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QUADRATIC VANISHING CYCLES, REDUCTION CURVES AND REDUCTION OF THE MONODROMY GROUP OF PLANE CURVE SINGULARITIES

NORBERT A'CAMPO

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Abstract. The geometric local monodromy of a plane curve singularity is a diffeomorphism of a compact oriented surface with non empty boundary. The monodromy diffeomorphism is a product of right Dehn twists, where the number of factors is equal to the rank of the first homology of the surface. The core curves of the Dehn twists are quadratic vanishing cycles of the singularity. Moreover, the monodromy diffeomorphism decomposes along reduction curves into pieces, which are invariant, such that the restriction of the monodromy on each piece is isotopic to a diffeomorphism of finite order. In this paper we determine the mutual positions of the core curves of the Dehn twists, which appear in the decomposition of the monodromy, together with the positions of the reduction curves of the monodromy.

1. Introduction. Let $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ be a polynomial mapping with $f(0) = 0$ having an isolated singularity at $0 \in \boldsymbol{C}^2$. Let $B \subset \boldsymbol{C}^2$ be a Milnor ball for the singularity. A disc $D \subset \boldsymbol{C}$ with center $0 \in \boldsymbol{C}$ is called *small relative to* B if for all $s \in D$ the intersection of ∂B with $\{p \in \boldsymbol{C}^2 \mid f(p) = s\}$ is a transversal intersection of smooth differential manifolds.

A *deformation of* f is a polynomial mapping $f : \boldsymbol{C}^2 \times \boldsymbol{C} \to \boldsymbol{C}$, $(p, t) \mapsto f_t(p)$, with $f_0(p) = f(p)$. A deformation of f is *small relative to the Milnor ball* B *and the disk* D if for all $s \in D$ and all $t \in \boldsymbol{C}$, $|t| \leq 1$, the intersection of ∂B with $\{p \in \boldsymbol{C}^2 \mid f_t(p) = s\}$ is a transversal intersection of smooth differential manifolds and moreover, the map f_t is regular on $f_t^{-1}(\partial D) \cap B$.

Let $d \in D$ be the intersection point of ∂D with the positive real axis. The fibers $f_t^{-1}(d) \cap B$, $|t| \leq 1$, are all diffeomorphic to the Milnor fiber $F := f^{-1}(d) \cap B$, and the diffeomorphism can be chosen to be unique up to isotopy if one follows the family $(f_{st}^{-1}(d))_{s \in [0,1]}$.

A small deformation f_t of f is a *morsification of* f if for all $t \in]0, 1]$, the map f_t in $B \cap f_t^{-1}(D)$ has only singularities of Morse type. *A real morsification* of a singularity with a real defining equation f is a morsification f_t with f_t real, such that all critical points of the maps $f_t : B \to \boldsymbol{C}$ are real and the level set $f_t^{-1}(0) \cap B$, $t \in]0, 1]$, contains all the saddle points of the restriction of f_t to $D^2 = B \cap \boldsymbol{R}^2$. The level set $f_t^{-1}(0) \cap D^2$ is then a divide for the singularity of f, see [AC2,G-Z].

Let f_t be a morsification, such that for some $\bar{t} \in]0, 1]$ the restriction to B has $\mu(f)$ distinct critical values. A simply closed essential curve c on the surface $f_{\bar{t}}^{-1}(d) \cap B$ is a *quadratic vanishing cycle* if there exists a continuously differentiable path $\gamma : [0, 1] \to D$

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such that $\gamma(0) = d$, $\gamma(u), u \in [0, 1[$, are regular values of the restriction of $f_{\bar{t}}$ to B and in $f_{\bar{t}}^{-1}([0, 1])$ the curve c is null homotopic. A simply closed essential curve c on the Milnor fiber F is a quadratic vanishing cycle if there exists a morsification f_t and $|\bar{t}| \leq 1, \bar{t} \neq 0$, such that c is mapped to a quadratic vanishing cycle on $f_{\bar{t}}^{-1}(d) \cap B$ by the natural diffeomorphism class.

The geometric monodromy group of the singularity of f is the image Γ_f of the geometric monodromy representation in the mapping class group of the surface with boundary $(F, \partial F)$ of the fundamental group of the complement (as germ) of the discriminant in the unfolding of the singularity of f [T2]. It follows from the results and constructions of Egbert Brieskorn [B] and a theorem of Helmut Hamm and Lê Dũng Tráng [H-L], that the geometric monodromy group is generated by the Dehn twists whose core curves are the curves of any distinguished system of vanishing cycles. Distinguished systems of vanishing cycles for isolated plane curve singularities have been constructed by [AC2, G-Z]. The set of quadratic vanishing cycles is an orbit of the action of the geometric monodromy group of the singularity. It follows that in principle the set of quadratic vanishing cycles is known.

A *reduction curve* for an isotopy class of diffeomorphisms T of a surface F is an essential simply closed curve a on F, such that for some integer $N > 0$ the curves a and $T^N(a)$ are homotopic and moreover, if $N > 0$ is chosen minimal, the curves $T^i(a), 0 \leq i \leq N-1$, can be made pairwise disjoint by an isotopy of T. The set $\{T^i(a) \mid 0 \leq i \leq N-1\}$ is then called a reduction system for T.

The set of reduction curves for the geometric monodromy of a singularity with one local branch is given by [AC1]. The purpose of the present paper is to describe the relative position of a distinguished base of vanishing cycles and the set of reduction curves for the monodromy diffeomorphism of isolated plane curve singularities.

Roughly speaking, we wish to give a picture of the Milnor fiber of a plane curve singularity, that shows both a distinguished system of quadratic vanishing cycles and the reduction curves of the geometric monodromy. The method uses the model of the Milnor fiber with monodromy coming from a divide of the singularity [AC2-5,G-Z]. It turns out that the divides constructed by Sabir Gusein-Zade are most suited for this purpose.

In Section 4 we will give an application of our constructions to the study of the geometric monodromy group of an irreducible plane curve singularity with several essential Puiseux pairs. It is proved that the geometric monodromy group of an irreducible plane curve singularity in a natural way contains a product of many copies of the geometric monodromy groups of its companion singularities.

I like to thank the referee for his valuable comments.

2. Building blocks and cabling. It was observed by René Thom that the Chebyshev polynomials $T : \boldsymbol{C} \to \boldsymbol{C}$ up to affine equivalence of functions are precisely the polynomial mappings from $\boldsymbol{C}$ to $\boldsymbol{C}$ with two or less critical values and with only quadratic singularities [T1]. The standard Chebyshev polynomial $T(p, z)$ of degree p has critical values $+1, -1$, the symmetry $T(p, z) = (-1)^p T(p, -z)$, and the coefficient of z^p is 2^{p-1}. For $p = 1$, the map $T(p, z)$ has no critical values and for $p = 2$, the map $T(p, z)$ has only the critical value -1. The Chebyshev polynomial $T(p, z)$ satisfies the identity $T(p, \cos(x)) = \cos(px)$, and its restriction to $[-1, 1]$ is defined by $T(p, t) = \cos(p \arccos(t))$.

