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CODIMENSION ONE LOCALLY FREE ACTIONS OF SOLVABLE LIE GROUPS

Dedicated to Professor Fuichi Uchida on his sixtieth birthday

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Abstract. Let G be a non-unimodular solvable Lie group which is a semidirect product of $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$. We consider a codimension one locally free volume preserving action of G on a closed manifold. It is shown that, under some conditions on the group G, such an action is homogeneous. It is also shown that such a group G has a homogeneous action if and only if the structure constants of G satisfy certain algebraic conditions.

Introduction. By a locally free action of a Lie group G, we mean an action all of whose isotropy subgroups are discrete. A locally free action Φ then induces a foliation $\mathcal{F}_\Phi$ whose leaves are given by the orbits of Φ. The primary purpose of this paper is to investigate the behavior of codimension one locally free actions of some solvable Lie groups on closed manifolds.

To begin with, let G be a nilpotent Lie group. Then, from the point of view of foliation theory, Hector, Ghys and Moriyama [8] proved that the codimension one foliation $\mathcal{F}_\Phi$ is *almost without holonomy*. That is, each non-compact leaf of $\mathcal{F}_\Phi$ has trivial leaf holonomy ([7, IV-2.11]). This implies that the qualitative structure of $\mathcal{F}_\Phi$ is comparatively simple.

When G is solvable but not nilpotent, the structure of $\mathcal{F}_\Phi$ is more complicated. Even in the case where G is the real affine group

$$\mathrm{Aff}^+(\boldsymbol{R}) = \left\{ \begin{pmatrix} e^t & x \\ 0 & 1 \end{pmatrix} \middle| \, t, x \in \boldsymbol{R} \right\},$$

which is the simplest non-nilpotent solvable Lie group, it is known ([5, Propositions II.1.4 and II.1.5]) that all leaves of $\mathcal{F}_\Phi$ are dense and there exists a leaf with non-trivial leaf holonomy. However, by assuming the existence of an invariant volume form, Ghys obtained the following remarkable result, which shows the smooth rigidity of codimension one locally free $\mathrm{Aff}^+(\boldsymbol{R})$-actions.

THEOREM ([5, Theorem B]). *Let G be $\mathrm{Aff}^+(\boldsymbol{R})$. Let $\Phi : G \times M \to M$ be a locally free G-action of class C^r $(r \geq 2)$ on a closed smooth 3-manifold M. Suppose that the action Φ preserves a volume form of class C^0. Then Φ is C^{r-1}-conjugate to a homogeneous action.*

To be precise, let Φ and Φ' be C^r-actions of a Lie group G on manifolds M and M', respectively. Then Φ and Φ' are said to be *C^s-conjugate* $(s \leq r)$ if there exist an isomorphism

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φ of G and a C^s-diffeomorphism f from M to M' such that $f \circ \Phi = \Phi' \circ (\varphi \times f)$. If a Lie group H contains G and a cocompact discrete subgroup Γ as well, then G acts on the compact homogeneous manifold H/Γ by left translations. Such an action is called a *homogeneous action*. Note that a homogeneous action preserves the natural volume form of H/Γ that is induced from a right and left invariant volume form of H.

Following the above theorem of Ghys, several rigidity results have since been obtained for actions of Lie groups other than $\mathrm{Aff}^+(\boldsymbol{R})$ ([1], [2] and [6]).

In this paper, we consider non-nilpotent solvable Lie groups G which are semidirect products of $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$, and study the rigidity of codimension one locally free volume preserving actions of G. To state our main results, we fix some notation.

For consistency with the case of $\mathrm{Aff}^+(\boldsymbol{R}) = \boldsymbol{R}_+ \ltimes \boldsymbol{R}$, we use the multiplicative notation for $\boldsymbol{R}^m$. Since the group structure of $G = \boldsymbol{R}^m_+ \ltimes \boldsymbol{R}^n$ is determined by a homomorphism $\psi : \boldsymbol{R}^m_+ \to \mathrm{Aut}(\boldsymbol{R}^n) \cong GL(n, \boldsymbol{R})$, we write the semidirect product by $\boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n$. We assume that ψ is diagonalizable. By changing the semidirect product structure of G if necessary, we may assume furthermore that ψ is locally injective, and in particular $m \leq n$ (Lemma 1.1).

Take a basis $\{\mathbf{e}_j \mid 1 \leq j \leq m\}$ of $\boldsymbol{R}^m$ and put $d\psi(\mathbf{e}_j) =: A_j \in M(n, \boldsymbol{R})$, where $d\psi$ is the differential of ψ. Then the matrices $\{A_j\}$ are simultaneously diagonalizable. Denote by λ_i^j the i-th diagonal element of the diagonalized form of A_j, and by Λ_ψ the $n \times m$-matrix whose i-th row vector is given by $\Lambda_i := (\lambda_i^1, \lambda_i^2, \dots, \lambda_i^m)$ $(1 \leq i \leq n)$. We call Λ_ψ the *structure matrix* of $G = \boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n$ (see Section 1.1). Put $\beta := \sum_{i=1}^n \Lambda_i$.

A main theorem of this paper is the following.

THEOREM 1. *Let $G = \boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n$ $(0 < m \leq n)$ and M an $(m + n + 1)$-dimensional connected closed orietable smooth manifold. Let $\Phi : G \times M \to M$ be a locally free smooth action preserving a volume form Ω of class C^0. Suppose that the homomorphism ψ is diagonalizable, locally injective and the structure matrix Λ_ψ of G satisfies*

$$\beta \notin \{a_i \Re \Lambda_i, b_j \Re \Lambda_j - \Re \Lambda_k \mid 0 \leq a_i, b_j \leq 1,\ 1 \leq i, j, k \leq n\}.$$

Then M is a solvmanifold and Φ is C^∞-conjugate to a homogeneous action.

The other result is the following theorem which gives a necessary and sufficient condition for $G = \boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n$ to have a codimension one homogeneous action. Two $n \times m$-matrices Λ and Λ' are said to be *equivalent* if $\Lambda' = K \Lambda P$, where K is an n-square matrix which exchanges rows of Λ and $P \in GL(m, \boldsymbol{R})$.

THEOREM 2. *Let $G = \boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n$ $(0 < m \leq n)$. Suppose that the homomorphism ψ is diagonalizable, locally injective and the structure matrix Λ_ψ of G satisfies*

$$\Lambda_i \neq \mathbf{0}\ (1 \leq i \leq n) \quad \text{and} \quad \beta \notin \{\pm\Lambda_i, \Lambda_i - \Lambda_j \mid 1 \leq i, j \leq n\}.$$

Then, G has a codimension one homogeneous action if and only if the $(n + 1) \times m$-matrix $(\Lambda_\psi^t, -\beta^t)^t$ is equivalent to a matrix $\hat{\Lambda}$ satisfying the following conditions.

(1) *There exist $u_k \times m$-matrices $\Lambda(k)$ $(1 \leq k \leq d)$ such that $\hat{\Lambda}^t = (\Lambda(1)^t, \Lambda(2)^t, \dots, \Lambda(d)^t)^t$.*

(2) *For each k $(1 \leq k \leq d)$, let $\lambda_i^j(k)$ be the (i, j)-element of $\Lambda(k)$. Then each number* $\exp(\pm\lambda_i^j(k))$ *is an algebraic integer, and there exists an algebraic integer α_k of degree u_k such that* $\exp\lambda_i^j(k) = \sigma_k^{(i)}(\exp\lambda_1^j(k)) \in \boldsymbol{Q}(\sigma_k^{(i)}(\alpha_k))$ $(1 \leq j \leq m, 1 \leq i \leq u_k)$. *Here* $\{\sigma_k^{(i)} \mid 1 \leq i \leq u_k, \sigma_k^{(1)} = \mathrm{id}\}$ *is the set of all conjugation mappings of* $\boldsymbol{Q}(\alpha_k)$.

The assumptions on the structure matrices in Theorems 1 and 2 depend only on their equivalence classes, thus, only on the isomorphism classes of the Lie groups G (Lemma 1.2 and Proposition 1.3). If $m < n$, then the set of isomorphism classes of $\{\boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n \mid \Lambda_\psi$ satisfies the assumptions of Theorems 1 and 2$\}$ has the cardinality of a continuum. Among them, only countably many Lie groups have codimension one homogeneous actions from Theorem 2, and hence, have codimension one locally free volume preserving actions on closed manifolds from Theorem 1 (Corollary 4.4). If $m = n$, then the group $\boldsymbol{R}_+^n \ltimes_\psi \boldsymbol{R}^n$ is isomorphic to $\mathrm{Aff}^+(\boldsymbol{R})^l \times \widetilde{\mathrm{Aff}}(\boldsymbol{C})^r$ for some non-negative integers l and r such that $l+2r = n$ (Proposition 1.4), where $\widetilde{\mathrm{Aff}}(\boldsymbol{C})$ denotes the universal covering group of the complex affine group. As a corollary of Theorem 2, it is shown that such a Lie group has a codimension one homogeneous action (Corollary 2.5).

This paper is organized as follows. In Section 1, we investigate fundamental properties of Lie groups of the form $\boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$. In Section 2, we study cocompact discrete subgroups of Lie groups of the form $\boldsymbol{R}_+^m \ltimes_\varphi \boldsymbol{R}^{n+1}$, and then prove Theorem 2 and Corollary 2.5. Section 3 and Section 4 are devoted to proving Theorem 1. The proof of Theorem 1 is given by improving the methods developed in [5], [2] and [6].

Throughout this paper, by manifolds we mean connected closed orientable smooth manifolds, and by actions we mean smooth actions unless otherwise specified. We use the following notation:

1. For $\mathbf{v} \in \boldsymbol{C}^n$, $\Re\mathbf{v}$ (resp. $\Im\mathbf{v}$) denotes the real (resp. imaginary) part of $\mathbf{v}$.
2. $\boldsymbol{R}_+$ denotes the multiplicative group of positive real numbers.
3. For $\mathbf{x}, \mathbf{y} \in \boldsymbol{C}^n$, $\mathbf{x} \cdot \mathbf{y}$ denotes the standard inner product $\mathbf{x}^t\bar{\mathbf{y}} = \sum_{i=1}^n x_i\bar{y}_i$.
4. For an n-row vector $\mathbf{u}$ and an n-column vector $\mathbf{v}$, the product $\mathbf{u}\bar{\mathbf{v}}$ as matrices is often written by the same notation $\mathbf{u} \cdot \mathbf{v}$.
5. E_n denotes the n-square identity matrix and J denotes the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
6. For n_i-square matrices A_i $(1 \leq i \leq k)$, we denote by $\mathrm{diag}(A_1, A_2, \ldots, A_k)$ the $(\sum_{i=1}^k n_i)$-square block-diagonal matrix.
7. $M(n, m, \boldsymbol{K})$ (resp. $M(n, \boldsymbol{K})$) denote the set of all $\boldsymbol{K}$-matrices of type $n \times m$ (resp. $n \times n$).

1. On the group $\boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$. In this section we study basic properties of Lie groups of the form $\boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$.

1.1 Structure matrix of the group $\boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$. Let $\mathbf{t} = (t_1, t_2, \cdots, t_m)^t \in \boldsymbol{R}^m$, $\mathbf{x} = (x_1, x_2, \cdots, x_n)^t \in \boldsymbol{R}^n$, and let $\exp\mathbf{t}$ be the vector $(e^{t_1}, e^{t_2}, \cdots, e^{t_m})^t \in \boldsymbol{R}_+^m$. We denote by $\boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$ the semidirect product group of $\boldsymbol{R}_+^m$ and $\boldsymbol{R}^n$ determined by a homomorphism $\psi : \boldsymbol{R}_+^m \to \mathrm{Aut}(\boldsymbol{R}^n) \cong GL(n, \boldsymbol{R})$. By definition, $\boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$ is the direct product $\boldsymbol{R}_+^m \times \boldsymbol{R}^n$

as a set, and the multiplication law is given as follows ([10, p. 18]):

$$(\exp \mathbf{t}, \mathbf{x})(\exp \mathbf{t}', \mathbf{x}') = (\exp(\mathbf{t} + \mathbf{t}'), \mathbf{x} + \psi(\exp \mathbf{t})(\mathbf{x}')), \quad \mathbf{t}, \mathbf{t}' \in \boldsymbol{R}^m, \mathbf{x}, \mathbf{x}' \in \boldsymbol{R}^n .$$

In this paper, we always assume that the homomorphism ψ is diagonalizable. That is, we assume that the matrix $d\psi(\mathbf{t})$ is diagonalizable over $\boldsymbol{C}$ for any $\mathbf{t} \in \boldsymbol{R}^m$, where $d\psi : \boldsymbol{R}^m \to M(n, \boldsymbol{R})$ denotes the differential of ψ.

Take a basis $\{\mathbf{e}_j \mid 1 \le j \le m\}$ of $\boldsymbol{R}^m$ and put $d\psi(\mathbf{e}_j) =: A_j$. Choose a complex n-square matrix U which simultaneously diagonalizes $\{A_j \mid 1 \le j \le m\}$, and let λ_i^j be the i-th diagonal element of $U^{-1} A_j U$. Let $\Lambda_\psi \in M(n, m, \boldsymbol{C})$ be the matrix whose (i, j)-element is λ_i^j. We call the matrix Λ_ψ the *structure matrix* (with respect to $\{\mathbf{e}_j\}$ and U) of the semidirect product group $G = \boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$. Denote by $\Lambda_i \in \boldsymbol{C}^m$ the i-th row vector of Λ_ψ. Note that, if $\Lambda_i \in \boldsymbol{C}^m \setminus \boldsymbol{R}^m$, then there exists a permutation $\sigma \in \mathcal{S}_n$ such that $\overline{\Lambda_i} = \Lambda_{\sigma(i)}$ $(1 \le i \le n)$.

