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SYMMETRY IN THE FUNCTIONAL EQUATION OF A LOCAL ZETA DISTRIBUTION

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Abstract. We examine the structure of the coefficient matrix in the functional equation of the zeta distribution of a self-adjoint prehomogeneous vector space over a non-Archimedean local field. Under a restrictive assumption on the generic stabilizers, we show that this matrix is block upper-triangular with almost symmetric blocks; this generalizes a result of Datskovsky and Wright for the space of binary cubic forms.

We call a matrix A *almost symmetric* if there is a non-singular diagonal matrix D such that DA is symmetric. In the situations that concern us here, A is a matrix of rational functions and we require the entries in D to be constant. Datskovsky and Wright [2] showed that the Γ-matrix in the functional equation for the zeta distributions on the space of binary cubic forms over a local field is almost symmetric. Later Datskovsky [1] wrote out the Γ-matrix for the space of binary quadratic forms over a field of odd residual characteristic explicitly (an earlier computation of Sato [15] gave the matrix in an elegant form) and thus revealed that, while it is not almost symmetric, it does have a block upper-triangular structure with almost symmetric blocks. (Care is required in interpreting Datskovsky's expression, since he uses the transpose of the standard Γ-matrix.)

Our purpose here is to understand these facts in a unified way and to generalize them. Although our generalization has strong hypotheses, it does apply to several interesting spaces for which the Γ-matrix is as yet unknown. The additional information it provides might prove helpful in determining the Γ-matrices for these spaces. The origin of the symmetry in the functional equation is the assumed self-adjointness of the underlying space. The true scope of this passage from self-adjointness to symmetry is presently hard to judge because the list of known Γ-matrices for self-adjoint spaces with more than one orbit is rather short. Further computations of explicit examples in order to gain insight into this and other open questions about the Γ-matrix would be very welcome. It is possible that the phenomenon discussed here is widespread or even completely general, but, if so, a less naive method of proof will have to be found.

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Let F be a non-Archimedean local field of characteristic zero with finite residue class field of cardinality q. Denote by $|\cdot|$ the normalized absolute value on F. Let V be a finite-dimensional F-vector space. For $g \in \mathrm{GL}(V)$ and $\Lambda \subset V$ a lattice, define

$$m(g; \Lambda) = \sup_{\xi \in \Lambda, \xi^* \in \Lambda^*} (|\xi^*(g\xi)|) ,$$

where $\Lambda^* \subset V^*$ is the dual lattice. We have $m(g; \Lambda) > 0$ and $m(u_1 g u_2; \Lambda) = m(g; \Lambda)$ for all $u_1, u_2 \in \mathrm{Aut}(\Lambda)$; in particular, $g \mapsto m(g; \Lambda)$ is continuous. Note that if $\xi_1, \ldots, \xi_n$ is a basis for Λ and $\xi_1^*, \ldots, \xi_n^*$ is the dual basis, then

$$m(g; \Lambda) = \max_{1 \le i, j \le n} (|\xi_i^*(g\xi_j)|) .$$

If ϕ and ψ are real-valued functions on a set S, then we shall say that ϕ and ψ are *of the same order* if there are positive constants c_1 and c_2 such that $c_1\phi(x) \le \psi(x) \le c_2\phi(x)$ for all $x \in S$.

LEMMA 1. *Let Λ_1, Λ_2 and Λ be lattices in V, $g_1, g_2 \in \mathrm{GL}(V)$ and $U_1, U_2 \subset \mathrm{GL}(V)$ be compact subgroups of $\mathrm{GL}(V)$. Then the following hold.*

(1) *We have*

$$m(g_2^{-1}; \Lambda)^{-1} m(g_1; \Lambda) \le m(g_1 g_2; \Lambda) \le m(g_1; \Lambda) m(g_2; \Lambda) .$$

(2) *The functions $g \mapsto m(g; \Lambda_1)$ and $g \mapsto m(g; \Lambda_2)$ are of the same order.*

(3) *The functions $g \mapsto m(g; \Lambda)$ and*

$$g \mapsto \sup_{u_1 \in U_1, u_2 \in U_2} m(u_1 g u_2; \Lambda)$$

are of the same order.

PROOF. Clear. □

For a lattice $\Lambda \subset V$ and $g \in \mathrm{GL}(V)$ we define

$$b(g; \Lambda) = \max\{\log_q(m(g; \Lambda)), \log_q(m(g^{-1}; \Lambda))\} .$$

It follows from (1) of Lemma 1 that

$$b(g_1 g_2; \Lambda) \le b(g_1; \Lambda) + b(g_2; \Lambda)$$

for all $g_1, g_2 \in \mathrm{GL}(V)$.

LEMMA 2. *Let Λ be a lattice in V and $X_1 \le X_2$ real numbers. Then the set*

$$\{g \in \mathrm{GL}(V) \mid q^{X_1} \le m(g; \Lambda) \le q^{X_2}\}$$

is compact.

PROOF. Choose a basis for Λ, give $\mathrm{GL}(V)$ the corresponding integral structure and let $K = \mathrm{GL}(V)(\mathcal{O})$. Let A be the diagonal torus with respect to the chosen basis. The claim follows from the Cartan decomposition $\mathrm{GL}(V) = KAK$ and (3) of Lemma 1. □

Now let $G \subset \mathrm{GL}(V)$ be a closed subgroup of $\mathrm{GL}(V)$. (Here, and in the following discussion, topological notions are relative to the classical topology on $\mathrm{GL}(V)$.) For a lattice Λ in V and a real number $X \geq 1$ define

$$G[X; \Lambda] = \{g \in G \mid q^{-X} \leq m(g; \Lambda) \leq q^{X}\}.$$

By Lemma 2, this is a compact subset of G.

LEMMA 3. *For all $g \in G$ we have*

$$G[X - b(g; \Lambda); \Lambda] \subset gG[X; \Lambda] \subset G[X + b(g; \Lambda); \Lambda].$$

PROOF. This follows at once from the definitions and (1) of Lemma 1. □

We call a subgroup H of G a *TC-group* (short for torus-compact-group) if H has commuting subgroups A and U such that A is an F-split torus, U is compact and AU has finite index in H. A TC-group is automatically closed and unimodular. If H is a TC-group, then we write $r(H)$ for the rank of any torus A as in the definition. This number is independent of the choice of A. The class of TC-groups includes all algebraic F-subgroups of G whose identity component is a (not necessarily split) torus, as well as all compact subgroups of G.

PROPOSITION 4. *Let $H \subset G$ be a TC-group and ν a non-zero Haar measure on H. Let Λ be a lattice in V and $g \in G$. Then there are non-zero constants $c(H, \nu)$ and C such that*

$$|\nu(gG[X; \Lambda] \cap H) - c(H, \nu)X^{r(H)}| \leq C[1 + |b(g; \Lambda)|]^{r(H)} X^{r(H)-1}.$$

The constant $c(H, \nu)$ depends only on the indicated data. The constant C depends only on H, ν and Λ.

