m\theta + o_1(r)) = 0\,, \tag{19}$$

$$\alpha_2 \ln r + \alpha_1\theta + \beta + 2k\pi - \frac{1}{r^m}(\sin m\theta + o_2(r)) = 0\,. \tag{20}$$

Then (19) and (20) yield

$$\begin{aligned}&\frac{\alpha_1}{m}\ln\frac{\alpha_1(\beta+2k\pi)+\alpha_2\ln|b|+(\alpha_1^2+\alpha_2^2)\theta}{\alpha_1\sin m\theta+\alpha_2\cos m\theta+o(r)}+\alpha_2\theta+\ln|b|\\ &\quad-(\cos m\theta+o(r))\frac{\alpha_1(\beta+2k\pi)+\alpha_2\ln|b|+(\alpha_1^2+\alpha_2^2)\theta}{\alpha_1\sin m\theta+\alpha_2\cos m\theta+o(r)}=0\,.\end{aligned} \tag{21}$$

Now, in (20) $m\theta$ is restricted in $[\pi/4, 3\pi/4]$, and r is restricted in $(0, r_0]$, r_0 being a very small constant number. If we choose a sufficiently large positive number k_0, when $k > k_0$, and $r = r_0$, then the left hand side of (20) will be negative for any fixed $m\theta \in [\pi/4, 3\pi/4]$. On the other hand, when r tends to zero, the left hand side of (20) will be positive. So, for any $m\theta \in [\pi/4, 3\pi/4]$ and $k > k_0$, there exists a number $r \in (0, r_0]$ satisfying (20). Choosing the maximum of such r's, we get a continuous function $r = r(\theta, k)$, and $r \to 0$ when $k \to +\infty$.

Moreover, in (21) the $m\theta$ is restricted in a small interval $[-\varepsilon + \pi/2, \varepsilon + \pi/2] \subset [\pi/4, 3\pi/4]$, which guarantees that

$$\left|\frac{\alpha_2}{\alpha_1}\cos m\theta\right| < \frac{1}{2}, \quad \sin m\theta > \frac{\sqrt{2}}{2}\,,$$

if $\alpha_1 \neq 0$. Now substitute $r = r(\theta, k)$ into the equation (21), and choose a sufficiently large $k_1 > k_0$ such that for every $k > k_1$ the left hand side of (21) is positive when $m\theta = \varepsilon + \pi/2$, and negative when $m\theta = -\varepsilon + \pi/2$. Then by the mean value theorem, given large positive integers $k^1, k^2, \cdots$ (tending to $+\infty$), there exist $m\theta_1, m\theta_2, \cdots \in [-\varepsilon+\pi/2, \varepsilon+\pi/2]$ satisfying (21) as well as (19). If $\alpha_1 = 0$, substitute $r = r(\theta, k)$ into (19), and take a sufficiently large $k_3 > k_2$ such that for $k > k_3$ the left hand side of (19) is negative when $m\theta = \pi/2 + \varepsilon$, and positive when $m\theta = \pi/2 - \varepsilon$. Then for sufficiently large integers $k^1, k^2, \cdots$ (tending to $+\infty$), by the mean value theorem, there exist $m\theta_1, m\theta_2, \cdots, \in [-\varepsilon + \pi/2, \varepsilon + \pi/2]$ satisfying (19). From the pair (θ_i, k^i), we get r_i, and consequently the desired points $z_i = r_i \exp\sqrt{-1}\theta_i$. It is easy to see that when k_i tends to $+\infty$, r_i and θ_i tends to 0, and $\theta_0 = \pi/(2m)$, respectively. The Lemma is now proved.

REMARK. In general case, if $a_0 = r_0 \exp\sqrt{-1}\beta_0$, then $\theta_0 = (\pi/2 + \beta_0)/m$.