Sabir Gusein-Zade has constructed real morsifications for real plane curve singularities with Chebyshev polynomials. The building blocks for his construction are the real morsification for the map $f(x, y) = 2^{p-1}x^p - 2^{q-1}y^q$ given by the family

$$f_s(x, y) = s^{pq}(T(p, x/s^q) - T(q, y/s^p)),\ s \in [0, 1]\,.$$

For each $s \in]0, 1]$ the function $f_s : \boldsymbol{C}^2 \to \boldsymbol{C}$ has $\mu_f = (p-1)(q-1)$ quadratic singularities all at points with real coordinates, its critical values are contained in $\{-2s^{pq}, 0, 2s^{pq}\}$ and $\lim_{s \to 0, s>0} f_s = f$.

If the exponents p and q are relatively prime to each other, the level set $\{f(x, y) = 0\}$ can be parametrized by the monomial map

$$t \in \boldsymbol{C} \mapsto (t^q/2^{p-1}, t^p/2^{q-1}) \in \boldsymbol{C}^2\,.$$

It is a miracle that in this case the Chebyshev polynomials can be used to parametrize the level sets $\{f_s(x, y) = 0\}$ as well. The map

$$t \in \boldsymbol{C} \mapsto (s^q T(q, t/s), s^p T(p, t/s)) \in \boldsymbol{C}^2$$

parametrizes the level set $\{f_s(x, y) = 0\}$. We recall that a divide is the image of a generic relative immersion of a compact 1-manifold in the unit disk D^2 in $\boldsymbol{R}^2$. For each $s \in]0, 1]$ the intersection $P_{p,q;s} := \{f_s(x, y) = 0\} \cap D^2$ with the unit disk D^2 is a divide for the singularity of $\{2^{p-1}x^p - 2^{q-1}y^q = 0\}$ at $0 \in \boldsymbol{C}^2$.

The curve $P_{p,q} := \{f_1(x, y) = 0\}$ can be drawn in a rectangular box as in Figure 1. As a first type of building block we will need the box $B := [-1, 1] \times [-1, 1]$ with the curve $P_{p,q}$. If $(p, q) = 1$ holds, the curve $P_{p,q}$ is the image of

$$T_{p,q} : [-1, 1] \to B, T_{p,q}(t) := (T(p, t), T(q, t))\,,$$

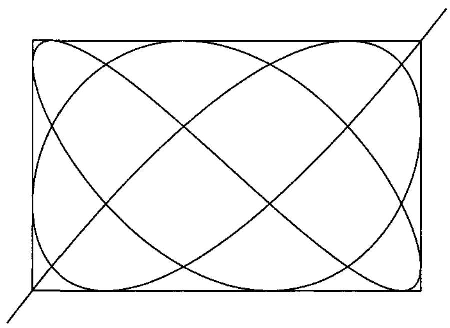

FIGURE 1. Divide in box $[-1, 1] \times [-1, 1]$ for $2^6x^7 - 2^4y^5 = 0$.

which leaves the box through the corners. In general the immersed curve has several components, which are immersions of the interval or of the circle. At most two components are immersions of the interval, which leave the box through the corners.

Let P be any divide having one branch given by an immersion $\gamma : [-1, 1] \to D^2$. We assume that the speed vector $\dot\gamma(t)$ and the position vector $\gamma(t)$ are proportional at $t = \pm 1$, i.e., the divide P meets ∂D^2 at right angles. Let $N\gamma : [-1, 1] \times [-1, 1] \to D$ be the corresponding immersion of a rectangular box, i.e., the restriction of $N\gamma$ to $[-1, 1] \times \{0\}$ is the immersion γ and the image of $N\gamma$ is in a small tubular neighborhood of P. For instance, for a small value of the parameter $\eta \in \boldsymbol{R}_{>0}$ the following expression defines an immersion $N\gamma : B \to \boldsymbol{R}^2$ of the rectangular box $B := [-1, 1] \times [-1, 1]$:

$$N\gamma(s,t) := \gamma(t) + s\eta \frac{J(\dot\gamma(t))}{\|\dot\gamma(t)\|},$$

where J is the rotation of $\boldsymbol{R}^2$ over $\pi/2$. The four corners $N\gamma(\pm 1, \pm 1)$ are on the circle of radius $\sigma := \sqrt{1+\eta^2}$. We finally define

$$N_\eta\gamma(s,t) := \frac{1}{\sigma}\left(\gamma(t) + s\eta \frac{J(\dot\gamma(t))}{\|\dot\gamma(t)\|}\right),$$

that will be an immersion $N\gamma : B \to D^2$ mapping the corners of the box B into ∂D^2.

We will denote by $P_{p,q} * P$ the divide in D^2, which is the image by $N\gamma : B \to D^2$ of $P_{p,q} \subset B$, see Figure 2. The number of double points $\delta(P_{p,q} * P)$ of $P_{p,q} * P$ is computed inductively from the number of double points $\delta(P)$ of P by:

$$\delta(P_{p,q} * P) = (p-1)(q-1)/2 + \delta(P)p^2 .$$

Let $R_\eta\gamma$ be the union of the image of $N_\eta\gamma$ with the two chordal caps at the endpoints of γ. The connected components of $D^2 \setminus R_\eta\gamma$ correspond via inclusion to the connected components of $D^2 \setminus P$. We declare a connected component of $D^2 \setminus P_{p,q} * P$ to be signed by $+$, if the component contains a component of $D^2 \setminus R_\eta\gamma$ that corresponds to a $+$ component of $D^2 \setminus P$. In this case we call the connected component of $D^2 \setminus R_\eta\gamma$ a *P_+-component.* Observe that there exists a chess board sign distribution for the components of $D^2 \setminus R_\eta\gamma$ that makes P_+-components indeed to $+$ components.