Two matrices $\Lambda, \Lambda' \in M(n, m, \boldsymbol{C})$ are said to be *equivalent* if $\Lambda' = K \Lambda P$, where K is an n-square matrix which exchanges rows of Λ and $P \in GL(m, \boldsymbol{R})$. It is easy to see that the equivalence class of the structure matrix Λ_ψ does not depend on the choice of $\{\mathbf{e}_j\}$ or U.

Denote by N_G the maximal connected nilpotent normal subgroup of G ([13, p. 2]).

LEMMA 1.1. (1) *If* $\boldsymbol{R}$-rank$(\Lambda_\psi) = m - s$ $(s > 0)$, *then the group* $G = \boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$ *has another semidirect product structure* $G = \boldsymbol{R}_+^{m-s} \ltimes_{\psi'} \boldsymbol{R}^{n+s}$, *where* $\boldsymbol{R}$-rank$(\Lambda_{\psi'}) = m - s$.

(2) $\boldsymbol{R}$-rank$(\Lambda_\psi) = m$ *if and only if* $N_G = \{\mathbf{1}\} \ltimes_\psi \boldsymbol{R}^n$.

PROOF. Suppose $\boldsymbol{R}$-rank$(\Lambda_\psi) = m - s$. Choose a basis $\{\mathbf{e}'_j \mid 1 \le j \le m\}$ of $\boldsymbol{R}^m$ such that the subset $\{\mathbf{e}'_j \mid m-s+1 \le j \le m\}$ spans the kernel of $d\psi : \boldsymbol{R}^m \to M(n, \boldsymbol{R})$. Define a homomorphism $\psi' : \boldsymbol{R}_+^{m-s} \to GL(n+s, \boldsymbol{R})$ by $\psi'(\exp(\mathbf{e}'_j)) := \mathrm{diag}(\psi(\exp(\mathbf{e}'_j)), E_s)$ $(1 \le j \le m-s)$, and consider the semidirect product $\boldsymbol{R}_+^{m-s} \ltimes_{\psi'} \boldsymbol{R}^{n+s}$. Then it is easy to see that the map $(\exp(\sum_{j=1}^m t_j \mathbf{e}'_j), (x_1, \dots, x_n)) \mapsto (\exp(\sum_{j=1}^{m-s} t_j \mathbf{e}'_j), (x_1, \dots, x_n, t_{m-s+1}, \dots, t_m))$ determines an isomorphism from $G = \boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$ to $\boldsymbol{R}_+^{m-s} \ltimes_{\psi'} \boldsymbol{R}^{n+s}$. Obviously the homomorphism ψ' is diagonalizable and $\boldsymbol{R}$-rank$(\Lambda_{\psi'}) = m - s$. Thus we have proved (1) and the sufficiency part of (2) because $N_G \supset \{\mathbf{1}\} \ltimes_{\psi'} \boldsymbol{R}^{n+s}$.

We prove the necessity in (2). Suppose $N_G \supsetneqq \{\mathbf{1}\} \ltimes_\psi \boldsymbol{R}^n$. Choose $(\exp \mathbf{s}, \mathbf{x}) \in N_G \setminus \{\mathbf{1}\} \ltimes_\psi \boldsymbol{R}^n$. Then the d-th iterated commutator of $(\exp a\mathbf{s}, \mathbf{x})$ $(a \in \boldsymbol{R})$ and $(\mathbf{1}, \mathbf{x}')$ $(\mathbf{x}' \in \boldsymbol{R}^n)$ is given by $[(\exp a\mathbf{s}, \mathbf{x}), \cdots, [(\exp a\mathbf{s}, \mathbf{x}), [(\exp a\mathbf{s}, \mathbf{x}), (\mathbf{1}, \mathbf{x}')]] \cdots] = (\mathbf{1}, (\psi(\exp a\mathbf{s}) - \mathrm{id})^d(\mathbf{x}'))$. Since N_G is nilpotent, there is $d > 0$ such that $(\psi(\exp a\mathbf{s}) - \mathrm{id})^d = 0$ for any $a \in \boldsymbol{R}$. This implies $\Lambda_\psi \mathbf{s} = \mathbf{0}$ and $\boldsymbol{R}$-rank$(\Lambda_\psi) < m$. □

Note that $\boldsymbol{R}$-rank$(\Lambda_\psi) = m$ if and only if ψ is *locally injective*, that is, $d\psi$ is injective. By Lemma 1.1(1), in considering a semidirect product $\boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$, we may assume that the homomorphism ψ is locally injective (and in particular, $m \le n$). Put $D(n, m) := \{\boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n \mid \psi$ is diagonalizable and locally injective$\}$, and let $\mathcal{D}(n, m)$ denote the set of isomorphism classes of $D(n, m)$. From Lemma 1.1, we obtain the following.

LEMMA 1.2. *Let G be a Lie group. Then $G \in \mathcal{D}(n,m)$ if and only if the following conditions are satisfied*: (1) $N_G \cong \boldsymbol{R}^n$ *and* $G/N_G \cong \boldsymbol{R}^m$. (2) *The natural exact sequence* $1 \xrightarrow{\iota} N_G \to G \to G/N_G \to 1$ *has a splitting* $\xi : G/N_G \to G$. (3) *The homomorphism* $\psi : G/N_G \to \mathrm{Aut}(N_G)$ *determined by* $\iota(\psi(h))(g) = \xi(h)\iota(g)\xi(h)^{-1}$ $(g \in N_G, h \in G/N_G)$ *is diagonalizable.*

Let $S(n,m) := \{\Lambda \in M(n,m,\boldsymbol{C}) \,|\, \boldsymbol{R}\text{-rank}(\Lambda) = m$ and $\bar{\Lambda} = K\Lambda$ for some row exchanging matrix $K\}$. The structure matrix of a semidirect product group $G = \boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n \in D(n,m)$ belongs to $S(n,m)$. Let $\mathcal{S}(n,m)$ denote the set of equivalence classes of matrices in $S(n,m)$.

PROPOSITION 1.3. *The map $\boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n \mapsto \Lambda_\psi$ induces a bijection from $\mathcal{D}(n,m)$ to $\mathcal{S}(n,m)$.*

PROOF. We show the well-definedness of the induced map. Suppose $G = G/N_G \ltimes_\psi N_G$ and $G' = G'/N_{G'} \ltimes_{\psi'} N_{G'}$ $(G, G' \in \mathcal{D}(n,m))$ are isomorphic by $\phi : G \to G'$. Then the isomorphism ϕ naturally induces two isomorphisms $\phi_0 : N_G \to N_{G'}$ and $\phi_1 : G/N_G \to G'/N_{G'}$, which satisfy the following condition:

$$\phi_0^{-1}(\psi'(\phi_1(\exp\mathbf{t}))(\phi_0(\mathbf{x}))) = \psi(\exp\mathbf{t})(\mathbf{x}), \quad \exp\mathbf{t} \in G/N_G, \ \mathbf{x} \in N_G.$$

It follows that the groups G and G' have equivalent structure matrices. The rest of the proof is easy and is omitted. □

Lemma 1.2 and Proposition 1.3 imply that $\mathcal{D}(n,m) \cap \mathcal{D}(n',m') = \emptyset$ if $(n,m) \neq (n',m')$, and the equivalence class of the structure matrix of $G \in \cup_{n,m}\mathcal{D}(n,m)$ is determined by its isomorphism class. It is easy to see that the assumptions on structure matrices in Theorems 1 and 2, and in the succeeding Propositions as well, depend only on their equivalence classes. By these reasons, as the structure matrix of a given Lie group $G \in \mathcal{D}(n,m)$ we may take any representative in its equivalence class.

1.2. Canonical coordinates. Let l and r be non-negative integers such that $l + 2r = n$. We say that $\Lambda \in S(n,m)$ is *of type* (l,r) if Λ has l real row vectors and $2r$ non-real row vectors. In that case, we say that Λ is *well-arranged* if the last $2r$ row vectors are non-real and $\overline{\Lambda_{l+2j-1}} = \Lambda_{l+2j}$ $(1 \leq j \leq r)$.

Let $G = \boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n \in D(n,m)$. We also say that G is of type (l,r) if its structure matrix $\Lambda_\psi = (\lambda_i^j)$ is of type (l,r). For such a G, up to equivalence, we may assume that Λ_ψ is well-arranged, and can take a coordinate $(\exp\mathbf{t}, \mathbf{x})$ of G so that the differential $d\psi(\mathbf{t})$ is given by the following real canonical form.

$$\begin{aligned} d\psi(\mathbf{t}) = \mathrm{diag}\Bigg(&\sum_{j=1}^m \lambda_1^j t_j, \dots, \sum_{j=1}^m \lambda_l^j t_j, \Big(\sum_{j=1}^m (\Re\lambda_{l+1}^j) t_j\Big) E_2 + \Big(\sum_{j=1}^m (\Im\lambda_{l+1}^j) t_j\Big) J, \dots, \\ &\Big(\sum_{j=1}^m (\Re\lambda_{l+2r-1}^j) t_j\Big) E_2 + \Big(\sum_{j=1}^m (\Im\lambda_{l+2r-1}^j) t_j\Big) J\Bigg). \end{aligned} \tag{1.1}$$

Such a coordinate of G will be called a *canonical coordinate*.

From now on, we assume that the structure matrix of $G \in \mathcal{D}(n,m)$ is well-arranged and G has a canonical coordinate, unless otherwise specified.

1.3. The case of $m = n$. Let $G_n(l,r)$ be the Lie group $\boldsymbol{R}^n_+ \ltimes_{\psi_n(l,r)} \boldsymbol{R}^n$ in $D(n,n)$, where the homomorphism $\psi_n(l,r)$ is defined by

$$d\psi_n(l,r)(\mathbf{t}) = \mathrm{diag}(t_1, \dots, t_l, (t_{l+1}E_2 + t_{l+r+1}J), \dots, (t_{l+r}E_2 + t_{l+2r}J)).$$

It is easy to see that the group $G_n(l,r)$ is isomorphic to $\mathrm{Aff}^+(\boldsymbol{R})^l \times \widetilde{\mathrm{Aff}}(\boldsymbol{C})^r$.

PROPOSITION 1.4. *Let $G = \boldsymbol{R}^n_+ \ltimes_\psi \boldsymbol{R}^n \in D(n,n)$ be of type (l,r). Then G is isomorphic to* $\mathrm{Aff}^+(\boldsymbol{R})^l \times \widetilde{\mathrm{Aff}}(\boldsymbol{C})^r$.

PROOF. Let

$$P := (\Lambda_1^t, \Lambda_2^t, \dots, \Lambda_l^t, \Re\Lambda_{l+1}^t, \Re\Lambda_{l+3}^t, \dots, \Re\Lambda_{l+2r-1}^t, \Im\Lambda_{l+1}^t, \Im\Lambda_{l+3}^t, \dots, \Im\Lambda_{l+2r-1}^t)^t.$$

Then it is easy to see that $\Lambda_\psi P^{-1} = \Lambda_{\psi_n(l,r)}$. From Proposition 1.3 it follows that G is isomorphic to $G_n(l,r)$, and hence to $\mathrm{Aff}^+(\boldsymbol{R})^l \times \widetilde{\mathrm{Aff}}(\boldsymbol{C})^r$. □

1.4. Lie algebra of $\boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n$. Let $G = \boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n \in D(n,m)$ be of type (l,r). Then the Lie algebra $\mathcal{G}$ of right invariant vector fields on G is generated by the following elements:

$$\begin{aligned}
X_i &= \frac{\partial}{\partial x_i} \quad (1 \le i \le n), \\
Y_j &= -\frac{\partial}{\partial t_j} - \sum_{k=1}^{l} \lambda_k^j x_k \left(\frac{\partial}{\partial x_k}\right) \\
&\quad - \sum_{k=1}^{r} (\Re\lambda_{l+2k-1}^j)\left(x_{l+2k-1}\left(\frac{\partial}{\partial x_{l+2k-1}}\right) + x_{l+2k}\left(\frac{\partial}{\partial x_{l+2k}}\right)\right) \\
&\quad + \sum_{k=1}^{r} (\Im\lambda_{l+2k-1}^j)\left(x_{l+2k}\left(\frac{\partial}{\partial x_{l+2k-1}}\right) - x_{l+2k-1}\left(\frac{\partial}{\partial x_{l+2k}}\right)\right) \quad (1 \le j \le m).
\end{aligned} \tag{1.2}$$

They satisfy the following commutation relations:

$$\begin{aligned}
&[X_i, X_{i'}] = [Y_j, Y_{j'}] = 0 \quad (1 \le i, i' \le n, 1 \le j, j' \le m), \\
&[Y_j, X_i] = \lambda_i^j X_i \quad (1 \le i \le l, 1 \le j \le m), \\
&[Y_j, X_{l+2k-1}] = (\Re\lambda_{l+2k-1}^j)X_{l+2k-1} + (\Im\lambda_{l+2k-1}^j)X_{l+2k}, \\
&[Y_j, X_{l+2k}] = -(\Im\lambda_{l+2k-1}^j)X_{l+2k-1} + (\Re\lambda_{l+2k-1}^j)X_{l+2k} \quad (1 \le j \le m, 1 \le k \le r).
\end{aligned} \tag{1.3}$$

For an element $g = (\exp \mathbf{t}, \mathbf{x})$ of G, the left translation L_g and the inner automorphism $\mathrm{Ad}(g) = L_g R_{g^{-1}}$ act on these vector fields according to the following formulas:

$$\begin{aligned}
&(L_g)_* X_i = e^{\Lambda_i \cdot \mathbf{t}} X_i \quad (1 \le i \le l), \\
&(L_g)_* \begin{pmatrix} X_{l+2j-1} \\ X_{l+2j} \end{pmatrix} = e^{(\Re\Lambda_{l+2j-1})\cdot\mathbf{t}} \exp((-(\Im\Lambda_{l+2j-1}) \cdot \mathbf{t})J) \begin{pmatrix} X_{l+2j-1} \\ X_{l+2j} \end{pmatrix} \\
&\qquad (1 \le j \le r),
\end{aligned} \tag{1.4}$$

$$(L_g)_*Y_j = Y_j + \sum_{k=1}^{l} \lambda_k^j x_k X_k + \sum_{k=1}^{r} (\Re\lambda_{l+2k-1}^j)(x_{l+2k-1}X_{l+2k-1} + x_{l+2k}X_{l+2k})$$
$$- \sum_{k=1}^{r} (\Im\lambda_{l+2k-1}^j)(x_{l+2k}X_{l+2k-1} - x_{l+2k-1}X_{l+2k}) \quad (1 \le j \le m).$$

1.5. Unimodular Lie group containing $\boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$. Let G be a Lie group. The *modular function* $\Delta : G \to \boldsymbol{R}_+$ is defined by $\Delta(g) = |\det(\mathrm{Ad}\, g)|$, which measures the deficiency of left invariance of the right invariant volume form of G. The Lie group G is said to be *unimodular* if $\Delta(G) = 1$. In particular, if G is connected, G is unimodular if and only if it has a biinvariant volume form. It is easy to see that a Lie group is unimodular if it contains a cocompact discrete subgroup.