We choose a sector domain S which contains an essential direction of the function

$$z^{-\sigma}\exp\zeta\,.$$

By using Lemma 3, we prove the following lemma:

LEMMA 4. *The holomorphic representation* $F : \widetilde{\Delta^*} \mapsto SL(2, \boldsymbol{C})$ *of the surface* $f : \Delta^* \mapsto H^3(-1)$ *can be chosen as*

$$F = \begin{pmatrix} t_1 X_1 & t_3 Y_1 \\ t_2 X_2 & t_4 Y_2 \end{pmatrix},$$

where t_i *are constant numbers.*

PROOF. In fact, we may first choose

$$F_1 = t_1 X_1, \quad F_3 = t_2 X_2,$$

and

$$F_2 = t_3 Y_1 + t_3' Y_2, \quad F_4 = t_4' Y_1 + t_4 Y_2.$$

Applying Lemma 3, there exist series $z_i^k \in S$ such that

$$(z_i^k)^{-2\sigma} \exp 2\zeta(z_i^k) = b_k,$$

where b_k $(k = 1, 2, \cdots, l)$ are different constant numbers.

Substitute these b_k into the equation $\det F = 1$, and note that $\lim z_i^k = 0, a + a' = 1 - n$. Then we get

$$t_1 t_4' b_k^{-1} - t_2 t_3' b_k + t_1 t_4 - t_2 t_3 = 0.$$

Since $t_1 \neq 0$ and $t_2 \neq 0$, this implies that $t_3' = t_4' = 0$ and $t_1 t_4 - t_2 t_3 = 0$. The lemma is proved.

In his paper [1], Bryant pointed out that the end of Enneper's cousin is asymptotic to every point at infinity. In fact, all of irregular ends have this property. One of our main results is the following

THEOREM 5. *The irregular end* $f : \Delta^* \mapsto H^3(-1)$ *tends to every point at infinity.*

PROOF. By Lemma 4, we get

$$f = F \cdot F^* = |z|^{2a} N \left(\frac{1}{N} \begin{pmatrix} t_1 h \\ t_2 h^{-1} \end{pmatrix} \overline{\begin{pmatrix} t_1 h & t_2 h^{-1} \end{pmatrix}} + |z|^l M \right),$$

where

$$h = z^\sigma \exp(-\zeta), \quad N = |h|^2 + |h|^{-2}, \quad l = \min\left(\frac{1}{2}, \mu\right),$$

and M is a bounded matrix. Recall that $\mu > 0$ and $a < 0$. Then we see that when $z \to 0$, $[t_1 h, t_2 h^{-1}]$ close to every point at infinity S^2_∞ of the hyperbolic space $H^3(-1)$. This completes the prove of Theorem 5.

Another main result is the following

THEOREM 6. *No irregular ends of CMC-1 surfaces are embedded.*

Before giving the proof, we recall some materials which are needed. The upperhalf space model $(\boldsymbol{R}^3_+, ds^2)$ of $H^3(-1)$ is given by

$$\boldsymbol{R}^3_+ = \{(X_1, X_2, X_3) \in \boldsymbol{R}^3 \mid X_3 > 0\},$$

$$ds^2 = (dX_1^2 + dX_2^2 + dX_3^2)/X_3^2.$$

This model is equivalent to the Minkowski model by (see [3])

$$\boldsymbol{L}^4 \leftrightarrow \boldsymbol{R}^3_+, \quad (x_0, x_1, x_2, x_3) \longleftrightarrow (X_1, X_2, X_3),$$

where

$$X_1 + \sqrt{-1}X_2 = (x_1 + \sqrt{-1}x_2)/(x_0 - x_3), \quad X_3 = 1/(x_0 - x_3). \tag{22}$$

On the other hand, from Theorem 2, we can choose a desired small sector domain S containing any direction θ, only by using suitable Φ and k in there. Without loss generality we may assume that $\sigma_0/(n-1) = 1$, and choose the sector domain S which contains the direction $\theta = 0$, that is, a part of the real axis (otherwise if $\sigma_0/(n-1) = r_0 \exp\sqrt{-1}\beta_0$, choose the sector domain S which contains the direction $\theta = \beta_0/(n-1)$). Note that Lemma 4 is not valid, since the sector may not contain an essential direction of the function $z^{-\sigma}\exp\zeta$. However, we can also choose F having the form

$$F = \begin{pmatrix} t_2X_2 & s_2Y_2 \\ t_1X_1 & s_1Y_1 \end{pmatrix}. \tag{23}$$

In fact, assume $F_3 = s_4Y_1 + s_2Y_2$, $F_4 = s_1Y_1 + s_3Y_2$. Note that $\det F = 1$, and when $z \to 0$ along the real axis (otherwise $\theta = \beta_0/(n-1)$), the function $z^{-2\sigma}\exp 2\zeta \to \infty$. Hence $t_1s_4 = t_2s_3 = 0$. But t_1, $t_2 \neq 0$, so $s_3 = s_4 = 0$.