The field $\Phi_{p,q}$ of cones on the box $B \subset \boldsymbol{R}^2$ is the subset in the tangent space of TB given by:

$$\Phi_{p,q} := \{(x,u) \in TB \mid |\langle u, e_1\rangle_{\boldsymbol{R}^2}| \geq \cos(\alpha(x))\|u\|\},$$

where $e_1 = (1, 0) \in \boldsymbol{R}^2$ and $\alpha : B \to \boldsymbol{R}$ is a function such that for every $(x, u) \in TB$ with $x \in P_{p,q}$ and $u \in T_x P_{p,q}$ we have the equality

$$|\langle u, e_1\rangle_{\boldsymbol{R}^2}| = \cos(\alpha(x))\|u\|.$$

Moreover, α has the boundary values $\alpha(\pm 1, t) = 0$ and $\alpha(s, \pm 1) = \pi/2$. We interpolate the function α on B by upper and lower convexity, i.e., such that $\partial^2\alpha/\partial t^2 < 0$ and $\partial^2\alpha/\partial s^2 > 0$. The definition of $\alpha(x)$ seems to be conflicting at the double points of the curve $P_{p,q}$; at a double point $x = (x_1, x_2)$ of the curve $P_{p,q}$ the two tangents lines to $P_{p,q}$ have opposite slopes $\tan(\alpha(x))$ and $-\tan(\alpha(x))$, since the curve $P_{p,q}$ is defined by the equation

$$T(q, x_1) - T(p, x_2) = 0$$

that separates the variables. For example, a nice such function α is given by:

$$\alpha(x_1, x_2) := \arctan\left(\frac{q\sqrt{1-x_2^2}}{p\sqrt{1-x_1^2}}\right).$$

The interest of the field $\Phi_{p,q}$ comes from the following lemma, that is immediate from the definitions. The differential $dN_\eta\gamma : TB \to T\boldsymbol{R}^2$ maps the field of sectors $\Phi_{p,q} \subset TB$ to a subset in $T\boldsymbol{R}^2$, which will be denoted by $\Phi_{\eta,p,q}\gamma$.

LEMMA. *Let the image of $\gamma : [-1, 1] \to D$ be a divide P, that meets ∂D^2 at right angles. For $\eta > 0$ small enough, the intersection $S^3 \cap \Phi_{\eta,p,q}\gamma$ of subsets in $T\boldsymbol{R}^2$ is a tubular neighborhood of the knot $L(P)$. The composition of $P_{p,q} : [-1, 1] \to B := [-1, 1]\times[-1, 1]$ and of $N\gamma : B \to D^2$ is again a divide, whose knot is a torus cable knot of the knot $L(P)$.*

For small η the image $\Phi_{\eta,p,q}\gamma$ contains those vectors that have feet near P and form a small angle with the tangent vectors of the divide P.

A sequence of pairs of integers $(a_i, b_i)_{1\leq i\leq k}$ is a *sequence of essential Puiseux pairs of an irreducible plane curve singularity* if the inequalities $2 \leq a_i < b_i$ and $b_i/a_1a_2\cdots a_i < b_{i+1}/a_1a_2\cdots a_ia_{i+1}$ are verified and if moreover, the integers b_i and $a_1a_2\cdots a_i$ are relatively prime. A sequence of essential Puiseux pairs defines a family of topologically equivalent singularities. A specific member $f_k(x, y)$ of this family is obtained from the Puiseux expansion with fractional and strictly increasing exponents

$$y = x^{b_1/a_1} + x^{b_2/a_1a_2} + \cdots + x^{b_k/a_1a_2\cdots a_k}$$

by the rule, which takes into account the ramification of $x^{1/a_1a_2\cdots a_k}$,

$$f_k(x, y) = \prod_\theta (y - \theta^{a_2\cdots a_k b_1} - \theta^{a_3\cdots a_k b_2} - \cdots - \theta^{b_k}),$$

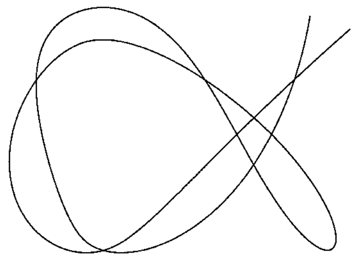

FIGURE 2. The divide $P_{2,9} * P_{2,3}$.

where θ runs over the $a_1 a_2 \cdots a_k$ roots of $z^{a_1 a_2 \ldots a_k} - x = 0$ in the algebraic closure of the field $\boldsymbol{C}((x))$. The coefficients of the polynomial $f_k(x, y)$ are integers.

For example, the Puiseux expansion $y = x^{3/2} + x^{7/4}$ leads to the polynomial $f_{(2,3),(4,7)} = (y^2 - x^3)^2 - 4x^5 y - x^7$ and the Puiseux expansion $y = x^{3/2} + x^{11/6}$ to the polynomial $(y^2 - x^3)^3 - 6x^7 y^2 - 2x^{10} - x^{11}$.

Let $\{f_{a,b}(x, y) = 0\}$ be a singularity having one branch and with essential Puiseux pairs $(a_i, b_i)_{1 \leq i \leq n}$. The theorem of Gusein-Zade [G-Z] very efficiently describes a divide for the singularity $\{f_{a,b}(x, y) = 0\}$ in a closed form, namely the iteratively composed divide

$$P_{a_n, b'_n} * \cdots * P_{a_2, b'_2} * P_{a_1, b_1} ,$$

where the numbers $b'_2, \ldots, b'_n$ can be computed recursively, as we will show here below. We denote by S_k, $1 \leq k \leq n$, the divide

$$P_{a_k, b'_k} * P_{a_{k-1}, b'_{k-1}} * \cdots * P_{a_2, b'_2} * P_{a_1, b_1} ,$$

and let $f_k(x, y)$ be a specific equation for a singularity with essential Puiseux pairs $(a_i, b_i)_{1 \leq i \leq k}$.

Remember that the product $a_1 a_2 \ldots a_k$ is the multiplicity of the singularity at 0 of the curve $\{f_k(x, y) = 0\}$ and that the linking number λ_k of $L(S_k)$ and $L(S_{k-1})$ in S^3 can be computed recursively by:

$$\lambda_1 = b_1 , \quad \lambda_{k+1} = b_{k+1} - b_k a_{k+1} + \lambda_k a_k a_{k+1} .$$

The linking number λ_k is equal to the intersection multiplicity

$$\dim \boldsymbol{C}[[x, y]]/(f_k(x, y), f_{k-1}(x, y))$$

at 0 of the curves $\{f_k(x, y) = 0\}$ and $\{f_{k-1}(x, y) = 0\}$. We have also for the linking number λ_k an interpretation in terms of divides

$$\lambda_k = \#(S_k \cap S_{k-1}) = 2a_k \delta(S_{k-1}) + b'_k .$$

To see the first equality, observe that the fiber over $\sqrt{-1}$ of the knot of a divide P, as constructed in [AC5], intersects the knot of a divide Q positively in $\#(P \cap Q)$ points, provided that the divides P and Q are mutually in general position. Remembering that we already have computed recursively the numbers $\delta(S_k)$ and $\#(S_k \cap S_{k-1})$, we conclude that b'_k too can be computed recursively.

For example, for the Puiseux expansion $y = x^{3/2} + x^{7/4}$ we have: $\lambda_1 = 3$, $\lambda_2 = 7 - 3 \cdot 2 + 3 \cdot 2 \cdot 2 = 13$, $b'_2 = 13 - 2 \cdot 2 \cdot 1 = 9$. Hence, the divide for the irreducible singularity with Puiseux expansion $y = x^{3/2} + x^{7/4}$ is the divide $P_{2,9} * P_{2,3}$, see Figure 2. For the Puiseux expansion $y = x^{3/2} + x^{11/6}$ we found: $\lambda_1 = 3$, $\lambda_2 = 11 - 3 \cdot 3 + 3 \cdot 2 \cdot 3 = 20$, $b'_2 = 20 - 2 \cdot 3 \cdot 1 = 14$. Hence, the divide for its singularity $\{(y^2 - x^3)^3 - 6x^7 y^2 - 2x^{10} - x^{11} = 0\}$ is $P_{3,14} * P_{2,3}$, see Figure 3.