Let $G = \boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n \in D(n, m)$. Denote by β the real row m-vector $\sum_{i=1}^n \Lambda_i$. From the formula (1.4), the modular function $\Delta : G \to \boldsymbol{R}_+$ is given by $\Delta(\exp \mathbf{t}, \mathbf{x}) = \exp(\sum_{i=1}^n \Lambda_i \cdot \mathbf{t}) = \exp(\beta \cdot \mathbf{t})$.

PROPOSITION 1.5. *Let $G = \boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n \in D(n, m)$. Suppose that the structure matrix Λ_ψ of G satisfies*

$$\Lambda_i \neq \mathbf{0} \ (1 \le i \le n) \quad \textit{and} \quad \beta \notin \{\pm\Lambda_i, \Lambda_i - \Lambda_j \mid 1 \le i, j \le n\}.$$

Then there exists uniquely an $(m+n+1)$-dimensional simply connected unimodular Lie group H which contains G as a subgroup.

PROOF. Consider the Lie group $\tilde{G} := \boldsymbol{R}_+^m \ltimes_{\tilde{\psi}} \boldsymbol{R}^{n+1} \in D(n+1, m)$, whose structure matrix $\Lambda_{\tilde{\psi}}$ is given by $(\Lambda_\psi^t, -\beta^t)^t$. Note that the group $\tilde{G}$ is unimodular and there is a natural embedding of G into $\tilde{G}$.

We prove the uniqueness. Let H be an $(m+n+1)$-dimensional simply connected unimodular Lie group which contains G. Suppose G is of type (l, r), and let $\{X_i \ (1 \le i \le n),\ Y_j \ (1 \le j \le m)\}$ be the basis of the Lie algebra $\mathcal{G}$ of G given in (1.2). From the assumption on Λ_ψ, we can take a vector $\mathbf{t} = (t_1, t_2, \dots, t_m)^t \in \boldsymbol{R}^m$ which satisfies

$$\begin{aligned}
&\beta \cdot \mathbf{t} \neq 0, \quad (\beta \pm \Lambda_i) \cdot \mathbf{t} \neq 0 \ (1 \le i \le l), \quad \Lambda_i \cdot \mathbf{t} \neq 0 \ (1 \le i \le l), \\
&(\Lambda_i - \beta) \cdot \mathbf{t} \neq \Lambda_j \cdot \mathbf{t} \ (1 \le i, j \le n), \quad (\Im\Lambda_{l+2k-1}) \cdot \mathbf{t} \neq 0 \ (1 \le k \le r).
\end{aligned} \tag{1.5}$$

For such a $\mathbf{t}$, put $Y := \sum_{j=1}^m t_j Y_j$. Then we have

$$\begin{aligned}
&[Y, Y_j] = 0 \ (1 \le j \le m), \quad [Y, X_i] = (\Lambda_i \cdot \mathbf{t}) X_i \ (1 \le i \le l), \\
&[Y, X_{l+2k-1} - \sqrt{-1}X_{l+2k}] = (\Lambda_{l+2k-1} \cdot \mathbf{t})(X_{l+2k-1} - \sqrt{-1}X_{l+2k}), \\
&[Y, X_{l+2k-1} + \sqrt{-1}X_{l+2k}] = (\Lambda_{l+2k} \cdot \mathbf{t})(X_{l+2k-1} + \sqrt{-1}X_{l+2k}) \ (1 \le k \le r).
\end{aligned} \tag{1.6}$$

Let $\mathcal{H}$ be the Lie algebra of H, and take $Z \in \mathcal{H} \setminus \mathcal{G}$. Since H is unimodular, we have $\mathrm{tr}(\mathrm{ad}\, Y) = 0$. Hence, from (1.6), the bracket $[Y, Z]$ is given by $-(\beta \cdot \mathbf{t})Z + \sum_{i=1}^n a_i X_i + \sum_{j=1}^m b_j Y_j$ for some $a_i, b_j \in \boldsymbol{R}$. Put $T := Z + \sum_{i=1}^n c_i X_i + \sum_{j=1}^m d_j Y_j$, where $d_j =$

$-b_j/(\beta\cdot\mathbf{t})$ $(1\le j\le m)$, $c_i=-a_i/((\beta+\Lambda_i)\cdot\mathbf{t})$ $(1\le i\le l)$, and

$$\begin{pmatrix} c_{l+2k-1} \\ c_{l+2k} \end{pmatrix} = \begin{pmatrix} (\Re\Lambda_{l+2k-1}+\beta)\cdot\mathbf{t} & -(\Im\Lambda_{l+2k-1})\cdot\mathbf{t} \\ (\Im\Lambda_{l+2k-1})\cdot\mathbf{t} & (\Re\Lambda_{l+2k-1}+\beta)\cdot\mathbf{t} \end{pmatrix}^{-1} \begin{pmatrix} -a_{l+2k-1} \\ -a_{l+2k} \end{pmatrix} \quad (1\le k\le r).$$

Then, an easy calculation shows that $[Y,T]=-(\beta\cdot\mathbf{t})T$.

Next we show $[X_i,T]=0$ $(1\le i\le n)$. From the Jacobi identity for the triple (Y,X_i,T), we have $[Y,[X_i,T]]=((\Lambda_i-\beta)\cdot\mathbf{t})[X_i,T]$. This shows that $(\Lambda_i-\beta)\cdot\mathbf{t}$ would be an eigenvalue of $\operatorname{ad}Y$ if $[X_i,T]\ne 0$. However, from (1.5) and (1.6), we see that it is not the case, and hence $[X_i,T]=0$. Replacing X_i by $X_{l+2k-1}\pm\sqrt{-1}X_{l+2k}$, we also obtain $[X_{l+2k-1},T]=[X_{l+2k},T]=0$.

Similarly, the Jacobi identity for the triple (Y,Y_j,T) gives the identity $[Y,[Y_j,T]]=-(\beta\cdot\mathbf{t})[Y_j,T]$. This shows that $[Y_j,T]=aT$ for some $a\in\boldsymbol{R}$. From (1.3) and $\operatorname{tr}(\operatorname{ad}Y_j)=0$, we conclude that $[Y_j,T]=-(\sum_{i=1}^n\lambda_i^j)T$.

The Lie algebra $\mathcal{H}$ is spanned by the vector fields $\{T,X_i\ (1\le i\le n),\ Y_j\ (1\le j\le m)\}$ whose bracket products are now completely determined. Since $\mathcal{H}$ is isomorphic to the Lie algebra of $\tilde{G}$, the simply connected Lie group H is isomorphic to $\tilde{G}$. □

2. Homogeneous actions. The object of this section is to prove Theorem 2.

2.1. Eigenvalues of commuting integer matrices. In this subsection, we investigate a relation between the eigenvalues of commuting integer matrices.

Let $\boldsymbol{Q}(\alpha)$ be an algebraic number field of degree u. Denote by $\mathcal{O}(\alpha)\subset\boldsymbol{Q}(\alpha)$ the subring of all algebraic integers in $\boldsymbol{Q}(\alpha)$. As a $\boldsymbol{Z}$-module, $\mathcal{O}(\alpha)$ has a $\boldsymbol{Z}$-basis consisting of u algebraic integers $w_1,w_2,\dots,w_u$. Such a basis is called an *integral basis*. Put $B=\{1,\alpha,\dots,\alpha^{u-1}\}$ and $B'=\{\omega_1,\dots,\omega_u\}$. Then both B and B' are $\boldsymbol{Q}$-bases of the $\boldsymbol{Q}$-vector space $\boldsymbol{Q}(\alpha)$. Let $\{\sigma^{(i)}\mid 1\le i\le u\}$ be the set of all conjugation mappings of $\boldsymbol{Q}(\alpha)$, where $\sigma^{(1)}$ is the identity map (see, e.g., [4]).

For an element $\gamma\in\boldsymbol{Q}(\alpha)$, define a linear transformation T_γ of the $\boldsymbol{Q}$-vector space $\boldsymbol{Q}(\alpha)$ by $T_\gamma(x)=\gamma x$ for all $x\in\boldsymbol{Q}(\alpha)$. Denote by $[T_\gamma]_{B''}$ the matrix of T_γ with respect to a basis B''. When $\gamma=\alpha$, the matrix $[T_\alpha]_{B''}$ has distinct eigenvalues $\sigma^{(1)}(\alpha)=\alpha,\sigma^{(2)}(\alpha),\dots,\sigma^{(u)}(\alpha)$, and is diagonalizable. Since each $\gamma\in\boldsymbol{Q}(\alpha)$ is expressed as a $\boldsymbol{Q}$-polynomial $\sum_{s=0}^{u-1}a_s\alpha^s$ $(a_s\in\boldsymbol{Q})$, the matrix $[T_\gamma]_{B''}=\sum_{s=0}^{u-1}a_s([T_\alpha]_{B''})^s$ is diagonalizable and has eigenvalues $\sigma^{(1)}(\gamma),\sigma^{(2)}(\gamma),\dots,\sigma^{(u)}(\gamma)$. If γ is a unit in $\mathcal{O}(\alpha)$, then the matrix $[T_\gamma]_{B'}$ lies in $GL(u,\boldsymbol{Z})$.

Let $h(x)=\sum_{s=0}^{u}q_sx^s$ be an irreducible monic $\boldsymbol{Q}$-polynomial of degree u. Consider the companion matrix $U(h)$ of $h(x)$:

$$U(h):=\left(\begin{array}{c|c} 0 & -q_0 \\ \hline & -q_1 \\ E_{u-1} & \vdots \\ & -q_{u-1} \end{array}\right).$$

If α is a root of $h(x)$, then $U(h)$ coincides with the matrix $[T_\alpha]_B$.

The following sublemma is well-known (see, e.g., [3, Proposition 6 in §5, Chapter VII]).

SUBLEMMA 2.1. *Let $C \in GL(n+1, \boldsymbol{Q})$ and $\chi_C(x)$ the eigenpolynomial of C. Let $\chi_C(x) = h_1(x)h_2(x)\cdots h_d(x)$ be the decomposition of $\chi_C(x)$ into irreducible monic $\boldsymbol{Q}$-polynomials. If C is diagonalizable, then there exists a non-singular rational matrix P such that*

$$P^{-1}CP = \mathrm{diag}(U(h_1), U(h_2), \dots, U(h_d)).$$

Now we prove the main lemma of this subsection.

LEMMA 2.2. *Let $\{A_j \mid 1 \le j \le m\}$ be commuting, diagonalizable real $(n+1)$-square matrices such that* $\exp A_j \in SL(n+1, \mathbf{Z})$ *for any j. Let* $\mathrm{diag}(\lambda_1^j, \lambda_2^j, \dots, \lambda_{n+1}^j)$ *be a simultaneousely diagonalized form of A_j $(1 \le j \le m)$, and let Λ be the matrix whose (i, j)-element is λ_i^j. Then there are an integer vector $\mathbf{t} = (t_1, t_2, \dots, t_m)^t \in \mathbf{Z}^m$ and a positive integer p for which the following hold.*

Put $A := p\sum_{j=1}^m t_j A_j$, and let $h_1(x)h_2(x)\dots h_d(x)$ be the $\mathbf{Z}$-irreducible decomposition of the eigenpolynomial of $\exp A$.

(1) *There exists a non-singular rational matrix P such that*

$$P^{-1}(\exp A)P = \mathrm{diag}(U(h_1), U(h_2), \dots, U(h_d)),$$
$$P^{-1}(\exp pA_j)P = \mathrm{diag}(B_j(1), B_j(2), \dots, B_j(d)) \quad (1 \le j \le m),$$

where $B_j(k) \in GL(u_k, \boldsymbol{Q})$ and $u_k = \deg h_k(x)$ $(1 \le k \le d)$.

(2) *For each j $(1 \le j \le m)$ and k $(1 \le k \le d)$, there exist $b_{jks} \in \boldsymbol{Q}$ $(0 \le s \le u_k - 1)$ such that $B_j(k) = \sum_{s=0}^{u_k-1} b_{jks} U(h_k)^s$. Hence, denoting by α_k a root of $h_k(x)$ and by $\sigma_k^{(1)} =$* id, *$\sigma_k^{(2)}, \dots, \sigma_k^{(u_k)}$ the conjugation mappings of $\boldsymbol{Q}(\alpha_k)$, the matrices $U(h_k)$ and $B_j(k)$ are simultaneously diagonalized to*

$$\mathrm{diag}(\sigma_k^{(1)}(\alpha_k), \sigma_k^{(2)}(\alpha_k), \dots, \sigma_k^{(u_k)}(\alpha_k)) \quad \textit{and} \quad \mathrm{diag}(\beta_1^j(k), \beta_2^j(k), \dots, \beta_{u_k}^j(k)),$$

respectively, where $\beta_1^j(k) = \sum_{s=0}^{u_k-1} b_{jks}(\alpha_k)^s$ and $\beta_i^j(k) = \sigma_k^{(i)}(\beta_1^j(k))$. Moreover each $\beta_i^j(k)$ is a unit in $\mathcal{O}(\alpha)$.