PROOF. If $H_0 \subset H$ is a subgroup of finite index in H, then

$$\nu(gG[X; \Lambda] \cap H) = \sum_{h \in H/H_0} \nu(h^{-1}gG[X; \Lambda] \cap H_0).$$

Now

$$|b(g; \Lambda)| - |b(h; \Lambda)| \leq |b(h^{-1}g; \Lambda)| \leq |b(g; \Lambda)| + |b(h; \Lambda)|$$

and so if we can prove the claim with H_0 in place of H, then the original claim will follow. We may thus assume that $H = AU$, where $A \subset H$ is an F-split torus, $U \subset H$ is compact and A and U commute.

Note the basic inequality $|(X + s)^r - X^r| \leq X^{r-1}(1 + |s|)^r$, valid for $X \geq 1$, r a natural number, and all s. Suppose that Λ_0 is a lattice in V. It follows from (2) of Lemma 1 that there is a constant s, depending only on Λ and Λ_0, such that

$$G[X - s; \Lambda_0] \subset G[X; \Lambda] \subset G[X + s; \Lambda_0].$$

Thus if we could verify the claim for Λ_0, then the claim for Λ would follow by using the basic inequality. It therefore suffices to use any convenient choice of Λ. By appealing to Lemma 3 and using the same argument, it also suffices to verify the claim when g is the identity.

We may decompose V into the direct sum of the eigenspaces associated with various characters of A. Each of these eigenspaces is stable under U and hence we may find a U-stable lattice in each eigenspace. The direct sum, Λ_0, of these lattices is a lattice in V. It is stable under U and there is a basis of Λ_0 on which A acts diagonally. The map $A \times U \to H$ given by multiplication has fibers of constant finite volume and hence there are Haar measures ν_A and ν_U on A and U, respectively, such that $\nu_U(U) = 1$ and (in the obvious sense) $\nu = \nu_A \otimes \nu_U$. Note also that $m(au; \Lambda_0) = m(a; \Lambda_0)$ for all $a \in A$ and $u \in U$. We are thus reduced to verifying that there is a constant $c > 0$ such that $\nu_A(A[X; \Lambda_0]) = cX^{r(H)} + O(X^{r(H)-1})$.

Let $r = r(H)$ be the rank of A, fix a basis $e_1, \ldots, e_n$ for Λ_0 on which A acts diagonally and let $\chi_1, \ldots, \chi_n$ be the characters of A such that $ae_i = \chi_i(a)e_i$ for each i. We are seeking to show that the ν_A-volume of the set

$$A[X; \Lambda_0] = \{a \in A \mid q^{-X} \leq |\chi_i(a)| \leq q^X \text{ for all } i\}$$

has the required form as a function of X. Choose coordinates $a_1, \ldots, a_r$ on A and write

$$\chi_i(a) = \prod_{j=1}^{r} a_j^{d_{ij}}$$

for $1 \leq i \leq n$. Note that, by construction, $a \in A$ is the identity if and only if $\chi_i(a) = 1$ for $1 \leq i \leq n$. It follows that the right kernel of the matrix $[d_{ij}]$ is $\{0\}$, and so the matrix $[d_{ij}]$ is of rank r. The volume of $A[X; \Lambda_0]$ is proportional, by a constant depending only on ν_A, to the number of $\boldsymbol{Z}^r$-points in the set

$$B[X] = \left\{ z \in \boldsymbol{R}^r \,\middle|\, -X \leq \sum_{j=1}^{r} d_{ij} z_j \leq X \text{ for all } i \right\}.$$

The set $B[1]$ is a bounded convex subset of $\boldsymbol{R}^r$ with non-empty interior and boundary contained in a union of proper affine subspaces. It is well-known that the number of $\boldsymbol{Z}^r$-points in $B[X] = XB[1]$ is then $\mathrm{vol}(B[1])X^r + O(X^{r-1})$ with $\mathrm{vol}(B[1]) > 0$, as required. $\square$

The significance of the class of TC-groups is that, when H is such a group, the leading term in the asymptotics of $\nu(gG[X; \Lambda] \cap H)$ as $X \to \infty$ is independent of g. This property does not extend to more general subgroups. Indeed, if H is $\mathrm{GL}(2)$ embedded in the upper left hand corner of $G = \mathrm{GL}(3)$, Λ is the standard lattice in F^3, $\varpi \in F$ is a uniformizer and

$$g_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varpi^{-m} & 0 & 1 \end{pmatrix}$$

for $m > 0$, then there is a constant $C > 0$, depending only on q and ν, such that

$$\nu(g_m G[X; \Lambda] \cap H) \sim Cq^{-m} X q^{2X}.$$

In order to proceed further, we position ourselves between the settings of Igusa [4] and Sato [15], the two main authorities for the functional equation of the zeta distribution of a prehomogeneous vector space over a non-Archimedean local field.

We use bold letters for varieties defined over F (which may be identified with their sets of $\bar{F}$-points if desired) and the corresponding non-bold letters for their sets of F-points. Let $(\boldsymbol{G}, \rho, \boldsymbol{V})$ be a prehomogeneous vector space defined over F, with singular set $\boldsymbol{S}$. We assume that $\boldsymbol{G}$ is connected and reductive and that the space is F-regular. Let $\boldsymbol{S}_1, \ldots, \boldsymbol{S}_n$ be the irreducible components of $\boldsymbol{S}$ over F, $P_1, \ldots, P_n \in F[\boldsymbol{V}]$ be relative invariants defining them, and $\chi_1, \ldots, \chi_n$ the corresponding rational characters of $\boldsymbol{G}$. These characters generate a free abelian group $X(\rho, F)$ of rank n which contains the character $\chi_0(g) = \det(\rho(g))^2$. We may thus write

$$\chi_0 = \prod_{i=1}^{n} \chi_i^{2\kappa_i}$$

for some uniquely determined row vector $\kappa = (\kappa_1, \ldots, \kappa_n) \in ((1/2)\boldsymbol{Z})^n$. Let

$$P_0 = \prod_{i=1}^{n} P_i^{2\kappa_i} ,$$

a non-zero relative invariant associated to the character χ_0. It is known that the locus $P_0 = 0$ is set-theoretically equal to the singular set $\boldsymbol{S}$ (see [8], Proposition 2.26).