An iterated $*$-composition of divides has to be evaluated from the right to the left.

Using [AC4-5], we can read off from this divide the Milnor fibration of the singularity $\{f_{a,b}(x, y) = 0\}$. In particular, we can describe the Milnor fiber with a distinguished base of quadratic vanishing cycles, see the next section. Using the above iterated cabling construction, in Section 4 we will also read off from the divide the reduction of the geometric monodromy of an irreducible plane curve singularity, as described in [AC1]. For instance, intersection numbers in the sense of Nielsen of quadratic vanishing cycles and reduction cycles can be computed.

In general, for an isolated singularity of a real polynomial $f(x, y)$ having several local branches, the divide $\{f_1(x, y) = 0\} \cap D$ of a real morsification $f_t(x, y)$ may have immersed circles as componants. The above cabling construction $P_{p,q} * P$ does not work if the divide P consists of an immersed circle. Of course, if one is willing to change the equation of the singularity to an equation, which defines a topologically equivalent singularity and which has only real local branches, one will only have to deal with divides consisting of immersed intervals. If we do not want to change the real equation, we will need a second type of building blocks for a cabling construction, see Figure 4.

These building blocks are the divides $L_{p,q}$ in the annular region $A := \{(x, y) \in D^2 \mid 1/4 \leq \sqrt{x^2 + y^2} \leq 3/4\}$. If for the integers $(p, q) = 1$ holds, the divide $L_{p,q}$ is the Lissajous

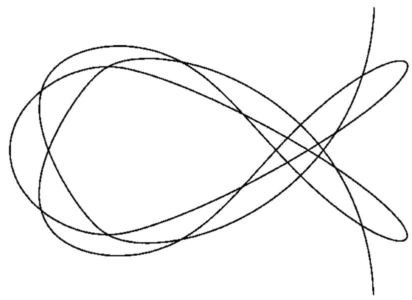

FIGURE 3. The divide $P_{3,14} * P_{2,3}$ for $(y^2 - x^3)^3 - 6x^7y^2 - 2x^{10} - x^{11}$.

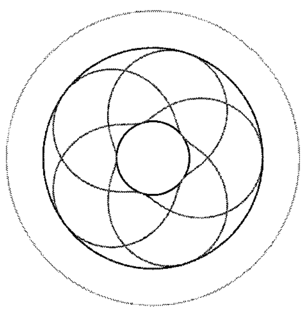

FIGURE 4. The building block $L_{3,5}$ of Lissajous type.

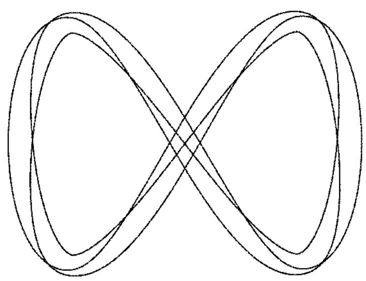

FIGURE 5. The cabling of $L_{3,5} * P_{2,4}$.

curve

$$s \in [0,1] \mapsto (1/2 + 1/4\sin(2\pi qs))(\sin(2\pi ps), \cos(2\pi ps))$$

in $A \subset D^2$. The curve $L_{p,q}$ has q-fold rotational symmetry. If $(p,q) = r > 1$, the divide $L_{p,q}$ will be defined as the union of r rotated copies of $L_{p/r,q/r}$ with rotations of angles $2\pi k/p$, $k = 0, \ldots, r-1$, of D^2. Again, the system of curves $L_{p,q}$ has a q-fold rotational symmetry.

The star-product $L_{p,q} * P$ can be defined as above if the divide P consists of one immersed circle. We leave many details to the reader. The two types of building blocks $P_{p,q}$ and $L_{p,q}$ together with the star-products $P_{p,q} * P$ and $L_{p,q} * P$ will allow one to describe the iterated cablings of real plane curve singularities in general. See Figure 5.

3. Visualization of the vanishing cycles for a divide. Let P be a connected divide and let $\pi_P : S^3 \setminus L(P) \to S^1$ be the fibration of the complement of the link $L(P)$ over S^1 as in [AC4-5]. The fibration map is given with the help of an auxiliary Morse function $f_P : D^2 \to \boldsymbol{R}$. The fiber $F_P := \pi_P^{-1}(1)$ above $1 \in S^1$ projects to the positive components of the complement of P in D^2. One has that the closure of

$$\{(x,u) \in TD^2 \mid f_P(x) > 0, (df_P)_x(u) = 0, \|x\|^2 + \|u\|^2 = 1\}$$

in $S^3 \setminus L(P)$ is the fiber surface F_P.

To each critical point of $f_P : D^2 \to \boldsymbol{R}$ corresponds a vanishing cycle on the surface F_P. In the case, where the divide P is a divide of a singularity, the surface F_P is a model for the Milnor fiber and the system of vanishing cycles on F_P is a model for a distinguished system of quadratic vanishing cycles of the singularity.

Let M be a maximum of f_P. The vanishing cycle δ_M is the non-oriented simply closed curve on the fiber F_P, see Figure 6,

$$\delta_M := \{u \in T_M D^2 \mid \|M\|^2 + \|u\|^2 = 1\}.$$

Let c be a crossing point of P. The point c is a saddle type singularity of f_P. The vanishing cycle is the non-oriented simply closed curve δ_c on F_P that results from the following construction. Put $P_+ := \{x \in D^2 \mid f_P(x) > 0\}$. Let $g_c \subset P_+ \cup \{c\}$ be the singular gradient

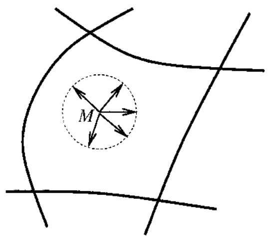

FIGURE 6. The vanishing cycle δ_M for a maximum.