(3) *Put $l(k) := \sum_{t=1}^{k-1} u_t$ $(1 \le k \le d)$. Then there exists a permutation $\tau \in \mathcal{S}_{n+1}$ such that $\beta_i^j(k) = \exp(p\lambda_{\tau(i+l(k))}^j)$ for any i, j and k.*

PROOF. Denote by Λ_i the i-th row vector of Λ. For each i, i', consider the subgroup $K_{ii'} = \{\mathbf{t} \in \mathbf{Z}^m \mid (\Lambda_i - \Lambda_{i'}) \cdot \mathbf{t} \in 2\pi\sqrt{-1}\boldsymbol{Q}\}$ of $\mathbf{Z}^m$. Take an integer vector $\mathbf{t} = (t_1, t_2, \dots, t_m)^t \in \left(\bigcup_{\mathrm{rank}\, K_{ii'} < m} K_{ii'}\right)^c$ and a positive integer p such that $p(\Lambda_i - \Lambda_{i'}) \in 2\pi\sqrt{-1}\mathbf{Z}^m$ whenever $\Lambda_i - \Lambda_{i'} \in 2\pi\sqrt{-1}\boldsymbol{Q}^m$. Put $A := p\sum_{j=1}^m t_j A_j$. Then the set of eigenvalues of A (resp. pA_j) is given by $\{p\Lambda_i \cdot \mathbf{t} \mid 1 \le i \le n+1\}$ (resp. $\{p\lambda_i^j \mid 1 \le i \le n+1\}$). It is easy to see that, for each i, i' $(1 \le i, i' \le n+1)$, the following three conditions are equivalent:

$$\text{(A) } \mathrm{rank}\, K_{ii'} = m, \quad \text{(B) } \exp(p\Lambda_i \cdot \mathbf{t}) = \exp(p\Lambda_{i'} \cdot \mathbf{t}),$$
$$\text{(C) } \exp(p\lambda_i^j) = \exp(p\lambda_{i'}^j) \ (1 \le j \le m).$$

From the implication (B) $\Rightarrow$ (C), it follows that each eigenvector of $\exp A$ is also an eigenvector of $\exp pA_j$ for each j.

On the other hand, from Sublemma 2.1, there is a non-singular rational matrix P such that $P^{-1}(\exp A)P = \mathrm{diag}(U(h_1), U(h_2), \dots, U(h_d))$. For each k ($1 \le k \le d$), take a matrix $V(k)$ diagonalizing $U(h_k)$, and put $V := \mathrm{diag}(V(1), V(2), \dots, V(d))$. Then the matrix PV diagonalizes $\exp A$, and hence $\exp pA_j$ for each j. This shows that there exist $B_j(k) \in GL(u_k, \boldsymbol{Q})$ such that $P^{-1}(\exp pA_j)P = \mathrm{diag}(B_j(1), B_j(2), \dots, B_j(d))$. Hence we have proved (1).

Let α_k be a root of $h_k(x)$ and take the basis $B = \{1, \alpha_k, (\alpha_k)^2, \dots, (\alpha_k)^{u_k-1}\}$ of the $\boldsymbol{Q}$-vector space $\boldsymbol{Q}(\alpha_k)$. Then we can define a linear map $T_j(k)$ of $\boldsymbol{Q}(\alpha_k)$ by $[T_j(k)]_B := B_j(k)$. Since $U(h_k)$ and $B_j(k)$ are commutative and $U(h_k) = [T_{\alpha_k}]_B$, the linear maps T_{α_k} and $T_j(k)$ are also commutative. Therefore we have, for each $x = \sum_{s=0}^{u_k-1} a_s(\alpha_k)^s \in \boldsymbol{Q}(\alpha_k)$,

$$T_j(k)\left(\sum_{s=0}^{u_k-1} a_s(\alpha_k)^s\right) = \sum_{s=0}^{u_k-1} a_s T_j(k)(T_{\alpha_k})^s(1) = \left(\sum_{s=0}^{u_k-1} a_s(\alpha_k)^s\right) T_j(k)(1)\,.$$

This shows that $T_j(k)$ is the linear map given by $T_j(k)(x) = T_j(k)(1)x$. Put $\beta_1^j(k) := T_j(k)(1)$. Then there exist $b_{jks} \in \boldsymbol{Q}$ ($0 \le s \le u_k - 1$) such that $\beta_1^j(k) = \sum_{s=0}^{u_k-1} b_{jks}(\alpha_k)^s$, and hence the matrix $[T_j(k)]_B = B_j(k)$ is of the form $\sum_{s=0}^{u_k-1} b_{jks}[T_{\alpha_k}]_B^s = \sum_{s=0}^{u_k-1} b_{jks}U(h_k)^s$. Since $\exp(\pm pA_j) \in SL(n+1, \boldsymbol{Z})$, each number $\beta_i^j(k)$ is a unit in $\mathcal{O}(\alpha_k)$. This proves (2).

The assertion (3) follows from the definition of the numbers $\{\beta_i^j(k)\}$. $\square$

2.2. Cocompact discrete subgroups of $\boldsymbol{R}_+^m \ltimes \boldsymbol{R}^{n+1}$. In this subsection we give a necessary and sufficient condition for a unimodular Lie group $H = \boldsymbol{R}_+^m \ltimes_\varphi \boldsymbol{R}^{n+1} \in D(n+1, m)$ to have a cocompact discrete subgroup.

LEMMA 2.3. *Let $H = \boldsymbol{R}_+^m \ltimes_\varphi \boldsymbol{R}^{n+1}$ be a group in $D(n+1, m)$. Let Γ be a cocompact discrete subgroup of H. Then the following hold.*

(1) *The intersection $\Gamma_0 := \Gamma \cap \boldsymbol{R}^{n+1}$ is a cocompact discrete subgroup of $\boldsymbol{R}^{n+1} \cong \{\mathbf{1}\} \ltimes_\psi \boldsymbol{R}^{n+1}$.*

(2) *The quotient $\Gamma_1 := \Gamma/\Gamma_0 \subset \boldsymbol{R}_+^m$ is a cocompact discrete subgroup of $\boldsymbol{R}_+^m \cong H/(\{\mathbf{1}\} \ltimes_\psi \boldsymbol{R}^{n+1})$.*

(3) *With respect to any generating sets $\{\exp \mathbf{e}_j \mid 1 \le j \le m\}$ of Γ_1 and $\{\mathbf{f}_i \mid 1 \le i \le n+1\}$ of Γ_0, the homomorphism $\varphi : \boldsymbol{R}_+^m \to \mathrm{Aut}(\boldsymbol{R}^{n+1}) \cong GL(n+1, \boldsymbol{R})$ is expressed as follows: Put $d\varphi(\mathbf{e}_j) =: A_j \in M(n+1, \boldsymbol{R})$ ($1 \le j \le m$). Then the matrices $\{A_j\}$ are commutative, $\exp A_j \in SL(n+1, \boldsymbol{Z})$ and $\varphi(\exp(\sum_{j=1}^m t_j \mathbf{e}_j)) = \exp(\sum_{j=1}^m t_j A_j)$.*

PROOF. Obviously, Γ_0 is discrete in $\boldsymbol{R}^{n+1}$. From Lemma 1.1(2), the normal subgroup $\boldsymbol{R}^{n+1}$ of H coincides with N_H. By a theorem of Mostow ([13, Theorem 3.4]), $N_H \cap \Gamma$ is cocompact in N_H. Hence (1) is proved.

The extension $1 \to \boldsymbol{R}^{n+1} \xrightarrow{\iota} H \xrightarrow{\pi} \boldsymbol{R}_+^m \to 1$ induces continuous maps $\boldsymbol{R}^{n+1}/\Gamma_0 \xrightarrow{\bar{\iota}} H/\Gamma \xrightarrow{\bar{\pi}} \boldsymbol{R}_+^m/\Gamma_1$. The quotient $\boldsymbol{R}_+^m/\Gamma_1$ is the continuous image of the compact space H/Γ

by $\bar{\pi}$ and hence is compact. Suppose Γ_1 is not discrete in $\boldsymbol{R}_+^m$. Then there is a sequence $\{\exp \mathbf{t}_k \mid k = 1, 2, \dots\}$ in Γ_1 such that $\lim_{k\to\infty} \exp \mathbf{t}_k = \exp \mathbf{t}_\infty \in \Gamma_1 \subset \boldsymbol{R}_+^m$ and $\exp \mathbf{t}_k \neq \exp \mathbf{t}_\infty$ for all k. From (1) there is a compact fundamental domain $K \subset \boldsymbol{R}^{n+1}$ for the subgroup $\Gamma_0 \subset \boldsymbol{R}^{n+1}$. So, for each k, we can take a lift $(\exp \mathbf{t}_k, \mathbf{x}_k) \in \Gamma$ of $\exp \mathbf{t}_k \in \Gamma_1$ such that $\mathbf{x}_k \in K$. Because the sequence $\{(\exp \mathbf{t}_k, \mathbf{x}_k)\} \subset \Gamma$ is both discrete and lies in a compact subset of $\boldsymbol{R}_+^m \times \boldsymbol{R}^{n+1}$, it is a finite set. This contradicts the choice of $\{\exp \mathbf{t}_k\}$. We have thus proved (2).

We now prove the third assertion. From (1) and (2), the group Γ_0 (resp. Γ_1) is isomorphic to $\boldsymbol{Z}^{n+1}$ (resp. $\exp(\boldsymbol{Z}^m)$). For any element $\exp \mathbf{t} \in \Gamma_1$ and its lift $(\exp \mathbf{t}, \mathbf{x}) \in \Gamma$, we have $(\exp \mathbf{t}, \mathbf{0})\Gamma_0(\exp \mathbf{t}, \mathbf{0})^{-1} = (\exp \mathbf{t}, \mathbf{x})\Gamma_0(\exp \mathbf{t}, \mathbf{x})^{-1} = \Gamma_0$. It follows that $\varphi(\exp \mathbf{t}) \in \mathrm{Aut}(\boldsymbol{R}^{n+1}, \Gamma_0) := \{f \in \mathrm{Aut}(\boldsymbol{R}^{n+1}) \mid f(\Gamma_0) = \Gamma_0\}$. The group $\mathrm{Aut}(\boldsymbol{R}^{n+1}, \Gamma_0)$ is identified with $GL(n+1, \boldsymbol{Z})$, whenever we choose a generating set of Γ_0. Obviously, $\varphi(\Gamma_1) \subset SL(n+1, \boldsymbol{Z})$. □

Now we are in a position to prove the main proposition of this subsection.

PROPOSITION 2.4. *Let $H = \boldsymbol{R}_+^m \ltimes_\varphi \boldsymbol{R}^{n+1}$ be a unimodular Lie group in $D(n+1, m)$. Then H contains a cocompact discrete subgroup if and only if the structure matrix Λ_φ of H is equivalent to a matrix $\hat{\Lambda}$ satisfying the following conditions.*

(1) *There exist $\Lambda(k) \in M(u_k, m, \boldsymbol{C})$ $(1 \le k \le d)$ such that $\hat{\Lambda}^t = (\Lambda(1)^t, \Lambda(2)^t, \dots, \Lambda(d)^t)^t$.*

(2) *For each k $(1 \le k \le d)$, there exists an algebraic integer α_k of degree u_k such that $\exp \lambda_i^j(k) = \sigma_k^{(i)}(\exp \lambda_1^j(k)) \in \boldsymbol{Q}(\sigma_k^{(i)}(\alpha_k))$ $(1 \le j \le m, 1 \le i \le u_k)$. Here $\{\sigma_k^{(i)} \mid 1 \le i \le u_k, \sigma_k^{(1)} = \mathrm{id}\}$ is the set of all conjugation mappings of $\boldsymbol{Q}(\alpha_k)$, and $\lambda_i^j(k)$ denotes the (i, j)-element of $\Lambda(k)$.*

(3) *Each number $\exp(\pm\lambda_i^j(k))$ is an algebraic integer.*

PROOF. Suppose that H contains a cocompact discrete subgroup. From (3) in Lemma 2.3, we can choose a basis $\{\mathbf{e}_j \mid 1 \le j \le m\}$ of $\boldsymbol{R}^m$ such that $\exp A_j \in SL(n+1, \boldsymbol{Z})$ for $A_j := d\varphi(\mathbf{e}_j)$. We apply Lemma 2.2 to these matrices $\{A_j\}$. Then the structure matrix $\Lambda_\varphi = \Lambda = (\lambda_i^j)$ of H is equivalent to a matrix $\hat{\Lambda}$ whose (i, j)-element is $p\lambda_{\tau(i)}^j$, where $\Lambda \in S(n+1, m)$, $p \in \boldsymbol{Z}$ and $\tau \in \mathcal{S}_{n+1}$ are as in the lemma. The matrix $\hat{\Lambda}$ satisfies the conditions (1)–(3) from Lemma 2.2.

Next we prove the sufficiency. From (2), for each j $(1 \le j \le m)$, we can define a linear automorphism $T_j(k)$ of $\boldsymbol{Q}(\alpha_k)$ by $T_j(k)(x) = (\exp \lambda_1^j(k))x$. Let $B'(k)$ be an integral basis of $\mathcal{O}(\alpha_k)$. Then, there exists $V(k) \in GL(u_k, \boldsymbol{C})$ such that $V(k)^{-1}[T_j(k)]_{B'(k)}V(k) = \mathrm{diag}(\exp \lambda_1^j(k), \exp \lambda_2^j(k), \dots, \exp \lambda_{u_k}^j(k))$. From (3), $[T_j(k)]_{B'(k)} \in GL(u_k, \boldsymbol{Z})$. Because H is unimodular, $\sum_{k=1}^d \sum_{i=1}^{u_k} \lambda_i^j(k) = 0$. It follows that the matrix $X_j := \mathrm{diag}([T_j(1)]_{B'(1)}, [T_j(2)]_{B'(2)}, \dots, [T_j(d)]_{B'(d)})$ is in $SL(n+1, \boldsymbol{Z})$. It is easy to see that there exist commuting, diagonalizable matrices $C_j \in M(n+1, \boldsymbol{R})$ such that (a) $\exp C_j = X_j$ and (b) the eigenvalues of C_j are $\{\lambda_i^j(k) \mid 1 \le i \le u_k, 1 \le k \le d\}$.