We assume that the prehomogeneous vector space under consideration is F-self-adjoint. That is, there is a non-degenerate F-bilinear form $\langle \cdot\,, \cdot \rangle$ on $\boldsymbol{V}$ and an anti-automorphism $g \mapsto g^\iota$ of period two of $\boldsymbol{G}$ over F such that $\langle \rho(g)x, y \rangle = \langle x, \rho(g^\iota)y \rangle$. We assume that $\langle \cdot\,, \cdot \rangle$ is either alternating or symmetric and denote its sign by η; that is, $\langle x, y \rangle = \eta \langle y, x \rangle$. Of course, if $\boldsymbol{V}$ is irreducible as a representation of $\boldsymbol{G}$, then this assumption is automatically fulfilled, but it need not be otherwise. We further assume that $\rho(G)$ contains the homothety $x \mapsto \eta x$ of V. There is necessarily some $J \in \mathrm{GL}(V)$ such that $\rho(g^\iota) = J\,{}^t\rho(g) J^{-1}$ and consequently $\chi_0(g^\iota) = \chi_0(g)$. We remark that the transpose appearing in this equation is ambiguous, as is the map J, since they depend upon a choice of basis for V. Later on, when we make further use of this equation, we shall make a convenient choice of basis. The representation of $\boldsymbol{G}$ on $\boldsymbol{V}$ given by $g \mapsto \rho(g^{-\iota})$ is equivalent to the contragredient ρ^*. It follows from this and the assumption that $(\boldsymbol{G}, \rho, \boldsymbol{V})$ is F-regular that the characters χ_i^* given by $\chi_i^*(g) = \chi_i(g^{-\iota})$ lie in $X(\rho, F)$ ([15], Lemma 1.1). Thus so too do the characters χ_i^ι defined by $g \mapsto \chi_i(g^\iota)$ and we may find a matrix $U = [u_{ij}] \in \mathrm{GL}(n, \boldsymbol{Z})$ such that

$$\chi_i^\iota = \prod_{j=1}^{n} \chi_j^{u_{ij}}$$

for $1 \leq i \leq n$. Directly from the definition we obtain $U^2 = I_n$, and from the fact that $\chi_0^\iota = \chi_0$ we obtain $\kappa U = \kappa$.

We also assume that $(\boldsymbol{G}, \rho, \boldsymbol{V})$ satisfies Sato's condition (A.2) ([15], p. 474). This condition states that $\boldsymbol{S}$ decomposes into a finite number of orbits under $\boldsymbol{G}$ and that if $\boldsymbol{O}$ is a $\boldsymbol{G}$-orbit in $\boldsymbol{S}$, then there is some $\chi \in X(\rho, F) - \{1\}$ such that $\boldsymbol{O}$ is a $\boldsymbol{G}^{(\chi)}$-orbit, where $\boldsymbol{G}^{(\chi)}$ denotes the kernel of χ. An equivalent formulation of the second part of the assumption is that for any $x \in \boldsymbol{S}$ there is some $\chi \in X(\rho, F) - \{1\}$ such that χ is non-trivial on the identity component of $\boldsymbol{G}_x$. We require this assumption only because it is currently a hypothesis for the functional equation (Proposition 5 below). If the functional equation is subsequently established under less restrictive hypotheses, then our result will correspondingly generalize.

With these assumptions in place, we are ready to introduce the remainder of the standard notation that we shall require below. The set $V - S$ is the union of finitely-many G-orbits. To verify this well-known fact, note that F is a non-Archimedean local field with finite residue class field, and hence of "type (F)" in the terminology of Serre ([17], p. 143). The required finiteness statement then follows from the theorem stated on p. 146 of [17]. We enumerate the orbits in $V - S$ as $O_1, \ldots, O_l$ and fix a base point $x_b \in O_b$ for $1 \leq b \leq l$. Denote by $\boldsymbol{G}_b$ the isotropy subgroup of x_b and recall that $\boldsymbol{G}_b$ is reductive.

Let $\boldsymbol{N} = \ker(\rho)$. If H is any subgroup of G that contains N, then let $\tilde{H} = H/N$. The group $\tilde{G}$ may be identified with a closed subgroup of $\mathrm{GL}(V)$, and we shall make this identification below. Note that the $\tilde{G}$-orbits in $V - S$ coincide with the G-orbits in $V - S$, and that the characters $\chi_1, \ldots, \chi_n$ and the map $g \mapsto g^\iota$ pass down to $\tilde{G}$. We shall abuse notation by retaining the same symbols for these objects on $\tilde{G}$ as we have been using on G.

Fix a non-zero Haar measure dy on V and define a measure λ on $V - S$ by

$$d\lambda(y) = |P_0(y)|^{-1/2} \cdot dy\,.$$

This measure is $\tilde{G}$-invariant. Since $\tilde{G}/\tilde{G}_a \approx O_a$, there is a measure μ_a on $\tilde{G}/\tilde{G}_a$ such that

$$\int_{\tilde{G}/\tilde{G}_a} \Phi(\rho(\dot{g})x_a)d\mu_a(\dot{g}) = \int_{O_a} \Phi(y)d\lambda(y) \tag{1}$$

for all $\Phi \in \mathrm{L}^1(O_a)$. Fix a non-zero Haar measure μ on $\tilde{G}$. Then there is a Haar measure ν_a on $\tilde{G}_a$ such that

$$\int_{\tilde{G}} f(g)d\mu(g) = \int_{\tilde{G}/\tilde{G}_a} \int_{\tilde{G}_a} f(\dot{g}h)\, d\nu_a(h)d\mu_a(\dot{g}) \tag{2}$$

for all $f \in \mathrm{L}^1(\tilde{G})$.

Let $\Omega(F^\times)$ be the group of continuous homomorphisms from $F^\times$ to $\mathbf{C}^\times$. If $\omega \in \Omega(F^\times)^n$, $x \in V$ and $g \in \tilde{G}$, then define

$$\omega(P(x)) = \prod_{i=1}^{n} \omega_i(P_i(x))$$

and

$$\omega(\chi(g)) = \prod_{i=1}^{n} \omega_i(\chi_i(g))$$

so that $\omega(P(\rho(g)x)) = \omega(\chi(g))\omega(P(x))$. The involution ι may be transported to $\Omega(F^\times)^n$ by defining $\omega^\iota = (\omega_1^\iota, \ldots, \omega_n^\iota)$, where

$$\omega_j^\iota = \prod_{i=1}^{n} \omega_i^{u_{ij}}\,.$$

With these definitions we have $\omega^\iota(\chi(g)) = \omega(\chi(g^\iota))$. We also set

$$\omega_0 = (|\cdot|^{\kappa_1}, \ldots, |\cdot|^{\kappa_n})$$

and note that the equation $\kappa U = \kappa$ implies that $\omega_0^\iota = \omega_0$.

Let $\mathcal{S}(V)$ be the space of Schwartz-Bruhat functions on V, let ψ be a non-trivial additive character of F and define a Fourier transform on $\mathcal{S}(V)$ by

$$\hat{\Phi}(y) = \int_V \Phi(x)\psi(\langle x, y\rangle)dx\,.$$

To the orbit O_a is associated a meromorphic family of distributions given by

$$Z_a(\omega, \Phi) = \int_{O_a} \omega(P(y))\Phi(y)d\lambda(y)$$

for $\omega \in \Omega(F^\times)^n$ and $\Phi \in \mathcal{S}(V)$ when the integral is absolutely convergent, and by meromorphic continuation otherwise. Note that the integral defining $Z_a(\omega, \Phi)$ is absolutely convergent when $\mathrm{re}(\omega_j) \geq \kappa_j$ for all $1 \leq j \leq n$.