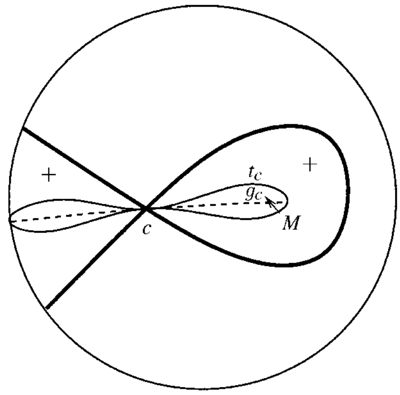

FIGURE 7. Tear for vanishing cycle δ_c for a saddle point.

line through c, for which the endpoints are a maximum of f_P or a point in ∂D^2. We splice g_c and get a double tear $t_c \subset P_+ \cup \{c\}$ as in Figure 7. The tear t_c is a closed curve that has at c a non-degenerate tangency with g_c from both sides. Moreover, t_c is perpendicular to g_c at the endpoints of g_c, if the endpoint is a maximum of f_P and else t_c has a tangency with ∂D^2. The vanishing cycle δ_c is the closure in F_P of the set

$$\{(x, u) \in F_P \mid x \in t_c,\ (df_P)_x \neq 0,\ u \text{ points to the inside of the tear } t_c\}.$$

Let m be a minimum of f_P. The following is a description of the vanishing cycle δ_m on F_P. The projection of δ_m in D^2 is a simply closed curve t_m in $P_+ \cup \{c_1, c_2, \ldots, c_k\}$, where $\{c_1, c_2, \ldots, c_k\}$ is the list of the double points of P that lie in the closure of the region of m, see Figure 8. The curve t_m and the singular gradient line g_{c_i}, $1 \leq i \leq k$, coincide in a neighborhood of c_i. Moreover, if the endpoint of g_{c_i} is a maximum M, the curve t_m leaves transversally the tear t_{c_i} at M and enters transversally the next tear $t_{c_{i\pm1}}$. If the endpoint of g_{c_i} is on the boundary of D, the curve which leaves the tear t_{c_i} becomes tangent to the boundary of D^2 and enters the next tear, see Figure 8. The vanishing cycle δ_m is the non-oriented simply

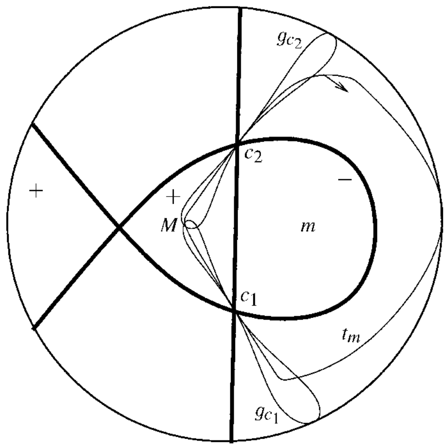

FIGURE 8. The vanishing cycle δ_M for a minimum.

closed curve on F_P, that is, the closure of

$$\{(x, u) \in F_P \mid x \in t_m,\ u \text{ points inwards to the disk bounded by } t_m\}.$$

The link of a divide P is naturally oriented by the following recipe. Let γ :] 0, 1[$\to D^2$ be a local regular parametrization of P. The orientation of $L(P)$ is such that the map $t \in]0,1[\mapsto (\gamma(t), \lambda(t)\dot{\gamma}(t)) \in L(P)$ is oriented. Here $\lambda(t)$ is a positive scalar function, which ensures that the map takes its values in $L(P)$. For a connected divide P we orient its fiber surface F_P such that the oriented boundary of $(F_P \cup L(P), L(P))$ coincides with the orientation of $L(P)$. Vanishing cycles δ_c, where c is a critical point of f_P, do not carry a natural orientation, since the third power of the geometric monodromy of the singularity $\{x^3 - y^2 = 0\}$ reverses the orientations of the vanishing cycles. The main use of a vanishing cycle δ_c in this paper is through the associated right Dehn twist Δ_c of F_P. This use does not require an orientation for the cycles δ_c, but requires an orientation of the surface F_P. Moreover, we orient the tangent space $TD = D \times \boldsymbol{R}^2$ and its unit sphere S^3 as boundary of its unit ball, such that the linking numbers of links $L(P_1)$ and $L(P_2)$ are positive for generic pairs of divides P_1 and P_2. In fact, the orientation TD^2 is opposite to the orientation as tangent space. We have $Lk_{S^3}(L(P_1), L(P_2)) = \#(P_1 \cap P_2)$, a fact which was already used in the previous section.

4. Reduction cycles and reduction tori. We consider a divide of the form $Q := P_{p,q} * P$, where the divide P is given by an immersion $\gamma : [-1, 1] \to D^2$. So, the divide Q is the image of $N\gamma \circ T_{p,q}$. If we change the immersion by a reparametrization $\gamma_1 := \gamma \circ \phi$, where ϕ is an oriented diffeomorphism of $[-1, 1]$, the divide $Q_1 := N\gamma_1 \circ \phi([-1, 1])$ is isotopic to Q by an transversal isotopy, which does not change the type of its knot. By choosing ϕ appropriately and η small, one can achieve that each double point of P corresponds to a

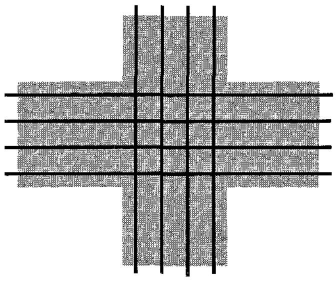

FIGURE 9. Manhattan: crossing of the box with 4 by 4 strands.

system of p^2 double points of the divide Q_1, which look like the intersection points of a system of p almost parallel lines with an other system of p almost parallel lines, see Figure 2, where $p = 2$, Figure 3, where $p = 3$ and Figure 9.

We may assume that the divide Q for each double point of P already has a grid of p^2 intersections.

We will construct reduction curves for the monodromy of the knot $L(Q)$ by the method of [AC1]. The reduction curves of the cabling $P_{p,q} *_\eta P$ are the intersection of the fiber F_Q over $1 \in S^1$ of the fibration on the complement of the knot $L(Q)$ with the boundary of a regular tubular neighborhood U of the closed tubular neighborhood V of the knot $L(P)$ for which $L(Q) \subset \partial V$ holds. The intersection $(F_Q \cap \partial U) \subset F_Q$ is indeed a system of reduction curves provided that the torus ∂U is transversal to the fibration of the knot $L(Q)$.

Assume that the tubular neighborhood V was constructed with the field $\Phi_{p,q}$ and a particular value of the parameter η. The same field of sectors, but a slightly bigger parameter value η' yields a tubular neighborhood U of V in S^3. The construction of the fibration will be done as in [AC5]. The main choice for the construction of the fibration for the knot $L(Q)$ is a Morse function $f_Q : D^2 \to \boldsymbol{R}$ with $f_Q^{-1}(0) = Q$. For our purpose here, where we must achieve the above transversality, we will choose f_Q as follows. First, after applying a regular transversal small isotopy, we may assume that the divide P has perpendicular rectilinear crossings. Next, we consider a Morse function $f_P : D^2 \to \boldsymbol{R}$ for the divide P that is Euclidian near its crossings. Let the fibration on the complement of the knot $L(P)$ be $\pi_{P,\eta} : S^3 \setminus L(P) \to S^1$, where $\pi_P(x,u) := \theta_P(x,u)/|\theta_P(x,u)|$ and

$$\theta_{P,\lambda}(x,u) := f_P(x) + i\lambda^{-1} df_P(x)(u) - \frac{1}{2}\lambda^{-2}\chi(x) H_{f_P}(x)(u,u)\,.$$

The function $\chi : D^2 \to \boldsymbol{R}$ is a bump function at the crossing points of P and λ is a big real parameter. We now choose a small positive real number v, such that $\{x \in D^2 \mid |f_P(x)| \le v\}$ is a regular tubular neighborhood of P, that meets each component of $\{x \in D^2 \mid \chi(x) = 1\}$. Next we choose $\eta' > 0$ such the corners of $N_{\eta'}\gamma(B)$ are in $\{|f_P(x)| = v\}$, i.e., $\eta'^2 = v$. We construct the torus knot $L(Q)$ with $Q := P_{p,q} *_\eta P$, where $0 < \eta < \eta'$. Since $Q \subset$

$\{|f_P(x)| < v\}$ holds, we can construct a Morse function $f_Q : D^2 \to \boldsymbol{R}$ for the divide Q, such that on $\{|f_P(x)| \geq v\}$ the function f_Q is constant along the level sets of f_P.