Now define a homomorphism $\varphi' : \boldsymbol{R}_+^m \to SL(n+1, \boldsymbol{R})$ by $\varphi'(\exp(t_1, \dots, t_m)^t) = \exp(\sum_{j=1}^m t_j C_j)$, and put $H' := \boldsymbol{R}_+^m \ltimes_{\varphi'} \boldsymbol{R}^{n+1}$. From (a), H' contains a cocompact discrete subgroup $\Gamma' := \exp(\boldsymbol{Z}^m) \ltimes_{\varphi'} \boldsymbol{Z}^{n+1}$. From (b), the structure matrix of H' is equivalent to $\hat{\Lambda}$, and hence to Λ_φ. Thus H' is isomorphic to H from Proposition 1.3. Consequently, H also has a cocompact discrete subgroup. □

It should be remarked that a general theorem of Mostow [11] gives a necessary and sufficient condition for a solvable Lie group to have a cocompact discrete subgroup. On the other hand, our conditions in Proposition 2.4 are for Lie groups of the form $H = \boldsymbol{R}_+^m \ltimes_\varphi \boldsymbol{R}^{n+1}$ and are more concrete.

2.3. Proof of Theorem 2. From Proposition 2.4, we can now prove Theorem 2 in Introduction.

PROOF OF THEOREM 2. Let H be an $(m+n+1)$-dimensional simply connected unimodular Lie group which contains G. From Proposition 1.5, H is isomorphic to $\tilde{G} = \boldsymbol{R}_+^m \ltimes_{\tilde{\psi}} \boldsymbol{R}^{n+1} \in D(n+1, m)$ whose structure matrix $\Lambda_{\tilde{\psi}}$ is given by $(\Lambda_\psi^t, -\beta^t)^t$. Thus the theorem follows from Proposition 2.4. □

In the case where $m = n$, we obtain a corollary of Theorem 2.

COROLLARY 2.5. *Let $G = \boldsymbol{R}_+^n \ltimes_\psi \boldsymbol{R}^n$ $(n \geq 2)$ be a group in $D(n, n)$. Then G has a codimension one homogeneous action.*

Note that, when $n \geq 2$, the asumptions that (1) $\Lambda_i \neq \boldsymbol{0}$ $(1 \leq i \leq n)$ and (2) $\beta \notin \{\pm\Lambda_i, \Lambda_i - \Lambda_j \mid 1 \leq i, j \leq n\}$ in Theorem 2 follow from the local injectivity of ψ. When $n = 1$, the condition (2) does not hold. But, in this case, the Lie group $G = \boldsymbol{R}_+^1 \ltimes_\psi \boldsymbol{R}^1$ is isomorphic to $\mathrm{Aff}^+(\boldsymbol{R})$ and the conclusion of the corollary is true (see, e.g., [5]).

To prove Corollary 2.5, we first show a lemma.

LEMMA 2.6. *For each integer $s \geq 2$ and a pair of non-negative integers (t, u) such that $t + 2u = s$, there exists an irreducible monic $\boldsymbol{Z}$-polynomial of degree s which has t real roots and $2u$ non-real roots.*

PROOF. For the given integer s, let $f_s(x) = (-1)^{s-1}(x-4)(x-4^2)\cdots(x-4^s)$, and $\{f_s(\alpha_i)\}$ $(\alpha_1 < \alpha_2 < \cdots < \alpha_{s-1})$ be the set of all local maxima and minima of $f_s(x)$. Obviously one has (1) $4^i < \alpha_i < 4^{i+1}$ $(1 \leq i \leq s-1)$ and (2) $f_s(\alpha_{2j-1}) > 0 > f_s(\alpha_{2j})$ $(j = 1, 2, \dots)$. Put $a_i := (4^i + 4^{i+1})/2$ $(1 \leq i \leq s+1)$. Then, by some calculations, one can show (3) $f_s(\alpha_1) \geq f_s(a_1) \geq 36$ and (4) $|f_s(a_{i+1})| \geq |f_s(a_i)| \geq 2|f_s(\alpha_{i-1})|$ $(2 \leq i \leq s)$. Let u_0 be the largest integer such that $2u_0 \leq s$. From (2), (3) and (4), we have (5):

$$0 < f_s(a_1) - 2 < f_s(\alpha_1) < f_s(a_3) - 2 < f_s(\alpha_3) < \cdots < f_s(a_{2u_0-1}) - 2 < f_s(\alpha_{2u_0-1}) < |f_s(a_{2u_0+1})| - 2 \,.$$

For each integer u $(0 \leq u \leq u_0)$, put $f_{s,u}(x) := f_s(x) - (|f_s(a_{2u+1})| - 2)$. Then the monic $\boldsymbol{Z}$-polynomial $f_{s,u}(x)$ is irreducible from Eisenstein's Irreducibility Criterion. Furthermore, from (5), the polynomial $f_{s,u}(x)$ has exactly $2u$ non-real roots. □

PROOF OF COROLLARY 2.5. Suppose G is of type (l, r). Then from Proposition 1.4, G is isomorphic to $G_n(l, r) = \boldsymbol{R}^n_+ \ltimes_{\psi_n(l,r)} \boldsymbol{R}^n \cong \mathrm{Aff}^+(\boldsymbol{R})^l \times \widetilde{\mathrm{Aff}}(\boldsymbol{R})^r$. From Lemma 2.6, there is a real algebraic integer α whose minimal polynomial has $(l+1)$ real roots and $2r$ non-real roots. Take a system of fundamental units $\Xi := \{\xi_j \mid 1 \le j \le l+r\}$ of $\boldsymbol{Q}(\alpha)$ (see, e.g., [4, IV.4]). As before, let $\{\sigma^{(1)} = \mathrm{id}, \sigma^{(2)}, \dots, \sigma^{(n+1)}\}$ be the set of all conjugation mappings of $\boldsymbol{Q}(\alpha)$. By rearranging the numbering if necessary, we can assume, for each j, that $\sigma^{(i)}(\xi_j)$ $(1 \le i \le l)$ and $\sigma^{(l+2r+1)}(\xi_j)$ are real numbers and the others satisfy $\overline{\sigma^{(l+2i-1)}(\xi_j)} = \sigma^{(l+2i)}(\xi_j)$ $(1 \le i \le r)$. Note that $\sum_{i=1}^{n+1} \log|\sigma^{(i)}(\xi_j)| = 0$ for each j.

Define the $(l+r)$-square matrix $\mathrm{Log}\,\Xi$ and the $r \times (l+r)$-matrix $\mathrm{Arg}\,\Xi$ by:

$$(i, j)\text{-element of } \mathrm{Log}\,\Xi = \begin{cases} \log|\sigma^{(i)}(\xi_j)| & \text{if } 1 \le i \le l\,, \\ \log|\sigma^{(2i-l-1)}(\xi_j)| & \text{if } l+1 \le i \le l+r\,, \end{cases}$$

$$(i, j)\text{-element of } \mathrm{Arg}\,\Xi = \arg(\sigma^{(l+2i-1)}(\xi_j)) \quad 1 \le i \le r\,.$$

Put

$$\Lambda_\Xi := 2\begin{pmatrix} \mathrm{Log}\,\Xi & \mathbf{0} \\ \mathrm{Arg}\,\Xi & \pi E_r \end{pmatrix}.$$

It is well-known that $\det \mathrm{Log}\,\Xi \ne 0$. Thus the matrix Λ_Ξ is non-singular. Consider the product matrix

$$\hat{\Lambda} := \begin{pmatrix} \Lambda_{\psi_n(l,r)} \\ -\beta \end{pmatrix} \Lambda_\Xi\,, \quad \text{where } \beta = (\underbrace{1, \dots, 1}_{l}, \underbrace{2, \dots, 2}_{r}, \underbrace{0, \dots, 0}_{r})\,.$$

Then the exponential of the (i, j)-element of $\hat{\Lambda}$ is $\sigma^{(i)}(\xi_j)^2$ if $1 \le j \le l+r$, and 1 if $l+r+1 \le j \le l+2r$, and lies in $\mathcal{O}(\sigma^{(i)}(\alpha))$. So the matrix $\hat{\Lambda}$ satisfies the conditions (1) and (2) in Theorem 2 with $d = 1$ and $\alpha_1 = \alpha$. The corollary follows from Theorem 2. □

In [14], the first author studied the classification of codimension one homogeneous actions of $\mathrm{Aff}^+(\boldsymbol{R})^n$.

3. Existence of an equivariant transverse vector field. Let $G = \boldsymbol{R}^m_+ \ltimes_\psi \boldsymbol{R}^n$ be a group in $D(n, m)$, and let M be an $(m+n+1)$-dimensional closed orientable manifold. The purpose of this section is to prove that, for a volume preserving locally free action Φ of G on M, there exists uniquely an equivariant transverse vector field of class C^0. That is, we prove Proposition 3.1 stated below.

3.1. Statement of the result. We have a natural homomorphism Φ^+ from the Lie algebra $\mathcal{G}$ of right invariant vector fields on G to the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on M, which is defined by

$$\Phi^+(X)_p(f) = \lim_{t \to 0} \frac{f(\Phi(\exp tX, p)) - f(p)}{t}\,, \quad X \in \mathcal{X}(M) \text{ and } f \in C^\infty_p(M)\,.$$

Here $C^\infty_p(M)$ is the set of germs of smooth functions at a point p in M. Since Φ is locally free, the vector field $\Phi^+(X)$ is nowhere zero if $X \ne 0$. To simplify the notation, we denote $\Phi^+(X)$ by X^*. For $g \in G$, we denote the diffeomorphism $\Phi(g,\ )$ of M by Φ_g and the induced homomorphism $(\Phi_g)_*$ (resp. $(\Phi_g)^*$) of $\mathcal{X}(M)$ (resp. $\bigwedge T^*(M)$) by g_* (resp. g^*). An element $g \in G$ acts on a vector field X^* as follows: $g_* X^* = g_*(\Phi^+(X)) = \Phi^+((L_g)_* X) = ((L_g)_* X)^*$.

Let $\{X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*\}$ be the vector fields on M which are induced from the basis $\{X_1, \dots, X_n, Y_1, \dots, Y_m\}$ of $\mathcal{G}$ given in Section 1.4. Recall the modular function $\Delta : G \to \boldsymbol{R}_+$ is given by $\Delta((\exp \mathbf{t}, \mathbf{x})) = \exp(\sum_{i=1}^n \Lambda_i \cdot \mathbf{t}) = \exp(\beta \cdot \mathbf{t})$. Here Λ_i is the i-th row vector of the structure matrix Λ_ψ and $\beta = \sum_{i=1}^n \Lambda_i$.

PROPOSITION 3.1. *Let $G = \boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$ $(0 < m \le n)$ be a group in $D(n, m)$, and let M be an $(m+n+1)$-dimensional connected closed orientable manifold. Let $\Phi : G \times M \to M$ be a locally free action which preserves a volume form Ω of class C^0. Suppose that the structure matrix Λ_ψ of G satisfies*

$$\beta \notin \{a_i \Re \Lambda_i, b_j \Re \Lambda_j - \Re \Lambda_k \mid 0 \le a_i, b_j \le 1, 1 \le i, j, k \le n\}.$$

Then there exists uniquely a vector field T of class C^0 on M such that

$$(1)\ \Omega(X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*, T) = 1 \quad \text{and} \quad (2)\ g_* T = \Delta(g)^{-1} T \text{ for any } g \in G.$$

3.2. Homothety equivariance. In this subsection, we show the following lemma.

LEMMA 3.2. *Let G, M, Φ and Ω be the same as in Proposition 3.1. Then there exists uniquely a vector field T of class C^0 satisfying the following conditions*:

$$\begin{aligned} &(1) \quad \Omega(X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*, T) = 1, \\ &(2) \quad (\exp \mathbf{t}, \mathbf{0})_* T = \Delta((\exp \mathbf{t}, \mathbf{0}))^{-1} T = e^{-\beta \cdot \mathbf{t}} T \quad \text{for any } \mathbf{t} \in \boldsymbol{R}^m. \end{aligned}$$

We prove the lemma through four steps.

Step 1. We assume G is of type (l, r), and use a canonical coordinate of G so that $d\psi(\mathbf{t})$ is described as in (1.1).

From a theorem of Ghys ([5, Theorem A]), the above volume form Ω is smooth. Take a smooth vector field Z on M satisfying $\Omega(X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*, Z) = 1$. Then for each $g = (\exp \mathbf{t}, \mathbf{x}) \in G$ we have, from (1.4),

$$\begin{aligned} 1 &= g^* \Omega(X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*, Z) \\ &= \Omega(g_* X_1^*, \dots, g_* X_n^*, g_* Y_1^*, \dots, g_* Y_m^*, g_* Z) \\ &= e^{\beta \cdot \mathbf{t}} \Omega(X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*, g_* Z). \end{aligned}$$

Thus we can write $g_* Z \equiv e^{-\beta \cdot \mathbf{t}} Z \,(\mathrm{mod}(X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*))$. Hence, for each $\mathbf{t} \in \boldsymbol{R}^m$, there are families of smooth functions $\{\phi_{\mathbf{t}}^k \mid 1 \le k \le m\}$ and $\{\psi_{\mathbf{t}}^k \mid 1 \le k \le n\}$ on M, indexed by $\mathbf{t}$, such that

$$(3.1) \qquad (\exp \mathbf{t}, \mathbf{0})_* Z = e^{-\beta \cdot \mathbf{t}} Z + \sum_{k=1}^m \phi_{\mathbf{t}}^k Y_k^* + \sum_{k=1}^n \psi_{\mathbf{t}}^k X_k^*.$$

These functions satisfy the following transition formulas.