Let Φ be the characteristic function of a compact open set and $\omega_s = (|\cdot|^{s_1}, \ldots, |\cdot|^{s_n})$. Then it is well known that $Z_a(\omega_s, \Phi)$ is a rational function in the variables q^{-s_j}. This rational function is regular at $(\kappa_1, \ldots, \kappa_n)$ and hence also in some neighborhood of this point. The integral defining $Z_a(\omega_s, \Phi)$ expands formally to the product of a Laurent monomial and a Taylor series in the variables q^{-s_j} with positive coefficients. By applying the usual "Landau Lemma" argument to this series (see the proof of Lemma 1 on p. 314 of [13], for example), we conclude that there are constants $c_j < \kappa_j$, depending on Φ, such that the integral defining $Z_a(\omega_s, \Phi)$ converges absolutely for $\mathrm{re}(s_j) > c_j$. The characteristic functions of compact open sets span $\mathcal{S}(V)$, and hence we may extend this conclusion to all $\Phi \in \mathcal{S}(V)$. Any $\omega \in \Omega(F^\times)^n$ is bounded componentwise by the character ω_s with $s = \mathrm{re}(\omega)$. Thus we may further extend the conclusion to say that there are $c_j < \kappa_j$ such that the integral defining $Z_a(\omega, \Phi)$ converges absolutely on the set $\mathrm{re}(\omega_j) > c_j$.

The zeta distributions Z_a enjoy the following functional equation with respect to the Fourier transform.

PROPOSITION 5. *There is a matrix $\Gamma(\omega) = [\Gamma_{ab}(\omega)]$ of rational functions on $\Omega(F^\times)^n$ such that*

$$Z_a(\omega, \hat{\Phi}) = \sum_{b=1}^{l} \Gamma_{ab}(\omega) Z_b(\omega_0\omega^{-\iota}, \Phi)$$

as meromorphic functions of ω for all $\Phi \in \mathcal{S}(V)$.

PROOF. This is a special case of Sato's Theorem $k_{\mathfrak{p}}$ [15], p. 477, translated into our current setting. □

We are now ready to state our main result, which concerns the structure of the Γ-matrix.

THEOREM 6. *With the above notation and assumptions, suppose that* $1 \leq a, b \leq l$ *are indices such that* $\tilde{G}_a$ *and* $\tilde{G}_b$ *are TC-groups. Let*

$$C_a = \frac{\nu_a(K \cap \tilde{G}_a)\lambda(\rho(K)x_a)}{c(\tilde{G}_a, \nu_a)}$$

and similarly for C_b. *Here* $c(\tilde{G}_a, \nu_a)$ *is the constant from Proposition* 4 *and* K *is any sufficiently small compact open subgroup of* $\tilde{G}$, *fixed throughout. Also define*

$$\xi(\omega) = \prod_{i=1}^{n} \omega_i(\eta)^{\deg(P_i)}$$

for $\omega \in \Omega(F^\times)^n$, *where* $\eta = \pm 1$ *is the sign of the bilinear form* $\langle \cdot\,,\, \cdot \rangle$. *Then the following hold.*

(A) *If* $r(\tilde{G}_a) < r(\tilde{G}_b)$, *then* $\Gamma_{ba}(\omega) = 0$ *for all* $\omega \in \Omega(F^\times)^n$.

(B) *If* $r(\tilde{G}_a) = r(\tilde{G}_b)$, *then*

$$C_a \Gamma_{ba}(\omega) = \xi(\omega) C_b \Gamma_{ab}(\omega^\iota)$$

for all $\omega \in \Omega(F^\times)^n$.

PROOF. Let $\mathcal{C} \subset \Omega(F^\times)^n$ be the intersection of the domains of absolute convergence of the integrals defining $Z_a(\omega, \Phi)$ for $1 \leq a \leq l$. Recall that $\Omega(F^\times)^n$ is the product of a discrete group and the complex polycylinder $(\mathbf{C}/2\pi\sqrt{-1}\log(q)\mathbf{Z})^n$. Let $\mathcal{X} \subset \Omega(F^\times)^n$ be a connected component and choose a basepoint $\alpha \in \mathcal{X}$ such that $\mathrm{re}(\alpha) = 0$. It follows from the remarks just before Proposition 5 that $\beta = \alpha\omega_0$ lies in the interior of $\mathcal{C} \cap \mathcal{X}$. Similarly, the character $\gamma = \alpha^\iota\omega_0$ is an interior point of $\mathcal{C} \cap \mathcal{X}^\iota$ and it follows that $\beta = \gamma^\iota$ is an interior point of $\mathcal{C}^\iota \cap \mathcal{X}$. Thus β is an interior point of $\mathcal{C} \cap \mathcal{C}^\iota \cap \mathcal{X}$. We deduce that the set $\mathcal{C} \cap \mathcal{C}^\iota \cap \mathcal{X}$ has non-empty interior for each connected component $\mathcal{X}$ of $\Omega(F^\times)^n$.

Let $\mathcal{X} \subset \Omega(F^\times)^n$ be a connected component and Λ a lattice in V. We noted above that there is an element $J \in \mathrm{GL}(V)$ such that $\rho(g^\iota) = J^t\rho(g)J^{-1}$. Observe that $\Lambda' = J(\Lambda)$ is also a lattice in V and that $m(g^\iota; \Lambda') = m(g; \Lambda)$ for all $g \in \tilde{G}$, provided that we choose to interpret the transpose with respect to a basis for Λ, as we may. It follows from this observation that $\tilde{G}[X; \Lambda]^\iota = \tilde{G}[X; \Lambda']$ for all $X \geq 1$.

The group $\tilde{G}$ has a neighborhood base at the identity consisting of compact open subgroups and the map $g \mapsto g^\iota$ is continuous. As a consequence of these facts we may find a compact open subgroup K of $\tilde{G}$ so small that the following conditions hold:

(a) $\rho(k)\Lambda = \Lambda$ and $\rho(k)\Lambda' = \Lambda'$ for all $k \in K$,

(b) $\rho(k^\iota)\Lambda = \Lambda$ and $\rho(k^\iota)\Lambda' = \Lambda'$ for all $k \in K$,

(c) $\omega(\chi(k)) = 1$ and $\omega(\chi(k^\iota)) = 1$ for all $k \in K$ and $\omega \in \mathcal{X}$.