PROOF OF THEOREM 6. We consider the problem in the Minkowski model and the upperhalf space model of $H^3(-1)$. Note that

$$z^{-\sigma}\exp\zeta(z) = \exp\frac{1}{r^{n-1}}(\cos(n-1)\theta + o_1(r) + \sqrt{-1}(\sin(1-n)\theta + o_2(r))),$$

where $z = r\exp\sqrt{-1}\theta$ and r is sufficiently small. From (22), (23), the above equation and

$$F \cdot F^* = \begin{pmatrix} x_0 + x_3 & x_1 + \sqrt{-1}x_2 \\ x_1 - \sqrt{-1}x_2 & x_0 - x_3 \end{pmatrix}.$$

We get the Norm $N(r,\theta)$, the Argument $\Theta(r,\theta)$ of the coordinates $X_1 + \sqrt{-1}X_2$, and X_3 of the surface in the upper space $\boldsymbol{R}^3_+$, respectively as

$$N(r,\theta) = |c_1(1 + o_3(r))|\exp\frac{2}{r^{n-1}}(\cos(n-1)\theta + o_1(r)), \tag{24}$$

$$\Theta(r,\theta) = \frac{2}{r^{n-1}}(\sin(1-n)\theta + o_4(r)), \tag{25}$$

$$X_3(r,\theta) = c_2(1+o_5(r))r^{-2a}\exp\frac{2}{r^{n-1}}(\cos(n-1)\theta + o_1(r)), \tag{26}$$

where c_1, c_2 are constant numbers, and $o_i = o(r^p)$ for a positive constant number $p > 0$. In a small sector domain defined by

$$\Delta_\varepsilon^* = \{z \in \boldsymbol{C} \mid 0 < |z| < \varepsilon,\ -\pi/2 + \varepsilon < (n-1)\theta < \pi/2 - \varepsilon\},$$

the factorial series of fundamental solutions (14) through (17) converge. Choose

$$[\theta_0, \theta_1] \subset [0, \pi/2(n-1) - \varepsilon/(n-1)],$$

$$[-\theta_1', -\theta_0'] \subset [-\pi/2(n-1) + \varepsilon/(n-1), 0],$$

and $\theta_0 > \theta_0' > 0$, $\theta_1 < \theta_1'$. We take sufficiently small $r < \varepsilon_0 < \varepsilon$. At present stage, X_3 may be very large, when $(r,\theta) \in (0,\varepsilon_0) \times ([\theta_0,\theta_1] \cup [-\theta_1', -\theta_0'])$.

First, consider functions $X_3(r,\theta)$ and $X_3(\tilde r, \tilde\theta)$, where

$$(r,\theta) \in (0,\varepsilon_0) \times [\theta_0,\theta_1], \quad (\tilde r,\tilde\theta) \in (0,\varepsilon_0) \times [-\theta_1', -\theta_0'].$$

Set $X_3(r,\theta) = X_3(\tilde r,\tilde\theta)$, that is

$$\begin{aligned} &r^{-2a}(1+o_5(r))\exp\frac{2}{r^{n-1}}(\cos(n-1)\theta + o_1(r)) \\ &= \tilde r^{-2a}(1+o_5(\tilde r))\exp\frac{2}{\tilde r^{n-1}}(\cos(n-1)\tilde\theta + o_1(\tilde r)). \end{aligned} \tag{27}$$

Let $\tilde r = r_0 < \varepsilon_0$, r_0 being a small constant number, and let $\tilde\theta$ vary in $[-\theta_1', -\theta_0']$. Then denote by M_0 the maximum of the right hand side of (27). Take $r \le r_1 < r_0$, r_1 being a constant number, and let θ vary in $[\theta_0,\theta_1]$. Then denote by $M_1$the minimum of the left hand side of (27). Fix r_1 such that $M_1 > M_0$. Then, for any $\theta, \tilde\theta$, and $r \le r_1$,

$$X_3(r_0,\tilde\theta) < M_0 < M_1 < X_3(r,\theta).$$

But $X_3(\tilde r,\tilde\theta) \to +\infty$ when $\tilde r \to 0$, and both side of (27) are continuous, so there exists $\tilde r \in (0, r_0]$ satisfying (27) for any $r \in (0, r_1)$, $\theta \in [\theta_0,\theta_1]$ and $\tilde\theta \in [-\theta_1', -\theta_0']$ (here we can choose the maximum of $\tilde r's$). Hence we get a continuous function $\tilde r = \tilde r(r,\theta,\tilde\theta)$, and $\tilde r \to 0$ when $r \to 0$.