SUBLEMMA 3.3.

$$\phi^k_{\mathbf{t}'+\mathbf{t}} = e^{-\beta\cdot\mathbf{t}}\phi^k_{\mathbf{t}'} + \phi^k_{\mathbf{t}} \circ \Phi_{(\exp(-\mathbf{t}'),\mathbf{0})} \quad (1 \le k \le m), \tag{3.2}$$

$$\psi^i_{\mathbf{t}'+\mathbf{t}} = e^{-\beta\cdot\mathbf{t}}\psi^i_{\mathbf{t}'} + e^{\Lambda_i\cdot\mathbf{t}'}\psi^i_{\mathbf{t}} \circ \Phi_{(\exp(-\mathbf{t}'),\mathbf{0})} \quad (1 \le i \le l), \tag{3.3}$$

$$\begin{aligned}\begin{pmatrix}\psi^{l+2j-1}_{\mathbf{t}'+\mathbf{t}}\\ \psi^{l+2j}_{\mathbf{t}'+\mathbf{t}}\end{pmatrix} &= e^{-\beta\cdot\mathbf{t}}\begin{pmatrix}\psi^{l+2j-1}_{\mathbf{t}'}\\ \psi^{l+2j}_{\mathbf{t}'}\end{pmatrix}\\ &+ e^{\Re\Lambda_{l+2j-1}\cdot\mathbf{t}'}\exp((\Im\Lambda_{l+2j-1}\cdot\mathbf{t}')J)\begin{pmatrix}\psi^{l+2j-1}_{\mathbf{t}} \circ \Phi_{(\exp(-\mathbf{t}'),\mathbf{0})}\\ \psi^{l+2j}_{\mathbf{t}} \circ \Phi_{(\exp(-\mathbf{t}'),\mathbf{0})}\end{pmatrix} \quad (1 \le j \le r).\end{aligned} \tag{3.4}$$

PROOF. From (3.1) and (1.4), the right hand side of

$$(\exp(\mathbf{t}'+\mathbf{t}),\mathbf{0})_* Z = (\exp\mathbf{t}',\mathbf{0})_* \circ (\exp\mathbf{t},\mathbf{0})_* Z$$

is calculated as

$$\begin{aligned}&e^{-\beta\cdot(\mathbf{t}'+\mathbf{t})}Z + e^{-\beta\cdot\mathbf{t}}\left\{\sum_{k=1}^m \phi^k_{\mathbf{t}'}Y^*_k + \sum_{k=1}^n \psi^k_{\mathbf{t}'}X^*_k\right\}\\ &+\sum_{k=1}^m \phi^k_{\mathbf{t}} \circ \Phi_{(\exp(-\mathbf{t}'),0)}Y^*_k + \sum_{i=1}^l \psi^i_{\mathbf{t}} \circ \Phi_{(\exp(-\mathbf{t}'),0)}e^{\Lambda_i\cdot\mathbf{t}'}X^*_i\\ &+\sum_{j=1}^r \psi^{l+2j-1}_{\mathbf{t}} \circ \Phi_{(\exp(-\mathbf{t}'),0)}e^{a_j}(\cos b_j X^*_{l+2j-1} + \sin b_j X^*_{l+2j})\\ &+\sum_{j=1}^r \psi^{l+2j}_{\mathbf{t}} \circ \Phi_{(\exp(-\mathbf{t}'),0)}e^{a_j}(-\sin b_j X^*_{l+2j-1} + \cos b_j X^*_{l+2j}),\end{aligned}$$

where we put $\Lambda_{l+2j-1}\cdot\mathbf{t}' = a_j + b_j\sqrt{-1}$ ($1 \le j \le r, a_j, b_j \in \boldsymbol{R}$). The lemma follows immediately from this identity. □

Let $C^0(M)$ denote the space of all continuous functions on M with the distance function d induced from the supremum norm $\| \ \|$. Any vector field T of class C^0 satisfying (1) in Lemma 3.2 is described as

$$T = Z + \sum_{k=1}^m F^k Y^*_k + \sum_{k=1}^n G^k X^*_k, \quad F^k, G^k \in C^0(M). \tag{3.5}$$

We show that, by choosing suitable continuous functions F^k and G^k, the vector field T satisfies the equivariance condition (2) in Lemma 3.2.

Step 2. In this step we choose the functions F^k ($1 \le k \le m$). From (3.5), (3.1) and (1.4), we have

$$(\exp\mathbf{t},\mathbf{0})_* T \equiv e^{-\beta\cdot\mathbf{t}}Z + \sum_{k=1}^m \phi^k_{\mathbf{t}}Y^*_k + \sum_{k=1}^m F^k \circ \Phi_{(\exp(-\mathbf{t}),\mathbf{0})}Y^*_k \quad (\mathrm{mod}(X^*_1,\cdots,X^*_n)).$$

Hence the vector field T satisfies the congruence $(\exp \mathbf{t}, \mathbf{0})_* T \equiv e^{-\beta\cdot\mathbf{t}} T (\mathrm{mod}(X_1^*, \ldots, X_n^*))$ if and only if each of the function F^k $(1 \leq k \leq m)$ satisfies the equality

$$F^k = e^{\beta\cdot\mathbf{t}}(\phi_{\mathbf{t}}^k + F^k \circ \Phi_{(\exp(-\mathbf{t}),\mathbf{0})}) . \tag{3.6}$$

For each k $(1 \leq k \leq m)$ and each $\mathbf{t} \in \boldsymbol{R}^m$, consider a continuous operator $U_{\mathbf{t}}^k$ from $C^0(M)$ to itself defined by

$$U_{\mathbf{t}}^k(F) := e^{\beta\cdot\mathbf{t}}(\phi_{\mathbf{t}}^k + F \circ \Phi_{(\exp(-\mathbf{t}),\mathbf{0})}) .$$

Obviously, we have $d(U_{\mathbf{t}}^k(F), U_{\mathbf{t}}^k(F')) = e^{\beta\cdot\mathbf{t}} d(F, F')$. So the operator $U_{\mathbf{t}_0}^k$ is Lipshitz contracting and has a unique fixed point if $\beta \cdot \mathbf{t}_0 < 0$. From the assumption on Λ_ψ, the vector β is non-zero, and hence such a vector $\mathbf{t}_0$ can be chosen. Furthermore, from the identity $\phi_{\mathbf{t}+\mathbf{t}'}^k = \phi_{\mathbf{t}'+\mathbf{t}}^k$ and (3.2), the family of operators $\{U_{\mathbf{t}}^k \mid \mathbf{t} \in \boldsymbol{R}^m\}$ is abelian. Thus if F_0^k is a fixed point of $U_{\mathbf{t}_0}^k$, then, for an arbitrary $\mathbf{t} \in \boldsymbol{R}^m$, the function $U_{\mathbf{t}}^k(F_0^k)$ is also a fixed point of $U_{\mathbf{t}_0}^k$. Consequently, from the uniqueness of the fixed point of $U_{\mathbf{t}_0}^k$, there exists uniquely a continuous function F_0^k on M which is a common fixed point of the operators $U_{\mathbf{t}}^k$ for any $\mathbf{t} \in \boldsymbol{R}^m$, and satisfies (3.6). Using this function F_0^k as F^k in (3.5), we obtain

$$(\exp \mathbf{t}, \mathbf{0})_* T \equiv e^{-\beta\cdot\mathbf{t}}\left(Z + \sum_{k=1}^m F^k Y_k^*\right) \quad (\mathrm{mod}(X_1^*, \ldots, X_n^*)) . \tag{3.7}$$

Step 3. Next we choose the functions G^i $(1 \leq i \leq l)$ in (3.5). From (3.7), (3.1) and (1.4) we have

$$(\exp \mathbf{t}, \mathbf{0})_* T \equiv e^{-\beta\cdot\mathbf{t}}\left(Z + \sum_{k=1}^m F^k Y_k^*\right) + \sum_{i=1}^l (\psi_{\mathbf{t}}^i + e^{\Lambda_i\cdot\mathbf{t}} G^i \circ \Phi_{(\exp(-\mathbf{t}),\mathbf{0})}) X_i^* \qquad (\mathrm{mod}(X_{l+1}^*, \ldots, X_{l+2r}^*)) .$$

For each i $(1 \leq i \leq l)$ and each $\mathbf{t} \in \boldsymbol{R}^m$, define an operator $V_{\mathbf{t}}^i$ on $C^0(M)$ by

$$V_{\mathbf{t}}^i(G) := e^{\beta\cdot\mathbf{t}}(\psi_{\mathbf{t}}^i + e^{\Lambda_i\cdot\mathbf{t}} G \circ \Phi_{(\exp(-\mathbf{t}),\mathbf{0})}) .$$

Then we have $d(V_{\mathbf{t}}^i(G), V_{\mathbf{t}}^i(G')) = e^{(\beta+\Lambda_i)\cdot\mathbf{t}} d(G, G')$. From the assumption on Λ_ψ, we can choose a vector $\mathbf{t}_0$ with $(\beta + \Lambda_i) \cdot \mathbf{t}_0 < 0$. The commutativity of the operators $\{V_{\mathbf{t}}^i\}$ follows from (3.3). Thus, as in Step 2, there exists uniquely a function G^i which is fixed by $V_{\mathbf{t}}^i$ for any $\mathbf{t} \in \boldsymbol{R}^m$. Using such a G^i in the expression (3.5) of T, we obtain

$$(\exp \mathbf{t}, \mathbf{0})_* T \equiv e^{-\beta\cdot\mathbf{t}}\left(Z + \sum_{k=1}^m F^k Y_k^* + \sum_{i=1}^l G^i X_i^*\right) \quad (\mathrm{mod}(X_{l+1}^*, \ldots, X_{l+2r}^*)) . \tag{3.8}$$

Step 4. Lastly, we consider the functions G^{l+2j-1}, G^{l+2j} $(1 \leq j \leq r)$ in (3.5). In this case we define an operator $W_{\mathbf{t}}^j$ on the product space $C^0(M) \times C^0(M)$ as follows:

$$W_{\mathbf{t}}^j \begin{pmatrix} G \\ G' \end{pmatrix} := e^{\beta\cdot\mathbf{t}} \left\{ \begin{pmatrix} \psi_{\mathbf{t}}^{l+2j-1} \\ \psi_{\mathbf{t}}^{l+2j} \end{pmatrix} + e^{\Re\Lambda_{l+2j-1}\cdot\mathbf{t}} \exp((\Im\Lambda_{l+2j-1}\cdot\mathbf{t})J) \begin{pmatrix} G \circ \Phi_{(\exp(-\mathbf{t}),\mathbf{0})} \\ G' \circ \Phi_{(\exp(-\mathbf{t}),\mathbf{0})} \end{pmatrix} \right\} .$$

Again we have

$$d\left(W_{\mathbf{t}}^{j}\begin{pmatrix}G\\G'\end{pmatrix}, W_{\mathbf{t}}^{j}\begin{pmatrix}H\\H'\end{pmatrix}\right) = e^{(\beta+\Re\Lambda_{l+2j-1})\cdot\mathbf{t}} d\left(\begin{pmatrix}G\\G'\end{pmatrix}, \begin{pmatrix}H\\H'\end{pmatrix}\right).$$

From the assumption on Λ_ψ, we can choose a vector $\mathbf{t}_0$ such that $(\beta + \Re\Lambda_{l+2j-1}) \cdot \mathbf{t}_0 < 0$. The commutativity of the operators $\{W_{\mathbf{t}}^j\}$ follows from (3.4). Thus, as in Steps 2 and 3, using the unique pair of functions $(G^{l+2j-1}, G^{l+2j})^t$ fixed by $W_{\mathbf{t}}^j$ for all $\mathbf{t} \in \boldsymbol{R}^m$, we obtain

$$(\exp\mathbf{t}, \mathbf{0})_* T = (\exp\mathbf{t}, \mathbf{0})_*\left(Z + \sum_{k=1}^{n} F^k Y_k^* + \sum_{k=1}^{n} G^k X_k^*\right) = e^{-\beta\cdot\mathbf{t}} T.$$

Through Steps 1 to 4, we have found a continuous vector field T which satisfies (1) and (2) in Lemma 3.2. By the construction, the vector field T is unique. This completes the proof of Lemma 3.2.

3.3. G-equivariance. Next we show that the vector field T in Lemma 3.2 is equivariant by any $g \in G = \boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$. Namely, we prove the following

LEMMA 3.4. *Let G, M, Φ and Ω be the same as in Proposition* 3.1. *Let T be a vector field satisfying* (1) *and* (2) *in Lemma* 3.2. *Then $g_* T = e^{-\beta\cdot\mathbf{t}} T$ for any $g = (\exp\mathbf{t}, \mathbf{x}) \in G$.*

For $g = (\exp\mathbf{t}, \mathbf{x}) \in G$, there exists a family of continuous functions μ_g^k and ν_g^k on M indexed by $g \in G$ such that

$$(3.9)\qquad (\exp\mathbf{t}, \mathbf{x})_* T = e^{-\beta\cdot\mathbf{t}} T + \sum_{k=1}^{m} \mu_g^k Y_k^* + \sum_{k=1}^{n} \nu_g^k X_k^*.$$

We prove the lemma by showing that the functions μ_g^k and ν_g^k are identically zero. By the assumption we have

$$(3.10)\qquad \mu_{(\exp\mathbf{t},\mathbf{0})}^k = \nu_{(\exp\mathbf{t},\mathbf{0})}^k = 0.$$

In the following, we omit the detail of calculations.