Suppose now that $1 \leq a, b \leq l$ are indices such that $\tilde{G}_a$ and $\tilde{G}_b$ are TC-groups. Let $A_{X,\Lambda}$ be the characteristic function of the set $\tilde{G}[X; \Lambda]$. Fix a point ω in the set $\mathcal{C} \cap \mathcal{C}^\iota \cap \mathcal{X}$ and, for $X \geq 1$ and $\varepsilon > 0$, consider the integral

$$I(X, \varepsilon) = \int_{\tilde{G},\, |\chi_0(g)| \geq \varepsilon} \omega(\chi(g)) A_{X,\Lambda}(g) \psi(\langle x_a, \rho(g)x_b \rangle) d\mu(g)\,.$$

In this integral we make the change of variable $g \mapsto k^\iota g$, use the defining property of ι, integrate the result over K, and use Fubini's Theorem to obtain

$$I(X, \varepsilon) = \mu(K)^{-1} \int_{|\chi_0(g)| \geq \varepsilon} \omega(\chi(g)) A_{X,\Lambda}(g) \int_K \psi(\langle \rho(k)x_a, \rho(g)x_b \rangle) d\mu(k) d\mu(g) .$$

Let $\Phi_a \in \mathcal{S}(V)$ be the characteristic function of the compact open set $\rho(K)x_a \subset V$. By using (2), we obtain, for any $y \in V$,

$$\int_K \psi(\langle \rho(k)x_a, y \rangle) d\mu(k) = \nu_a(K \cap \tilde{G}_a) \int_{K\tilde{G}_a/\tilde{G}_a} \psi(\langle \rho(\dot{g})x_a, y \rangle) d\mu_a(\dot{g}) .$$

Applying (1) to this expression, we find that it is equal to

$$\begin{aligned} &\nu_a(K \cap \tilde{G}_a) \int_{\rho(K)x_a} \psi(\langle x, y \rangle) d\lambda(x) \\ &\qquad = \nu_a(K \cap \tilde{G}_a) |P_0(x_a)|^{-1/2} \int_{\rho(K)x_a} \psi(\langle x, y \rangle) dx \\ &\qquad = \nu_a(K \cap \tilde{G}_a) |P_0(x_a)|^{-1/2} \hat{\Phi}_a(y). \end{aligned}$$

Consequently,

$$I(X, \varepsilon) = D_a \int_{|\chi_0(g)| \geq \varepsilon} \omega(\chi(g)) A_{X,\Lambda}(g) \hat{\Phi}_a(\rho(g)x_b) d\mu(g) ,$$

where

$$D_a = \mu(K)^{-1} \nu_a(K \cap \tilde{G}_a) |P_0(x_a)|^{-1/2} .$$

Now observe that χ_i is trivial on $\tilde{G}_b$ for $1 \leq i \leq n$. In light of this, we can apply (2) to the last expression for $I(X, \varepsilon)$ to conclude that

$$I(X, \varepsilon) = D_a \int_{\tilde{G}/\tilde{G}_b,\, |\chi_0(\dot{g})| \geq \varepsilon} \omega(\chi(\dot{g})) \hat{\Phi}_a(\rho(\dot{g})x_b) \nu_b(\dot{g}^{-1}\tilde{G}[X; \Lambda] \cap \tilde{G}_b) d\mu_b(\dot{g}) .$$

Let us define a function $B_{X,\Lambda,b}$ on O_b by setting

$$B_{X,\Lambda,b}(y) = \nu_b(g^{-1}\tilde{G}[X; \Lambda] \cap \tilde{G}_b)$$

for any $g \in \tilde{G}$ such that $y = \rho(g)x_b$. This is well defined because of the left invariance of the Haar measure ν_b. By applying (1) to the previous expression for $I(X, \varepsilon)$, we arrive at

$$I(X, \varepsilon) = D_a \omega(P(x_b))^{-1} \int_{O_b,\, |P_0(y)| \geq \varepsilon |P_0(x_b)|} \omega(P(y)) \hat{\Phi}_a(y) B_{X,\Lambda,b}(y) d\lambda(y) .$$

The support of $\hat{\Phi}_a$ is a compact subset of V. The set $\{y \in V \mid |P_0(y)| \geq \varepsilon |P_0(x_b)|\}$ is closed in V and contained in $V - S$, and O_b is closed in $V - S$. It follows that the set

$$Y = \{y \in \operatorname{supp}(\hat{\Phi}_a) \cap O_b \mid |P_0(y)| \geq \varepsilon |P_0(x_b)|\}$$

is compact. We may thus find a compact set $R \subset \tilde{G}$ such that $Y \subset \rho(R)x_b$. From this and Proposition 4 we draw two conclusions. First, the family of functions

$$\{X^{-r(\tilde{G}_b)} B_{X,\Lambda,b} \mid X \geq 1\}$$

is uniformly bounded on Y and, secondly,

$$\lim_{X\to\infty} X^{-r(\tilde{G}_b)} B_{X,\Lambda,b}(y) = c(\tilde{G}_b, \nu_b)$$

for all $y \in Y$. By the choice of ω, the integral obtained from the last expression for $I(X,\varepsilon)$ by removing the factor $B_{X,\Lambda,b}(y)$ from the integrand and taking $\varepsilon = 0$ is absolutely convergent. By the Dominated Convergence Theorem, it follows that

$$\lim_{X\to\infty} X^{-r(\tilde{G}_b)} I(X,\varepsilon) \\ = c(\tilde{G}_b, \nu_b) D_a \omega(P(x_b))^{-1} \int_{O_b,\, |P_0(y)|\geq \varepsilon |P_0(x_b)|} \omega(P(y)) \hat{\Phi}_a(y) d\lambda(y)$$

and hence

$$\lim_{\varepsilon\to 0}\lim_{X\to\infty} X^{-r(\tilde{G}_b)} I(X,\varepsilon) = c(\tilde{G}_b, \nu_b) D_a \omega(P(x_b))^{-1} Z_b(\omega, \hat{\Phi}_a)\,.$$

On the other hand, Proposition 5 gives

$$\begin{aligned} Z_b(\omega, \hat{\Phi}_a) &= \sum_{c=1}^{l} \Gamma_{bc}(\omega) Z_c(\omega_0 \omega^{-\iota}, \Phi_a) \\ &= \omega^{\iota}(P(x_a))^{-1} \mathrm{vol}(\rho(K)x_a) \Gamma_{ba}(\omega)\,, \end{aligned}$$

where vol denotes the volume with respect to the chosen Haar measure on V. In deriving the last equality we used the fact that $\omega_0(P(x)) = |P_0(x)|^{1/2}$. Combining these two evaluations, and using the fact that $|P_0(x_a)|^{-1/2}\mathrm{vol}(\rho(K)x_a) = \lambda(\rho(K)x_a)$, we obtain

$$\lim_{\varepsilon\to 0}\lim_{X\to\infty} X^{-r(\tilde{G}_b)} I(X,\varepsilon) \\ = \mu(K)^{-1} c(\tilde{G}_b, \nu_b) \nu_a(K\cap \tilde{G}_a) \lambda(\rho(K)x_a) \omega(P(x_b))^{-1} \omega^{\iota}(P(x_a))^{-1} \Gamma_{ba}(\omega).$$