The following theorem follows directly from Lemme 2, [AC1, page 153] and the above construction.

THEOREM 1. *The torus $\partial \Phi_{\eta', p, q} \gamma$ is transversal to the fibration deduced from f_Q on the complement of the knot $L(Q)$. The intersection*

$$\partial \Phi_{\eta', p, q} \gamma \cap F_Q$$

is a system of p closed curves on the fiber F_Q, which is a reduction of the monodromy of $L(Q)$.

With a few examples, we now explain how to depict in the Milnor fiber a distinguished system of vanishing cycles and the reduction curves for the monodromy of a singularity, for which a divide of the form $Q = P_{p,q} * P$ is given.

The fiber F_Q with a distinguished system of vanishing cycles is already constructed in Section 3.

For a (p, q) cabling the reduction system consists of p simply closed curves on the fiber F_Q. Each of them cuts out from F_Q a surface diffeomorphic to the fiber F_P of the divide P. The p copies of F_P in F_Q are cyclicly permuted by the monodromy T_Q.

One of those copies can be visualized more easily as follows. Let $\{x \in D^2 \mid f_P(x) \geq v\}$. For each double point c of P we connect the two components of $\{x \in D^2 \mid f_P(x) \geq v\}$, that are incident with the double point c, by a special bridge which projects diagonally through the Manhattan part of the divide Q, that corresponds to c. The projection of the bridge is a twisted strip S_c in D^2, that realizes a boundary connected sum of the P_+-components. The twist points of the strip S_c are precisely the critical points of f_Q, that lie on the diagonal. The boundary of S_c consists of two smooth curves, that intersect each other transversally and that also intersect the divide Q transversally.

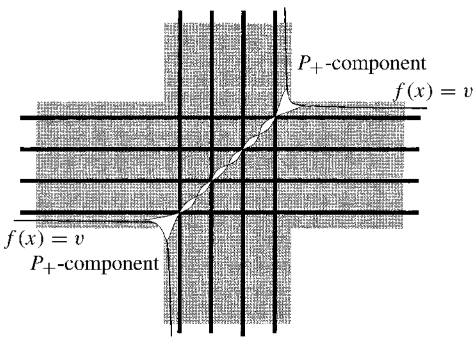

FIGURE 10. Bridge through Manhattan.

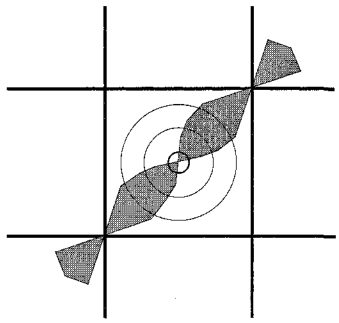

FIGURE 11. Bridge through a block of Manhattan.

Let C be the union of the projections S_c of the bridges with $\{x \in D^2 \mid f_P(x) \geq v\}$, see Figure 10. In Figure 11, we have zoomed out one block to show more details. The copy $F_{P,Q}$ of the fiber of the knot $L(P)$ is the closure in the fiber F_Q of the knot $L(Q)$ of the set

$$\{(x,u) \in TD^2 \mid x \in C, (df_Q)_x \neq 0, (df_Q)_x(u) = 0\} \cap S^3 .$$

The first reduction curve is the boundary of the surface $R := \partial F_{P,Q}$ of the surface $F_{P,Q} \subset F_Q$.

The reduction system is the orbit $\{R, T_Q(R), T_Q^2(R), \dots\}$ under the monodromy T_Q of the singularity with divide Q.

Our first example is the singularity with two essential Puiseux pairs $(x^3 - y^2)^2 - 4x^5y - x^7$, whose link is a two stage iterated torus knot. Its divide $Q = P_{2,9} * P_{2,3}$, (see Figure 2), has two P_+-components, where $P = P_{2,3}$, (see Figure12, where the projection of the reduction curve R is drawn). In this case Manhattan consists of one block. The reduction curve R is the pre-image in the fiber F_Q of its projection $\text{proj}(R) \subset D^2$ under the map $(x,u) \mapsto x$ a drawn in Figure 12. That means R is the closure in F_Q of the set

$$\{(x,u) \in TD^2 \mid x \in \text{proj}(R), (df_Q)_x \neq 0, (df_Q)_x(u) = 0, ||x||^2 + ||u||^2 = 1\} .$$

The reduction system is $\{R, T_Q(R)\}$.

The curve R is homologically trivial in F_Q. It turns out that the power T_Q^{156} of the monodromy is the composition of the right Dehn twists, whose core curves are $\{R, T_Q(R)\}$. The power T_Q^{156} is a product of $2496 = 16 \times 156$ Dehn twists, since T_Q is the product of those Dehn twists whose core curves are the system of distinguished quadratic vanishing cycles of the real morsification with divide Q. It turns out that the expression as product of Dehn twists is far from being as short as possible. In fact, the right Dehn twist Δ_R with core curve R can be written as a product of 36 right Dehn twists that have core curves coming from the morsification with divide Q. More precisely, the Dehn twist Δ_R factors as

$$\Delta_R = (\Delta_M \circ \Delta_b \circ \Delta_m \circ \Delta_a \circ \Delta_m^{-1} \circ \Delta_b^{-1})^6 .$$

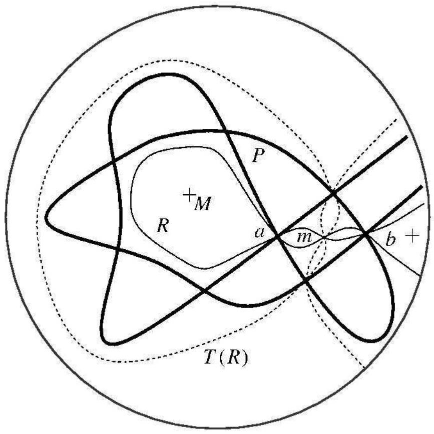

FIGURE 12. Divide $Q = P_{2,9} * P_{2,3}$ with reduction curves R and $T(R)$ (dotted).