3.3.1. Nullity of μ_g^k. From (3.9) and (1.4), for $g = (\exp\mathbf{t}, \mathbf{x})$ and $h = (\exp\mathbf{t}', \mathbf{x}') \in G$, the following congruence is derived.

$$(hg)_* T \equiv e^{-\beta\cdot(\mathbf{t}+\mathbf{t}')} T + \sum_{k=1}^{m} (e^{-\beta\cdot\mathbf{t}} \mu_h^k + \mu_g^k \circ \Phi_{h^{-1}}) Y_k^* \quad (\mathrm{mod}(X_1^*, \cdots, X_n^*)).$$

Thus the following transition formula holds.

$$(3.11)\qquad \mu_{hg}^k = e^{-\beta\cdot\mathbf{t}} \mu_h^k + \mu_g^k \circ \Phi_{h^{-1}}.$$

Let $\mathbf{f}_i$ denote the i-th unit vector in $\boldsymbol{R}^n$. Then from (3.10) and (3.11), the following equalities are derived.

$$(3.12)\qquad \mu_{(1, e^{\Lambda_i\cdot\mathbf{t}} x\mathbf{f}_i)}^k = \mu_{(\exp\mathbf{t},\mathbf{0})(1,x\mathbf{f}_i)(\exp(-\mathbf{t}),\mathbf{0})}^k = e^{\beta\cdot\mathbf{t}} \mu_{(1,x\mathbf{f}_i)}^k \circ \Phi_{(\exp(-\mathbf{t}),\mathbf{0})} \quad (1 \le i \le l),$$

$$\mu^k_{(1,e^{\Re\Lambda_{l+2j-1}\cdot\mathbf{t}}\exp((\Im\Lambda_{l+2j-1}\cdot\mathbf{t})J)(x_1\mathbf{f}_{l+2j-1}+x_2\mathbf{f}_{l+2j}))} = e^{\beta\cdot\mathbf{t}}\mu^k_{(1,x_1\mathbf{f}_{l+2j-1}+x_2\mathbf{f}_{l+2j})}\circ\Phi_{(\exp(-\mathbf{t}),\mathbf{0})} \quad (1\le j\le r), \tag{3.13}$$

$$\mu^k_{(1,\mathbf{x}+\mathbf{x}')} = \mu^k_{(1,\mathbf{x})} + \mu^k_{(1,\mathbf{x}')}\circ\Phi_{(1,-\mathbf{x})}, \tag{3.14}$$

$$\mu^k_{(\exp\mathbf{t},\mathbf{x})} = \mu^k_{(\mathbf{1},\mathbf{x})(\exp\mathbf{t},\mathbf{0})} = e^{-\beta\cdot\mathbf{t}}\mu^k_{(\mathbf{1},\mathbf{x})}. \tag{3.15}$$

Hence we obtain the following relations on the supremum norms of μ^k_g.

$$\|\mu^k_{(1,e^{\Lambda_i\cdot\mathbf{t}}x\mathbf{f}_i)}\| = e^{\beta\cdot\mathbf{t}}\|\mu^k_{(1,x\mathbf{f}_i)}\| \quad (1\le i\le l), \tag{3.16}$$

$$\|\mu^k_{(1,e^{\Re\Lambda_{l+2j-1}\cdot\mathbf{t}}\exp((\Im\Lambda_{l+2j-1}\cdot\mathbf{t})J)(x_1\mathbf{f}_{l+2j-1}+x_2\mathbf{f}_{l+2j}))}\| = e^{\beta\cdot\mathbf{t}}\|\mu^k_{(1,x_1\mathbf{f}_{l+2j-1}+x_2\mathbf{f}_{l+2j})}\| \quad (1\le j\le r), \tag{3.17}$$

$$\|\mu^k_{(1,\mathbf{x}+\mathbf{x}')}\| \le \|\mu^k_{(1,\mathbf{x})}\| + \|\mu^k_{(1,\mathbf{x}')}\|. \tag{3.18}$$

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^t \in \boldsymbol{R}^n$. Then from (3.18), we have

$$\|\mu^k_{(\mathbf{1},\mathbf{x})}\| \le \sum_{i=1}^{l}\|\mu^k_{(\mathbf{1},x_i\mathbf{f}_i)}\| + \sum_{j=1}^{r}\|\mu^k_{(\mathbf{1},x_{l+2j-1}\mathbf{f}_{l+2j-1}+x_{l+2j}\mathbf{f}_{l+2j})}\|.$$

From this inequality and (3.15), to prove $\mu^k_{(\exp\mathbf{t},\mathbf{x})} = 0$, it is sufficient to show $\mu^k_{(1,x_i\mathbf{f}_i)} = \mu^k_{(1,x_{l+2j-1}\mathbf{f}_{l+2j-1}+x_{l+2j}\mathbf{f}_{l+2j})} = 0$ $(1\le i\le l, 1\le j\le r)$. For notational convenience, we put $(\mathbf{x})_j := x_{l+2j-1}\mathbf{f}_{l+2j-1} + x_{l+2j}\mathbf{f}_{l+2j}$. For a fixed k $(1\le k\le m)$, define non-decreasing functions τ^k_i and σ^k_j on $\boldsymbol{R}_+\cup\{0\}$ by

$$\tau^k_i(d) := \sup_{|x_i|\le d}\|\mu^k_{(\mathbf{1},x_i\mathbf{f}_i)}\| \quad (1\le i\le l) \quad\text{and}\quad \sigma^k_j(d) := \sup_{\|(\mathbf{x})_j\|\le d}\|\mu^k_{(\mathbf{1},(\mathbf{x})_j)}\| \quad (1\le j\le r).$$

We first show $\tau^k_i = 0$ $(1\le i\le l)$, and hence $\mu^k_{(\mathbf{1},x_i\mathbf{f}_i)} = 0$.

SUBLEMMA 3.5. (1) *For each i $(1\le i\le l)$ and $\mathbf{t}\in\boldsymbol{R}^m$, we have*

$$\tau^k_i(e^{\Lambda_i\cdot\mathbf{t}}r) = e^{\beta\cdot\mathbf{t}}\tau^k_i(r) \quad \textit{for any } r>0. \tag{3.19}$$

(2) *For each i $(1\le i\le l)$ and $\mathbf{t}\in\boldsymbol{R}^m$ such that $\Lambda_i\cdot\mathbf{t}\ne\mathbf{0}$, we have*

$$\tau^k_i(d) = d^{\frac{\beta\cdot\mathbf{t}}{\Lambda_i\cdot\mathbf{t}}}\tau^k_i(1) \quad \textit{for any } d>0. \tag{3.20}$$

(3) *For each i $(1\le i\le l)$, we have*

$$\tau^k_i(d) \le (d+1)\tau^k_i(1) \quad \textit{for any } d>0. \tag{3.21}$$

PROOF. The first assertion follows directly from (3.16). The second assertion follows from (3.19) by putting $r=1$ and $d=e^{\Lambda_i\cdot\mathbf{t}}$.

It follows from (3.18) that $\tau^k_i(d+d') \le \tau^k_i(d) + \tau^k_i(d')$. For $d>0$, choose a positive integer a such that $d\le a<d+1$. Then we have $\tau^k_i(d)\le\tau^k_i(a)\le a\tau^k_i(1)\le(d+1)\tau^k_i(1)$. We have thus proved the third assertion. □

If $\Lambda_i=\mathbf{0}$, then from (1) in Sublemma 3.5 we have $\tau^k_i(r) = e^{\beta\cdot\mathbf{t}}\tau^k_i(r)$. Hence, from the assumption $\beta\ne\mathbf{0}$, we obtain $\tau^k_i(r)=0$ for any $r>0$. When $\Lambda_i\ne\mathbf{0}$, we first suppose

$\beta \neq a_i \Lambda_i$ ($a_i > 0$). Then we can choose a vector $\mathbf{t} \in \boldsymbol{R}^m$ such that $\Lambda_i \cdot \mathbf{t} < 0 < \beta \cdot \mathbf{t}$. For such a $\mathbf{t}$, we have

$$\tau_i^k(e^{\Lambda_i \cdot \mathbf{t}} d) \leq \tau_i^k(d) \leq e^{\beta \cdot \mathbf{t}} \tau_i^k(d).$$

Thus, from (3.19) we obtain $\tau_i^k(d) = e^{\beta \cdot \mathbf{t}} \tau_i^k(d)$ and hence $\tau_i^k(d) = 0$.

Next suppose $\beta = a_i \Lambda_i$ ($a_i > 0$). Then, from the assumption on Λ_ψ, a_i is larger than 1. So we can choose a vector $\mathbf{t} \in \boldsymbol{R}^m$ such that $0 < \Lambda_i \cdot \mathbf{t} < \beta \cdot \mathbf{t}$. Put $b := (\beta \cdot \mathbf{t})/(\Lambda_i \cdot \mathbf{t})$ (> 1). Then from (3.20) and (3.21) we have

$$d^b \tau_i^k(1) = \tau_i^k(d) \leq (d+1)\tau_i^k(1) \quad \text{for any } d > 0.$$

So we obtain $\tau_i^k(1) = 0$, and hence $\tau_i^k(d) = 0$. Thus we have proved $\tau_i^k(d) = 0$ ($1 \leq k \leq m, 1 \leq i \leq l$).

Similarly, one can prove $\sigma_j^k(d) = 0$ ($1 \leq k \leq m, 1 \leq j \leq r$), using (3.17) instead of (3.16). From the nullity of τ_i^k and σ_j^k, we have the required result $\mu_g^k = 0$ for any k ($1 \leq k \leq m$).

3.3.2. Nullity of ν_g^k. The nullity of ν_g^k ($1 \leq k \leq n$) is proved in a fashon similar to the case of μ_g^k. So we only remark the formulas corresponding to (3.11), (3.12) and (3.13), but omit the detail of the proof. All the formulas are given under the assumption that $\mu_g^k = 0$ ($1 \leq k \leq m$). As before, we put $g = (\exp \mathbf{t}, \mathbf{x})$, $h = (\exp \mathbf{t}', \mathbf{x}')$. We continue to use the notation $(\mathbf{x})_j = x_{l+2j-1}\mathbf{f}_{l+2j-1} + x_{l+2j}\mathbf{f}_{l+2j}$.

SUBLEMMA 3.6. (1) *Case of* $1 \leq k \leq l$.

$$\nu_{hg}^k = e^{-\beta \cdot \mathbf{t}} \nu_h^k + e^{\Lambda_k \cdot \mathbf{t}'} \nu_g^k \circ \Phi_{h^{-1}},$$

$$\nu^k_{(\mathbf{1}, e^{\Lambda_i \cdot \mathbf{t}} x_i \mathbf{f}_i)} = e^{(\beta + \Lambda_k)\cdot \mathbf{t}} \nu^k_{(\mathbf{1}, x_i \mathbf{f}_i)} \circ \Phi_{(\exp(-\mathbf{t}), \mathbf{0})} \quad (1 \leq i \leq l),$$

$$\nu^k_{(\mathbf{1}, e^{\Re \Lambda_{l+2j-1} \cdot \mathbf{t}} \exp((\Im \Lambda_{l+2j-1} \cdot \mathbf{t})J)(\mathbf{x})_j)} = e^{(\beta + \Lambda_k)\cdot \mathbf{t}} \nu^k_{(\mathbf{1}, (\mathbf{x})_j)} \circ \Phi_{(\exp(-\mathbf{t}), \mathbf{0})} \quad (1 \leq j \leq r).$$

(2) *Case of* $l+1 \leq k \leq l+2r$. *Let* $k' := [(k-l+1)/2] =$ *the largest integer not greater than* $(k-l+1)/2$.

$$\begin{pmatrix} \nu_{hg}^{l+2k'-1} \\ \nu_{hg}^{l+2k'} \end{pmatrix} = e^{-\beta \cdot \mathbf{t}} \begin{pmatrix} \nu_h^{l+2k'-1} \\ \nu_h^{l+2k'} \end{pmatrix} + e^{a(\mathbf{t}')} \exp(b(\mathbf{t}')J) \begin{pmatrix} \nu_g^{l+2k'-1} \circ \Phi_{h^{-1}} \\ \nu_g^{l+2k'} \circ \Phi_{h^{-1}} \end{pmatrix},$$

$$\begin{pmatrix} \nu^{l+2k'-1}_{(\mathbf{1}, e^{\Lambda_i \cdot \mathbf{t}} x_i \mathbf{f}_i)} \\ \nu^{l+2k'}_{(\mathbf{1}, e^{\Lambda_i \cdot \mathbf{t}} x_i \mathbf{f}_i)} \end{pmatrix} = e^{\beta \cdot \mathbf{t}} e^{a(\mathbf{t})} \exp(b(\mathbf{t})J) \begin{pmatrix} \nu^{l+2k'-1}_{(\mathbf{1}, x_i \mathbf{f}_i)} \circ \Phi_{(\exp(-\mathbf{t}), \mathbf{0})} \\ \nu^{l+2k'}_{(\mathbf{1}, x_i \mathbf{f}_i)} \circ \Phi_{(\exp(-\mathbf{t}), \mathbf{0})} \end{pmatrix} \quad (1 \leq i \leq l),$$

$$\begin{pmatrix} \nu^{l+2k'-1}_{(\mathbf{1}, e^{c(\mathbf{t})} \exp(d(\mathbf{t})J)(\mathbf{x})_j)} \\ \nu^{l+2k'}_{(\mathbf{1}, e^{c(\mathbf{t})} \exp(d(\mathbf{t})J)(\mathbf{x})_j)} \end{pmatrix} = e^{\beta \cdot \mathbf{t}} e^{a(\mathbf{t})} \exp(b(\mathbf{t})J) \begin{pmatrix} \nu^{l+2k'-1}_{(\mathbf{1}, (\mathbf{x})_j)} \circ \Phi_{(\exp(-\mathbf{t}), \mathbf{0})} \\ \nu^{l+2k'}_{(\mathbf{1}, (\mathbf{x})_j)} \circ \Phi_{(\exp(-\mathbf{t}), \mathbf{0})} \end{pmatrix} \quad (1 \leq j \leq r),$$

where $\Lambda_{l+2k'-1} \cdot \mathbf{t} = a(\mathbf{t}) + b(\mathbf{t})\sqrt{-1}$ *and* $\Lambda_{l+2j-1} \cdot \mathbf{t} = c(\mathbf{t}) + d(\mathbf{t})\sqrt{-1}$ ($a(\mathbf{t}), b(\mathbf{t}), c(\mathbf{t}), d(\mathbf{t}) \in \boldsymbol{R}$).