On the other hand, from (24) it is easily seen that $N \to +\infty$ as $r \to 0$. Consider two functions $(X_1 + \sqrt{-1}X_2)(r,\theta)$ and $(X_1 + \sqrt{-1}X_2)(\tilde r,\tilde\theta)$. Set

$$(X_1 + \sqrt{-1}X_2)(r,\theta) = (X_1 + \sqrt{-1}X_2)(\tilde r,\tilde\theta), \tag{28}$$

and substitute $\tilde r = \tilde r(r,\theta,\tilde\theta)$ into (28). By (24) we obtain

$$N(r,\theta) = |c_1(1+o_3(r))|\exp\frac{2}{r^{n-1}}(\cos(n-1)\theta + o_1(r)),$$

and

$$N(\tilde r,\tilde\theta) = |c_1(1+o_3(\tilde r))|\exp\frac{2}{\tilde r^{n-1}}(\cos(n-1)\tilde\theta + o_1(\tilde r)).$$

Moreover, from (27) we have

$$\frac{\exp 2(\cos(n-1)\theta + o_1(r))/r^{n-1}}{\exp 2(\cos(n-1)\tilde{\theta} + o_1(\tilde{r}))/\tilde{r}^{n-1}} = \frac{\tilde{r}^{-2a}(1+o_5(\tilde{r}))}{r^{-2a}(1+o_5(r))}. \tag{29}$$

Define the function $K(r, \theta, \tilde{\theta})$ by

$$K(r, \theta, \tilde{\theta}) = \frac{\tilde{r}^{-2a}(1+o_5(\tilde{r}))}{r^{-2a}(1+o_5(r))} - \frac{|1+o_3(\tilde{r})|}{|1+o_3(r)|}.$$

Taking logarithm of (29), and noting that if $r \to 0$, then $\tilde{r} \to 0$, $r^{n-1}\ln r \to 0$, and $\tilde{r}^{n-1}\ln\tilde{r} \to 0$, we get

$$\lim_{r\to 0+} \frac{r^{n-1}}{\tilde{r}^{n-1}} = \frac{\cos(n-1)\theta}{\cos(n-1)\tilde{\theta}}. \tag{30}$$

By (30), when $\tilde{\theta} = -\theta_0'$ or $-\theta_1'$, it follows that

$$\lim_{r\to 0+} \frac{r^{n-1}}{\tilde{r}^{n-1}} = \frac{\cos(n-1)\theta}{\cos(n-1)\tilde{\theta}} \le \frac{\cos(n-1)\theta_0}{\cos(n-1)\theta_0'} < 1,$$

$$\left(\text{or} \ge \frac{\cos(n-1)\theta_1}{\cos(n-1)\theta_1'} > 1\right).$$

Hence, when r is small enough (it is $r < r_2 < r_1$, r_2 being a small constant number), for any $\theta \in [\theta_0, \theta_1]$ the continuous function $K(r, \theta, \tilde{\theta})$ has the properties $K(r, \theta, \theta_0') < 0$, $K(r, \theta, \theta_1') > 0$. Then, by the mean value theorem, there exists a $\tilde{\theta} \in [-\theta_1', -\theta_0']$ such that $K(r, \theta, \tilde{\theta}) = 0$. Choose $\tilde{\theta}$ as the minimum of those $\tilde{\theta}'s$. Consequently, we get a continuous function $\tilde{\theta} = \tilde{\theta}(r, \theta)$, and this implies that

$$X_3(r, \theta) = X_3(\tilde{r}, \tilde{\theta}), \quad N(r, \theta) = N(\tilde{r}, \tilde{\theta}),$$

hold for $r < r_2$ and $\theta \in [\theta_0, \theta_1]$. On the other hand, the Argument of $X_1 + \sqrt{-1}X_2$ are respectively

$$\Theta(r, \theta) = \frac{2}{r^{n-1}}(\sin(1-n)\theta + o_4(r)),$$

and

$$\Theta(\tilde{r}, \tilde{\theta}) = \frac{2}{\tilde{r}^{n-1}}(\sin(1-n)\tilde{\theta} + o_4(\tilde{r})).$$