We have assumed that there is some $z \in \tilde{G}$ such that $\rho(z)x = \eta x$ for all $x \in V$. A calculation shows that $\omega(\chi(z)) = \xi(\omega)$. In the original integral defining $I(X,\varepsilon)$ we make the change of variable $g \mapsto zg$, use the identity $\eta\langle x_a, \rho(g)x_b\rangle = \langle x_b, \rho(g^{\iota})x_a\rangle$, and then make the change of variable $g \mapsto g^{\iota}$. At the end of this process, we obtain

$$I(X,\varepsilon) = \xi(\omega) \int_{|\chi_0(g)|\geq\varepsilon} \omega^{\iota}(\chi(g)) A_{X,\Lambda'}(g) \psi(\langle x_b, \rho(g)x_a\rangle) d\mu(g)\,.$$

This integral has the same form as the original one, but with a and b interchanged and ω^{ι} in place of ω. Thus

$$\lim_{\varepsilon\to 0}\lim_{X\to\infty} X^{-r(\tilde{G}_a)} I(X,\varepsilon) \\ = \xi(\omega)\mu(K)^{-1} c(\tilde{G}_a, \nu_a) \nu_b(K\cap \tilde{G}_b) \lambda(\rho(K)x_b) \omega^{\iota}(P(x_a))^{-1} \omega(P(x_b))^{-1} \Gamma_{ab}(\omega^{\iota}).$$

If $r = r(\tilde{G}_a) = r(\tilde{G}_b)$, then equating these two evaluations of

$$\lim_{\varepsilon\to 0}\lim_{X\to\infty} X^{-r} I(X,\varepsilon)$$

gives the formula stated in part (B). If $r(\tilde{G}_a) < r(\tilde{G}_b)$, then the existence of the limit $\lim_{X\to\infty} X^{-r(\tilde{G}_a)} I(X, \varepsilon)$ forces the limit $\lim_{X\to\infty} X^{-r(\tilde{G}_b)} I(X, \varepsilon)$ to be 0 and so $\Gamma_{ba}(\omega) = 0$, as required for part (A). Initially, these conclusions hold for all ω in $\mathcal{C} \cap \mathcal{C}^\iota \cap \mathcal{X}$. However, we have seen that this set has non-empty interior and $\Gamma_{ab}(\omega^\iota)$ and $\Gamma_{ba}(\omega)$ are rational functions on $\Omega(F^\times)^n$. Thus the conclusions hold for all $\omega \in \mathcal{X}$ and, since $\mathcal{X}$ was an arbitrary connected component of $\Omega(F^\times)^n$, it follows that they are true generally. $\square$

If $(\boldsymbol{G}, \rho, \boldsymbol{V})$ satisfies all the above assumptions and, in addition, $\tilde{G}_a$ is a TC-group for all $1 \le a \le l$, then it is a consequence of Theorem 6 that the matrix Γ is block upper-triangular provided that we arrange the orbits in such a way that $r(\tilde{G}_a)$ is non-decreasing with a. If we restrict ourselves to characters ω such that $\omega^\iota = \omega$, then the diagonal blocks will be almost symmetric (if $\xi(\omega) = 1$) or almost antisymmetric (if $\xi(\omega) = -1$).

Before we discuss some examples, we would like to raise the following question.

QUESTION. Let $(\boldsymbol{G}, \rho, \boldsymbol{V})$ be a prehomogeneous vector space satisfying the above conditions. Is there a way to attach to each $1 \le a \le l$ an element $r(\tilde{G}_a)$ of some totally ordered set and a constant C_a in such a way that $r(\tilde{G}_a) < r(\tilde{G}_b)$ implies that $\Gamma_{ba}(\omega) = 0$ and $r(\tilde{G}_a) = r(\tilde{G}_b)$ implies that $C_a\Gamma_{ba}(\omega) = \xi(\omega)C_b\Gamma_{ab}(\omega^\iota)$?

The reason for allowing $r(\tilde{G}_a)$ to lie in a totally ordered set, rather than just $\boldsymbol{N}$, is that an appropriate generalization of Proposition 4 is likely to invoke functions such as $q^{r_1 X} X^{r_2}$ that depend on more than one parameter. The order on the set of parameters would then express the relative asymptotic magnitude of the corresponding functions.

We now discuss two examples to illustrate that, despite the strong restriction that the isotropy subgroups of generic points be TC-groups, Theorem 6 does give non-trivial information about the Γ-matrix of a number of interesting spaces. As concerns general notation, Aff^n denotes affine n-space regarded as a representation of $\mathrm{GL}(n)$ in the natural way, $e_1, \ldots, e_n$ is the distinguished basis of Aff^n, $\vee^2$ denotes the symmetric square, $\wedge^2$ denotes the exterior square, $V' = V - S$, and $\boldsymbol{\varepsilon}$ is the fully alternating tensor defined by

$$\boldsymbol{\varepsilon}^{i_1\ldots i_n} = \begin{cases} 1 & \text{if } i_1, \ldots, i_n \text{ is an even rearrangement of } 1, \ldots, n, \\ -1 & \text{if } i_1, \ldots, i_n \text{ is an odd rearrangement of } 1, \ldots, n. \end{cases}$$

EXAMPLE 1. Let (G, V) be

$$(\mathrm{GL}(3) \times \mathrm{GL}(3) \times \mathrm{GL}(2), \mathrm{Aff}^3 \otimes \mathrm{Aff}^3 \otimes \mathrm{Aff}^2)\,.$$

This is essentially (12) on the Sato-Kimura list [16] of regular reduced irreducible prehomogeneous vector spaces. It is discussed from an arithmetical point of view in [18]. The orbits over an algebraically closed field of characteristic zero are enumerated in Table 1 of [11]. From this data it is a routine, though tedious, exercise to verify that this space satisfies condition (A.2). We identify the space with the space of pairs $[M_1, M_2]$ of 3-by-3 matrices under the action of G given by

$$(g_1, g_2, h)[M_1, M_2] = [g_1 M_1 g_2^t, g_1 M_2 g_2^t]h^t\,.$$

The bilinear form

$$\langle [M_1, M_2], [M_1', M_2'] \rangle = \mathrm{tr}(M_1 M_1' + M_2 M_2')$$

is non-degenerate and symmetric. It satisfies the identity

$$\langle (g_1, g_2, h)x, y \rangle = \langle x, (g_2^t, g_1^t, h^t) y \rangle$$

and so we set $(g_1, g_2, h)^\iota = (g_2^t, g_1^t, h^t)$. There is a single relative invariant P of degree 12, associated to the character $\chi(g_1, g_2, h) = \det(g_1)^4 \det(g_2)^4 \det(h)^6$. The G_F-orbits in V_F' are in one-to-one correspondence with separable cubic F-algebras [18] and the isotropy group of a point in an orbit is isomorphic to $\mathrm{GL}(1)$ times the multiplicative group of the corresponding algebra. Let R_a denote the cubic F-algebra corresponding to the orbit of x_a. We have

$$r(\tilde{G}_a) = \begin{cases} 1 & \text{if } R_a \text{ is a field,} \\ 2 & \text{if } R_a \cong F \oplus E \text{ with } E \text{ a field,} \\ 3 & \text{if } R_a \cong F \oplus F \oplus F. \end{cases}$$