The factors are right Dehn twists whose core curves are among the quadratic vanishing cycles $\delta_m, \delta_a, \delta_M, \delta_b$ of the divide Q as indicated in Figure 11, δ_M is the vanishing cycle of a P_+-region, δ_m of the maximum of Manhattan, and δ_a, δ_b of street corners of Manhattan. It follows that T_Q^{156} can also be written as a composition of 72 Dehn twists with core curves among the vanishing cycles of the divide Q. The composition $\Delta_b \circ \Delta_m \circ \Delta_a \circ \Delta_m^{-1} \circ \Delta_b^{-1}$ is the Dehn twist with core curve $\bar{a} := \Delta_b(\Delta_m(a))$.

The reduction curve R cuts off from F_Q a piece $F_{P,Q}$ of genus one (see also Figure 1 on page 159 of [AC1]), and the Dehn twists Δ_M and $\Delta_{\bar{a}}$ act only on this piece, since the curves δ_M and $\bar{a}$ lie entirely in this piece; in this piece, that is a copy of the fiber F_P, they generate the geometric monodromy group of the accompanying singularity $x^3 - y^2 = 0$ with divide $P_{2,3}$.

Our second example is the singularity with two branches $(x^3 - y^2)(y^3 - x^2)$. Its homological monodromy is of infinite order [AC1]. Each branch is a torus knot. Again Manhattan consists of one block. In Figure 13 we have drawn the projections of the curves R, R' and S, S', that together are the boundary components of the two diagonals through Manhattan. In this case the curves R and R' are isotopic to each other, as are the curves S and S'. A complete reduction system for the geometric monodromy is the system $\{R, S\}$. Each component of this system carries a non-trivial homology class. The isotopy classes of the curves R and S are permuted by the monodromy T_P, and hence the system $\{R, S\}$ is invariant under the monodromy.

Let h be the action of T_P on the homology $H_1(F_P, \mathbf{Z})$ of the the fiber F_P. Let $\delta_a, \delta_b, \delta_c, \delta_d$ be the vanishing cycles of the double points, that are the corners of Manhattan of P, and let δ_m be the vanishing cycle of the maximum in the center of Manhattan.

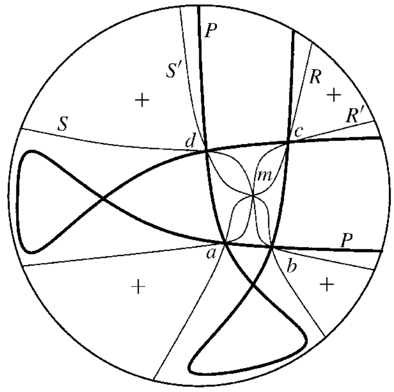

FIGURE 13. Divide P for $(x^3 - y^2)(y^3 - x^2)$ with reduction system $R \cup S$.

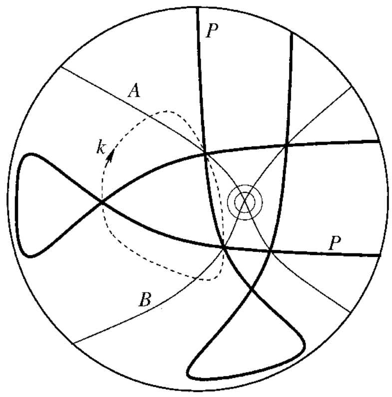

FIGURE 14. Divide P for $(x^3 - y^2)(y^3 - x^2)$ with reduction system $A \cup B$.

If one chooses the orientations appropriately, one has

$$[R] = [\delta_a] + [\delta_m] + [\delta_c],\ [S] = [\delta_b] + [\delta_m] + [\delta_d],\ h([R]) = -[S],\ h([S]) = -[R],$$

and hence also $h([R] - [S]) = [R] - [S]$. Let $[k]$ be any cycle on F_P, that is, carried by a simple oriented curve k and intersects the curves R and S each transversally in one point. One has $h^{10}([k]) = [k] \pm ([R] + [S])$, which shows that the homological monodromy h is not of finite order. We have drawn in Figure 13 the oriented projection of such a cycle k, that intersects the curves A and B. The curve A is halfway in between the curves R and R' on the cylinder they cut out. Let B be the curve halfway in between S and S'. The curves A and B are the reduction curves of Figure 14 on page 167 of [AC1]. The reduction system A, B is much easier to draw, see Figure 14, where are drawn the projections in D^2. The projections meet transversally at the maximum in Manhattan of f_P. The curve δ_m intersects transversally in two points each curve R and S. One has $h^{10}([\delta_m]) = [\delta_m] \pm 2([R] + [S])$.

The power T_P^{10} of the geometric monodromy, that is, a word of length 110 in the Dehn twists of the divide P is equal to the composition of those right Dehn twists, whose core curves are R and S. So, the power T_P^{10} also can be written as the much shorter word

$$\Delta_c \circ \Delta_m \circ \Delta_a \circ \Delta_m^{-1} \circ \Delta_c^{-1} \circ \Delta_d \circ \Delta_m \circ \Delta_b \circ \Delta_m^{-1} \circ \Delta_d^{-1} .$$

REMARK. The curves A and $\Delta_m(\Delta_c(\delta_a))$ are isotopic, where Δ_m and Δ_c are the right Dehn twists with core curves δ_m and δ_c. It follows that the reduction system A, B consists of quadratic vanishing cycles of the singularity of $\{(x^3 - y^2)(y^3 - x^2) = 0\}$ with two branches. In contrast, a reduction curve of a singularity with only one branch can not be a quadratic vanishing cycle, since all reduction curves are zero in the homology, see [L,AC1]. It follows that the action of the monodromy of an irreducible plane curve singularity on homology is of finite order [L].

5. Geometric monodromy group and reduction system. Let the polynomial $f_{(a,b)}$ be an equation for an irreducible plane curve singularity with n essential Puiseux pairs $(a_i, b_i)_{1\le i\le n}$. The number of simply closed curves contained in a complete reduction system R for the monodromy of f is

$$a_n a_{n-1} \cdot \ \ldots\ \cdot a_2 + a_{n-1} a_{n-2} \cdot \ \ldots\ \cdot a_2 + \cdots + a_3 a_2 + a_2 .$$

Let $\Gamma_{f,\mathrm{red}}$ be the subgroup of the geometric monodromy group of Γ_f of f of those elements $\gamma \in \Gamma_f$ that up to isotopy fix each component of R. Let $\Gamma^0_{f,\mathrm{red}}$ be the subgroup of Γ which is generated by the Dehn twist, whose core curves are quadratic vanishing cycles and do not intersect any component of R. Obviously, one has $\Gamma^0_{f,\mathrm{red}} \subset \Gamma_{f,\mathrm{red}}$, but we do not know if this inclusion is strict. A component of $F \setminus R$ is called a *top-component* if its closure in F meets only one component of R. Let $\Gamma_{f,\mathrm{top}}$ be the subgroup of Γ_f of those monodromy transformations, which induce the identity in each component of $F \setminus R$ that is not a top-component. Let $\Gamma^0_{f,\mathrm{top}}$ be the intersection $\Gamma_{f,\mathrm{top}} \cap \Gamma^0_{f,\mathrm{red}}$. We have