Note that $\sin(1-n)\theta < 0$ and $\sin(1-n)\tilde{\theta} > 0$, and that $\Theta(r, \theta)$ tends to $-\infty$ and $\Theta(\tilde{r}, \tilde{\theta})$ tends to $+\infty$, as $r \to 0$. Hence, for any $\theta \in [\theta_0, \theta_1]$, there exists a r such that

$$\Theta(\tilde{r}, \tilde{\theta}) - \Theta(r, \theta) = 2k\pi,$$

where k is a sufficiently large integer. Then we have found the points $z = r\exp\sqrt{-1}\theta$ and $\tilde{z} = \tilde{r}\exp\sqrt{-1}\tilde{\theta}$ such that

$$(X_1 + \sqrt{-1}X_2)(r, \theta) = (X_1 + \sqrt{-1}X_2)(\tilde{r}, \tilde{\theta}),$$

and

$$X_3(r, \theta) = X_3(\tilde{r}, \tilde{\theta}).$$

Hence the irregular end must self-intersect. Theorem 6 is now proved.

4. An explicit representation of a surface. In their paper [4], Umehara and Yamada constructed a null immersion from a Riemann surface into $SL(2, \boldsymbol{C})$. Let $M = \boldsymbol{C}^* = \boldsymbol{C} \setminus \{0\}$, and set

$$g = z^{-3}, \quad \omega = -dz.$$

The pair (g, ω) determine a null immersion $F : M \to SL(2, \boldsymbol{C})$, and consequently a CMC-1 surface $f : M \to H^3(-1)$. It has two ends; the one corresponding to $z = \infty$ is regular and embedded, and the other corresponding to $z = 0$ is an irregular end.

Now we compute the explicit representation of the surface. F_1 and F_3 satisfy

$$X'' - \frac{3}{z^4} X = 0,$$

and the fundamental solutions are

$$X_1 = z \exp \frac{\sqrt{3}}{z}, \quad X_2 = z \exp -\frac{\sqrt{3}}{z}.$$

F_2 and F_4 satisfy

$$Y'' + \frac{6}{z} Y' - \frac{3}{z^4} Y = 0,$$

and the fundamental solutions are

$$Y_1 = z^{-2}(1 - \sqrt{3}z + z^2) \exp \frac{\sqrt{3}}{z},$$

and

$$Y_2 = z^{-2}(1 + \sqrt{3}z + z^2) \exp -\frac{\sqrt{3}}{z}.$$

We can easily check that Bryant's holomorphic representation is given by

$$F = \begin{pmatrix} z \exp \sqrt{3}/z & -z^{-2}(1 - \sqrt{3}z + z^2) \exp \sqrt{3}/z \\ -z \exp -\sqrt{3}/z & z^{-2}(1 + \sqrt{3}z + z^2) \exp -\sqrt{3}/z \end{pmatrix}.$$

By a proposition in [7], using the inverse matrix F^{-1}, we get a dual surface

$$f^{\#} : M \to H^3(-1),$$

$$f^{\#} = (F^{-1}) \cdot (F^{-1})^*,$$

which is a complete CMC-1 surface. The Weierstrass data of the dual surface $f^{\#}$ are

$$G = -\frac{1 - z}{1 + z} \exp(2z), \quad \Omega = 2 \left(\frac{1 + z}{z} \right)^2 (\exp -2z) dz.$$

The pair (G, Ω) generates a new complete minimal surface in the Euclidean 3-space, its Weierstrass representation is

$$
\begin{aligned}
x_1 &= \operatorname{Re}\left(-\frac{1}{2}(\exp 2z + \exp -2z) + \frac{1}{z}(\exp 2z - \exp -2z)\right), \\
x_2 &= \operatorname{Re}\sqrt{-1}\left(\frac{1}{2}(\exp 2z - \exp -2z) - \frac{1}{z}(\exp 2z + \exp -2z)\right), \\
x_3 &= \operatorname{Re} 2\left(z + \frac{1}{z}\right).
\end{aligned}
$$

This minimal surface has two ends, and the one end $(z = 0)$ is an embedded planar end.

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