Let us organize the orbits in a list so that $r(\tilde{G}_a)$ increases along the list. Theorem 6 then implies that the Γ-matrix of (G, V) takes the form

$$\Gamma = \begin{pmatrix} \Delta_1 & * & * \\ 0 & \Delta_2 & * \\ 0 & 0 & \gamma \end{pmatrix} .$$

Here Δ_1 is an almost symmetric matrix of size equal to the number of cubic extensions of F, Δ_2 is an almost symmetric matrix of size equal to the number of quadratic extensions of F, and γ is a single rational function corresponding to the unique orbit with $r(\tilde{G}_a) = 3$.

EXAMPLE 2. Let (G, V) be

$$(\mathrm{GL}(5) \times \mathrm{GL}(3), \wedge^2 \mathrm{Aff}^5 \otimes \mathrm{Aff}^3 \oplus \mathrm{Aff}^3) .$$

This space is essentially (7) in the classification of regular 2-simple prehomogeneous vector spaces of type I [9]. The orbits of the space $(\mathrm{GL}(5) \times \mathrm{GL}(3), \wedge^2 \mathrm{Aff}^5 \otimes \mathrm{Aff}^3)$ over an algebraically closed field of characteristic zero, together with the isotropy algebra of a particular point in each orbit, are enumerated at length in Section 11 of [7]. This data, together with the remarks on the space (G, V) to be found in the proof of Theorem 4.30 of [10], allow one to confirm that (G, V) has finitely-many orbits, and that the second part of condition (A.2) is also satisfied. For this purpose, the alternate formulation of the second part of the condition is the more convenient one.

There is much less information about the arithmetic properties of (G, V) available in the literature than was the case with Example 1 and so we shall sketch the relevant details. We may identify V with the space of 4-tuples $[M_1, M_2, M_3, y]$, where the M_i are 5-by-5 alternating matrices and y is a 3-by-1 matrix. The action of G on this model of the space is given by

$$(g, h)[M_1, M_2, M_3, y] = [(gM_1 g^t, gM_2 g^t, gM_3 g^t)h^t, hy] .$$

Define a bilinear form on V by

$$\langle [M_1, M_2, M_3, y], [M_1', M_2', M_3', y'] \rangle = \mathrm{tr}(M_1M_1' + M_2M_2' + M_3M_3') + y^t y' .$$

It is easy to verify that this form is symmetric and non-degenerate and that it satisfies $\langle (g,h)x, y\rangle = \langle x, (g,h)^\iota y\rangle$ with $(g,h)^\iota = (g^t, h^t)$. Note that (G, V) has an obvious $\boldsymbol{Z}$-structure, to which we shall refer below.

Let E_{ij} be the 5-by-5 alternating matrix with 1 in the (i, j)-entry, -1 in the (j, i)-entry and 0 elsewhere. The point

$$w = [E_{12} + E_{34}, E_{23} + E_{45}, E_{13} + E_{25}, e_2]$$

is generic. The identity component of its stabilizer is the image of the 1-parameter subgroup

$$\alpha(t) = (\mathrm{diag}(1, t^{-1}, t, t^{-2}, t^2)\, , \mathrm{diag}(t, 1, t^{-1}))$$

and the component group of the stabilizer is generated by the class of

$$\tau = \left(\begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right),$$

where $i \in \bar{F}$ satisfies $i^2 = -1$. It easily follows that $G_w^\circ \cong \mathrm{GL}(1)$ and $G_w/G_w^\circ \cong \mu_4$ as group schemes over F, where μ_4 denotes the group scheme of fourth roots of unity. Now $H^1(G) = \{1\}$, where H^1 denotes the Galois cohomology set with respect to F, and it follows from a basic theorem of Igusa [5] that $G_F\backslash V_F'$ may be identified with $H^1(G_w)$. Since $H^1(G_w^\circ) = \{1\}$, $H^1(G_w)$ may be regarded as a subset of $H^1(G_w/G_w^\circ)$, and the fact that the homomorphism $G_w \to G_w/G_w^\circ$ is split over F (by the map $(\tau G_w^\circ)^k \mapsto \tau^k$) implies that $H^1(G_w)$ coincides with $H^1(G_w/G_w^\circ)$. It is well-known that $H^1(\mu_4) = F^\times/(F^\times)^4$ and so $G_F\backslash V_F'$ may be identified with $F^\times/(F^\times)^4$. In order to make this identification concrete, we now consider the basic relative invariants of V.

The space V has two basic relative invariants P_1 and P_2. The first has degree 15 and is associated to the character $\chi_1(g, h) = \det(g)^6 \det(h)^5$. Indeed, P_1 is simply the basic relative invariant polynomial of the space $(\mathrm{GL}(5) \times \mathrm{GL}(3), \wedge^2\mathrm{Aff}^5 \otimes \mathrm{Aff}^3)$. This relative invariant was first constructed by Gyoja [3] and subsequently considered by Ochiai [14]. The second has degree 12 and is associated with the character $\chi_2(g, h) = \det(g)^4 \det(h)^4$. All relative invariants of 2-simple prehomogeneous vector spaces of type I have recently been constructed by Kogiso et al. [12]. For the reader's convenience, we give simple uniform expressions for both of these relative invariants, using the notation of tensor invariant theory. The construction is based upon a relatively equivariant map $Z : V \to \vee^2\mathrm{Aff}^3$ of degree 5 given by

$$Z_{jk} = \frac{1}{160} \sum \boldsymbol{\varepsilon}^{i_3 i_4 i_5} \boldsymbol{\varepsilon}^{\alpha_1\beta_1\alpha_4\beta_4\alpha_3} \boldsymbol{\varepsilon}^{\beta_3\alpha_2\beta_2\alpha_5\beta_5} x_{\alpha_1\beta_1 j} x_{\alpha_2\beta_2 k} x_{\alpha_3\beta_3 i_3} x_{\alpha_4\beta_4 i_4} x_{\alpha_5\beta_5 i_5}\, ,$$