This completes the proof of Lemma 3.4 and Proposition 3.1.

4. Proof of Theorem 1. In this section we first prove Proposition 4.1 which states that the vector field T in Proposition 3.1 is smooth, and then complete the proof of Theorem 1. Let $\{X_1, \dots, X_n, Y_1, \dots, Y_m\}$ be the basis of the Lie algebra of G given in Section 1.4.

PROPOSITION 4.1. *Let $G = \boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$ $(0 < m \leq n)$ be a group in $D(n, m)$ and let M be an $(m+n+1)$-dimensional connected closed orientable manifold. Let $\Phi : G \times M \to M$ be a locally free action which preserves a volume form Ω of class C^0. Suppose that the structure matrix Λ_ψ of G satisfies*

$$\beta \notin \{a_i \Re \Lambda_i \mid -1 \leq a_i \leq 0,\ 1 \leq i \leq n\}.$$

Suppose furthermore that there exists a C^0-vector field T on M such that $\Omega(X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*, T) = 1$ and $g_* T = \Delta(g)^{-1} T$ for any $g \in G$. Then the vector field T is smooth.

For the proof we use the invariant manifold theory of hyperbolic diffeomorphisms ([9]). Let $S := \{\mathbf{t} \in \boldsymbol{R}^m \mid -\beta \cdot \mathbf{t} > 0, -\beta \cdot \mathbf{t} \neq \Re \Lambda_i \cdot \mathbf{t}\ (1 \leq i \leq n)\}$. Choose $\mathbf{t} \in S$, and define $u(\mathbf{t}) := \{i \in \{1, \dots, n\} \mid -\beta \cdot \mathbf{t} < \Re \Lambda_i \cdot \mathbf{t}\}$. Put $F := \Phi_{(\exp \mathbf{t}, 0)}$. Then we have an F-invariant continuous splitting $T(M) = E_1 \oplus E_2$, where E_1 (resp. E_2) is generated by the vector fields $\{X_i^*\ (i \in u(\mathbf{t})), T\}$ (resp. $\{X_i^*\ (i \notin u(\mathbf{t})), Y_j^*\ (1 \leq j \leq m)\}$). Let $\rho > 1$ be a real number such that $\max\{|e^{\Lambda_i \cdot \mathbf{t}}| \mid i \notin u(\mathbf{t})\} < \rho < e^{-\beta \cdot \mathbf{t}}$. The splitting $E_1 \oplus E_2$ satisfies the following property.

LEMMA 4.2. *There exists a smooth Riemannian metric $|\ |$ of M such that $0 \neq v \in E_1 \Rightarrow |F_*(v)| > \rho |v|$, and $0 \neq v \in E_2 \Rightarrow |F_*(v)| < \rho |v|$.*

PROOF. For each $\delta > 0$ choose a smooth vector field T_δ on M such that $\Omega(X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*, T_\delta) = 1$ and $\lim_{\delta \to 0} T_\delta = T$ in the C^0-topology. Let $|\ |_0$ (resp. $|\ |_\delta$) be the C^0- (resp. C^∞-) Riemannian metric of M such that the vectors $\{(X_i^*)_p, (Y_j^*)_p, T_p\}$ (resp. $\{(X_i^*)_p, (Y_j^*)_p, (T_\delta)_p\}$) are orthonormal at any point $p \in M$. Then, for each $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that (1) $(1 - \varepsilon(\delta))|v|_0 \leq |v|_\delta \leq (1 + \varepsilon(\delta))|v|_0$ for all $v \in T(M)$ and (2) $\lim_{\delta \to 0} \varepsilon(\delta) = 0$.

Let ρ_1 be a positive number such that $\rho < \rho_1 < e^{-\beta \cdot \mathbf{t}}$. From the formula (1.4) and $F_* T = e^{-\beta \cdot \mathbf{t}} T$, it is easy to see that $0 \neq v \in E_1 \Rightarrow |F_*(v)|_0 > \rho_1 |v|_0$. Then we have

$$|F_*(v)|_\delta \geq (1 - \varepsilon(\delta))|F_*(v)|_0 > (1 - \varepsilon(\delta))\rho_1 |v|_0 \geq \frac{1 - \varepsilon(\delta)}{1 + \varepsilon(\delta)} \rho_1 |v|_\delta .$$

So, if we choose $\delta_1 > 0$ small enough so that $((1 - \varepsilon(\delta))\rho_1)/(1 + \varepsilon(\delta)) \geq \rho$ for any $\delta \leq \delta_1$, then we have $|F_*(v)|_\delta > \rho |v|_\delta$ $(\delta \leq \delta_1)$. Similarly we can choose $\delta_2 < \delta_1$ so that the metric $|\ |_\delta$ also satisfies the second condition if $\delta \leq \delta_2$. □

By Lemma 4.2, the diffeomorphism F is *ρ-pseudo hyperbolic* ([9], §5). From Theorem (5.5) in [9], the continuous plane field E_1 is uniquely integrable and is tangent to a C^0-foliation, denoted by $\mathcal{W}(\mathbf{t})$, with C^∞-leaves.

LEMMA 4.3. *The foliation $\mathcal{W}(\mathbf{t})$ is preserved by the action Φ and is smooth.*

PROOF. For each $g \in G$, we have $g_* E_1 = E_1$. So the action Φ preserves the foliation $\mathcal{W}(\mathbf{t})$.

Let G_2 be the subgroup of G defined, with respect to the canonical coordinate of G, by

$$G_2 = \{(\exp \mathbf{s}, (x_1, \dots, x_n)^t) \in G \mid \mathbf{s} \in \boldsymbol{R}^m,\ x_i = 0 \text{ for } i \in u(\mathbf{t})\}.$$

Consider the restricted action $\Phi|_{G_2} : G_2 \times M \to M$. Then the action $\Phi|_{G_2}$ is locally free and preserves the foliation $\mathcal{W}(\mathbf{t})$. Furthermore the orbit foliation $\mathcal{F}_{\Phi|_{G_2}}$ is of complementary dimension and is transverse to $\mathcal{W}(\mathbf{t})$. In other words, the foliation $\mathcal{W}(\mathbf{t})$ is a *transversely* G_2 *foliation* ([7, p. 152]).

We will show the smoothness of $\mathcal{W}(\mathbf{t})$ from this fact. Let $n_1 = \dim E_1$ and let D be the unit disc in $\boldsymbol{R}^{n_1}$. Take an arbitrary point $p \in M$. Since each leaf of $\mathcal{W}(\mathbf{t})$ is of class C^∞, there is a smooth embedding $f_0 : D \to M$ such that $f_0(\mathbf{0}) = p$ and $f_0(D)$ is contained in a leaf of $\mathcal{W}(\mathbf{t})$. Define a C^∞-map $f : D \times G_2 \to M$ by $f(x, g) = \Phi_g(f_0(x))$. Then there exists in G_2 a neighbourhood V of the identity element such that $f|_{D\times V} : D \times V \to M$ is an into diffeomorphism. For each $g \in G_2$, the image $f(D \times \{g\})$ is contained in a leaf of $\mathcal{W}(\mathbf{t})$ because $\Phi|_{G_2}$ preserves $\mathcal{W}(\mathbf{t})$. This shows that the foliation $\mathcal{W}(\mathbf{t})$ has a smooth distinguished chart $f|_{D\times V}$ at p. Since p is arbitrary, the foliation $\mathcal{W}(\mathbf{t})$ is smooth on M. □

PROOF OF PROPOSITION 4.1. From the assumption on the structure matrix, for each i $(1 \le i \le n)$, there exists $\mathbf{t}_i \in S$ such that $-\beta \cdot \mathbf{t}_i > \Re \Lambda_i \cdot \mathbf{t}_i$. Then we have $\bigcap_{i=1}^n u(\mathbf{t}_i) = \emptyset$, and $\mathcal{T} = \bigcap_{i=1}^n \mathcal{W}(\mathbf{t}_i)$ is a one dimensional foliation tangent to T. Note that the foliation $\mathcal{T}$ is smooth from Lemma 4.3. Since T is tangent to $\mathcal{T}$ and satisfies $\Omega(X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*, T) = 1$ for the smooth volume form Ω (see Step 1 in Section 3.2), the vector field T is smooth. □

We are now in a position to prove Theorem 1 in Introduction. Note that the assumption on the structure matrix in Proposition 4.1 follows from that in Proposition 3.1.

PROOF OF THEOREM 1. By Proposition 3.1, there exists a continuous vector field T on M such that $\Omega(X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*, T) = 1$ and $g_*T = \Delta(g)^{-1}T$ for any $g \in G$. From Proposition 4.1, the vector field T is smooth.

Let $\{\phi_t \mid t \in \boldsymbol{R}\}$ be the flow of M generated by the C^∞ vector field T. Let $g \in G$. Because $g_*T = \Delta(g)^{-1}T$, we have

$$\Phi_g \circ \phi_t \circ \Phi_{g^{-1}} = \phi_{\Delta(g)^{-1}t}. \tag{4.1}$$

Let $\hat{G} = G \ltimes_{\Delta^{-1}} \boldsymbol{R}$ be the semidirect product of G and $\boldsymbol{R}$ determined by the homomorphism $\Delta^{-1} : G \to \boldsymbol{R}_+ \subset GL(1, \boldsymbol{R})$. From (4.1) we can define a smooth action $\hat{\Phi}$ of $\hat{G}$ on M by $\hat{\Phi}(g, t) = \phi_t \circ \Phi_g$. Since the flow ϕ_t is transverse to the foliation $\mathcal{F}_\Phi$ and Φ is locally free, the action $\hat{\Phi}$ is also locally free. A locally free action $\hat{\Phi}$ of an $(m+n+1)$-dimensional Lie group $\hat{G}$ on an $(m+n+1)$-dimensional connected manifold has a single orbit, and hence $\hat{\Phi}$ is homogeneous. It follows that the action Φ, which is the restriction of $\hat{\Phi}$ to the subgroup $G \subset \hat{G}$, is homogeneous. Since $\hat{G}$ is solvable, M is a solvmanifold. □

REMARK. The group $\hat{G}$ in the proof of Theorem 1 is naturally isomorphic to the Lie group $\tilde{G}$ constructed in the proof of Proposition 1.5.

From Theorems 1 and 2, we have the following corollary.

COROLLARY 4.4. *Let $G = \boldsymbol{R}_+^m \ltimes_\psi \boldsymbol{R}^n$ be a group in $D(n,m)$. Suppose that the structure matrix Λ_ψ of G satisfies*

$$\Lambda_i \neq \mathbf{0} \ (1 \leq i \leq n) \quad \textit{and} \quad \beta \notin \{a_i \Re \Lambda_i, b_j \Re \Lambda_j - \Re \Lambda_k \mid 0 \leq a_i, b_j \leq 1, 1 \leq i, j, k \leq n\}.$$

If the matrix $(\Lambda_\psi^t, -\beta^t)^t$ is not equivalent to a matrix $\hat{\Lambda}$ satisfying the conditions (1) *and* (2) *in Theorem* 2, *then G has no codimension one locally free volume preserving action on a closed manifold.*

When $m = n \geq 2$, the structure matrix of $G \in D(n,n)$ always satisfies the assumption on Λ_ψ in Theorem 1. Thus, from Proposition 1.5, Corollary 2.5 and Theorem 1, we also obtain the following concluding corollary.

COROLLARY 4.5. *Let $G = \boldsymbol{R}_+^n \ltimes_\psi \boldsymbol{R}^n$ $(n \geq 2)$ be a group in $D(n,n)$. Then we have the following.*

(1) *There exists uniquely a simply connected unimodular Lie group which contains G as a subgroup.*

(2) *G has a codimension one homogeneous action.*

(3) *If G acts on a $(2n+1)$-dimensional connected closed orientable manifold locally freely and preserves a volume form of class C^0, then the action is C^∞-conjugate to a homogeneous action.*

REFERENCES

[1] M. BELLIART, Actions de groupes de Lie sur les variétés compactes, Thesis, Université de Valenciennes et du Hainaut-Cambrésis, 1995.

[2] M. BELLIART AND O. BIREMBAUX, Actions localement libres de groupes résolubles, Ann. Inst. Fourier 44 (1994), 1519–1537.

[3] N. BOURBAKI, Algebra II, Elements of Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 1988.

[4] A. FRÖLICH AND M. J. TAYLOR, Algebraic number theory, Cambridge Stud. Adv. Math. 27, Cambridge Univ. Press, Cambridge, 1991.

[5] E. GHYS, Actions localement libres du groupe affine, Invent. Math. 82 (1985), 479–526.

[6] E. GHYS AND A. VERJOVSKY, Locally free holomorphic actions of the complex affine group, Proceedings of geometric study of foliations, 201–217, World Scientific, Singapore, 1994.

[7] C. GODBILLON, Feuilletages, Progr. Math. 98, Birkhäuser Verlag, Basel-Boston-Berlin, 1991.

[8] G. HECTOR, E. GHYS AND Y. MORIYAMA, On codimension one nilfoliations and a theorem of Malcev, Topology 28 (1989), 197–210.

[9] M. W. HIRSH, C. C. PUGH AND M. SHUB, Invariant manifolds, Lecture Notes in Math. 583, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[10] A. KIRILLOV, Elements of the Theory of Representations, Grundlehren Math. Wiss. 20, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[11] G. D. MOSTOW, Representative functions on discrete groups and solvable arithmetic subgroups, Amer. J. Math. 92 (1970), 1–32.

[12] A. L. ONISHCHIK AND E. B. VINBERG, Lie groups and Lie algebras III, Encyclopaedia Math. Sci. 41, Springer-Verlag, Berlin-Heidelberg-New York, 1994.

[13] M. S. RAGHUNATHAN, Discrete subgroups of Lie groups, Ergeb. Math. Grenzgeb. 68, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[14] A. YAMAKAWA, Equivalence classes of codimension one locally free actions of n-product of $\mathrm{Aff}^+(\boldsymbol{R})$, ICU Math. Preprint series, 2 (1998).

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