where we identify $\vee^2\mathrm{Aff}^3$ with the space of symmetric 3-by-3 matrices. In this expression, $x_{\alpha\beta i}$ is the (α, β)-entry in the matrix M_i, the greek indices run from 1 to 5 and the roman indices from 1 to 3, and the summation convention is in force. The reader may see [6] for a more detailed discussion of this notation and its interpretation. The entries in the matrix Z are polynomials in the variables $x_{\alpha\beta i}$ with integer coefficients and

$$Z(w) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

With respect to G, Z transforms via the equation

$$Z((g,h)x) = \det(g)^2 \det(h) h Z(x) h^t .$$

We may set $P_1(x) = \det(Z(x))$; thus normalized, $P_1 \in \mathbf{Z}[V]$ and $P_1(w) = 1$. We may also define

$$P_2(x) = -\frac{1}{2} \sum \boldsymbol{\varepsilon}^{i_1 i_3 j_1} \boldsymbol{\varepsilon}^{i_2 i_4 j_2} Z(x)_{i_1 i_2} Z(x)_{i_3 i_4} y_{j_1} y_{j_2} ,$$

where $y = (y_1, y_2, y_3)^t$; thus normalized, $P_2 \in \mathbf{Z}[V]$ and $P_2(w) = 1$. In classical terminology, $P_2(x)$ is simply the bordered determinant of $Z(x)$ and y. It is now routine to verify that the bijection between $G_F \backslash V'_F$ and $F^\times/(F^\times)^4$ that was derived above from Galois cohomology is given concretely by $G_F x \mapsto P_2(x)(F^\times)^4$. We choose orbital representatives x_a for $G_F \backslash V'_F$ labelled by the classes in $F^\times/(F^\times)^4$ in such a way that the class of $P_2(x_a)$ is a.

The last issue that must be considered in order to interpret the result of Theorem 6 in this case is the determination of the invariant $r(\tilde{G}_a)$, where G_a is the stabilizer of a point x_a. Note that the kernel of the representation in this example is $\{(\pm I_5, I_3)\}$ and so it is sufficient to determine the F-rank of G_a itself. Since $\mathrm{GL}(3)$ acts transitively on $\mathrm{Aff}^3 - \{0\}$ over F, we may assume that every orbital representative x_a has the form $[*, e_2]$. The map $G_w^\circ \to \mathrm{SL}(3)$ given by projection onto the second factor is injective and it follows that the same is true of the map $G_a^\circ \to \mathrm{SL}(3)$ for any a. Thus it suffices to determine the F-rank of the identity component of the isotropy group of the point $[Z(x_a), e_2]$ in the space $\vee^2 \mathrm{Aff}^3 \oplus \mathrm{Aff}^3$ with its natural $\mathrm{SL}(3)$ action. The identity component of the isotropy group of this point is easily seen to be isomorphic to $\mathrm{SO}(\Psi_a)$, where Ψ_a is the binary quadratic form with matrix

$$\Psi_a = \begin{pmatrix} Z_{11}(x_a) & Z_{13}(x_a) \\ Z_{13}(x_a) & Z_{33}(x_a) \end{pmatrix} .$$

A computation shows that $P_2(x_a)$ is precisely the discriminant of Ψ_a. The F-rank of $\mathrm{SO}(\Psi_a)$ is 1 if this discriminant is a square in F and 0 otherwise. Thus we have

$$r(G_a) = \begin{cases} 1 & \text{if } \ a \in (F^\times)^2/(F^\times)^4 , \\ 0 & \text{if } \ a \notin (F^\times)^2/(F^\times)^4 . \end{cases}$$

Note that $\chi_i^\iota = \chi_i$ for $i = 1, 2$ and so $\omega^\iota = \omega$ in this case. Also $\xi(\omega) = 1$ for all ω because $\eta = 1$. If we arrange the orbits so that those with $r(G_a) = 0$ precede those with $r(G_a) = 1$, then Theorem 6 implies that the Γ-matrix of (G, V) takes the form

$$\Gamma = \begin{pmatrix} \Delta_1 & * \\ 0 & \Delta_2 \end{pmatrix} ,$$

where Δ_1 and Δ_2 are almost symmetric matrices of the appropriate sizes.

REFERENCES

[1] B. A. DATSKOVSKY, On Dirichlet series whose coefficients are class numbers of binary quadratic forms, Nagoya Math. J. 142 (1996), 95–132.

[2] B. A. DATSKOVSKY AND D. J. WRIGHT, The adelic zeta function associated to the space of binary cubic forms. II. Local theory, J. Reine Angew. Math. 367 (1986), 27–75.

[3] A. GYOJA, Construction of invariants, Tsukuba J. Math. 14 (1990), 437–457.

[4] J.-I. IGUSA, Some results on p-adic complex powers, Amer. J. Math. 106 (1984), 1013–1032.

[5] J.-I. IGUSA, On a certain class of prehomogeneous vector spaces, J. Pure Appl. Algebra 47 (1987), 265–282.

[6] A. C. KABLE AND A. YUKIE, A construction of quintic rings, Nagoya Math. J. 173 (2004), 163–203.

[7] T. KIMURA, The b-functions and holonomy diagrams of irreducible regular prehomogeneous vector spaces, Nagoya Math. J. 85 (1982), 1–80.

[8] T. KIMURA, Introduction to prehomogeneous vector spaces, Transl. Math. Monogr. 215, Amer. Math. Soc., Providence, R.I., 2003.

[9] T. KIMURA, S.-I. KASAI, M. INUZUKA AND O. YASUKURA, A classification of 2-simple prehomogeneous vector spaces of type I, J. Algebra 114 (1988), 369–400.

[10] T. KIMURA, S.-I. KASAI AND O. YASUKURA, A classification of the representations of reductive algebraic groups which admit only a finite number of orbits, Amer. J. Math. 108 (1986), 643–692.

[11] T. KIMURA AND M. MURO, On some series of regular irreducible prehomogeneous vector spaces, Proc. Japan Acad. Ser. A Math. Sci. 55 (1979), 384–389.

[12] T. KOGISO, G. MIYABE, M. KOBAYASHI AND T. KIMURA, Explicit construction of relative invariants for regular 2-simple prehomogeneous vector spaces of type I, preprint.

[13] S. LANG, Algebraic number theory, Graduate Texts in Mathematics 110, Springer, New York, 1986.

[14] H. OCHIAI, Quotients of some prehomogeneous vector spaces, J. Algebra 192 (1997), 61–73.

[15] F. SATO, On functional equations of zeta distributions, Automorphic forms and geometry of arithmetic varieties, Adv. Stud. Pure Math. 15, 465–508, Academic Press, Boston, Mass., 1989.

[16] M. SATO AND T. KIMURA, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J. 65 (1977), 1–155.

[17] J.-P. SERRE, Galois cohomology, Springer, Berlin, 1997.

[18] D. J. WRIGHT AND A. YUKIE, Prehomogeneous vector spaces and field extensions, Invent. Math. 110 (1992), 283–314.

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