THEOREM 2. *Let $f = f_{(a,b)}$ be an irreducible singularity with $n \ge 2$ essential Puiseux pairs $(a_i, b_i)_{1\le i\le n}$. Let $g = f_{(a',b')}$ be a singularity with the $n-1$ essential Puiseux pairs $(a', b') = (a_i, b_i)_{1\le i\le n-1}$. The group Γ_f contains the product of a_n copies of the group Γ_g.*

THEOREM 3. *Let $f_{(a,b)}$ be an irreducible singularity with $n \ge 2$ essential Puiseux pairs. The group $\Gamma^0_{f,\mathrm{top}}$ is isomorphic to the product of $a_n a_{n-1} \cdot \ldots \cdot a_2$ copies of the geometric monodromy group of the singularity $y^{a_1} - x^{b_1} = 0$.*

PROOF OF THEOREM 2. Let P be the divide $P_{a_{n-1},b'_{n-1}} * \cdots * P_{a_2,b'_2} * P_{a_1,b_1}$ for the singularity of g and let $Q = P_{a_n,b'_n} * P$ be the divide for the singularity of f. A copy $F_{P,Q}$ of the fiber F_P is constructed as a subset of the fiber F_Q. Remember, that F_P is obtained by connecting with strips the sets $\{(x,u) \in TD^2 \mid f_P(x) > 0, (df_P)_x(u) = 0\}$, where $f_P : D^2 \to \boldsymbol{R}$ is a

Morse function for the divide P. For each double point of P there are two connecting strips. To each $+$-component of P corresponds a P_+-component of Q with the same topology and to each double point of P corresponds a Manhattan grid of Q, in which we have drawn diagonally the projection of the strips that connect $\{(x,u) \in TD^2 \mid x \in Q_{P,+}, (df_Q)_x(u) = 0\}$. Here, $Q_{P,+}$ denotes the union of the P_+-components of the complement of the divide Q. From the divide P is deduced a distinguished base of quadratic vanishing cycles for the singularity of f. Let B_P be the union of the curves of this base. This base can be drawn on the fiber F_P, see Section 3.

In order to prove the theorem, we will construct inside $F_{P,Q}$ a system of simply closed curves with union $B_{P,Q}$, each of them being a quadratic vanishing cycle for the singularity g, such that the pairs (F_P, B_P) and $(F_{P,Q}, B_{P,Q})$ are diffeomorphic. This finishes the proof, since the Dehn twist, whose cores are the quadratic vanishing cycles of $B_{P,Q}$, generate a copy of Γ_g in Γ_f. By acting with the geometric monodromy T of the singularity f, one obtains a_n commuting copies of Γ_g in Γ_f.

To each $+$-region of P corresponds one P_+-region of Q. The maximum of f_P, say at M in the region, is also a maximum of f_Q. The quadratic vanishing cycle $\delta_M := \{(M,u) \in TD^2 \mid ||M||^2 + ||u||^2 = 1\}$ of F_Q lies in F_P and also in $F_{P,Q}$. For each double point c of P the quadratic vanishing cycle $\delta_c \subset F_P$ projects in D^2 to a tear splicing the gradient line g_c of f_P through c. The endpoints of g_c are maxima of f_P or points on ∂D^2. The function f_Q has exactly one gradient line $g_{Q,c}$ that has the same endpoints as g_c and coincides with g_c in a neighborhood of the common endpoints. The gradient line $g_{Q,c}$ runs along a diagonal through the Manhattan grid corresponding to c. Let $g_{Q,c}$ be the simply closed curve on F_Q, that projects to a tear $t_{Q,c}$ equal to $g_{Q,c}$, except above a neighborhood of its endpoints where $t_{Q,c}$ equals t_c. We remark that $\delta_{Q,c}$ is a cycle in $F_{P,Q}$. Let $c_1, \ldots, c_p$ be the $p := a_n$ double points of Q that occur along the $g_{Q,c}$ and let $M_2, \ldots, M_p$ along $g_{Q,c}$ be the maxima. Let δ_{Q,c_1} be the quadratic vanishing cycle of the singularity f that corresponds to c_1. One verifies that the cycles $\delta_{Q,c}$ and

$$\Delta_{c_p} \circ \Delta_{M_p} \circ \cdots \circ \Delta_{c_2} \circ \Delta_{M_2}(\delta_{Q,c_1})$$

are isotopic. Here Δ_{c_i} or Δ_{M_i} stands for the right Dehn twist of F_Q whose core curve is the quadratic vanishing cycle δ_{c_i} or δ_{M_i} of the singularity f. Hence $\delta_{Q,c} \subset F_{P,Q}$ is a quadratic vanishing cycle for the singularity f. So far, we have constructed for each maximum and for each saddle point of f_P a simply closed curve on $F_{P,Q}$ that is a quadratic vanishing cycle of the singularity f. These cycles intersect on $F_{P,Q}$ as do the corresponding quadratic vanishing cycles of the singularity g on F_P.

We now wish to construct for each minimum of f_P a vanishing cycle on $F_{P,Q}$. We have to handle two cases: p odd, see Figure 15, and p even, see Figure 16.

If p is odd, a minimum m of f_P will also be a minimum of f_Q. Let $\delta_{Q,m}$ be the vanishing cycle on F_Q corresponding to m, see Figure 15. The projection of $\delta_{Q,m}$ into D^2 is a smooth simply closed curve s' transversal to Q, that surrounds the $-$ region of m through its neighboring $+$ regions of Q. One needs to take care that in each neighboring $+$ component the projection runs through the maximum of f_Q in that region. The points of s correspond to pairs (x, u) with $x \in s'$ and u pointing inwards to m. Let r be a simply closed cycle on $F_{P,Q}$ that projects into D^2 upon the curve r', which now surrounds the $-$ region of m through the P_+-components of Q, see Figure 15. In the Manhattan grids r' is just a diagonal, again r' runs through the maxima of the regions or touches ∂D. On r we only allow pairs (x, u), where u points inwards to m. It is clear that the cycle r on $F_{P,Q}$ intersects the cycles of the previous construction as the vanishing cycle to the minimum of f_P intersects the vanishing cycles of the critical points of f_P. It remains, however, to check that the cycle r is a quadratic vanishing cycle of the singularity of g. By applying to $r \subset F_Q$ the Dehn twist corresponding to the critical points of f_Q that are in between the curves r' and s', one can transform the isotopy class of the curve $\delta_{Q,m}$ to the class of the curve r. This proves that r is indeed a quadratic vanishing cycle of the singularity of g.

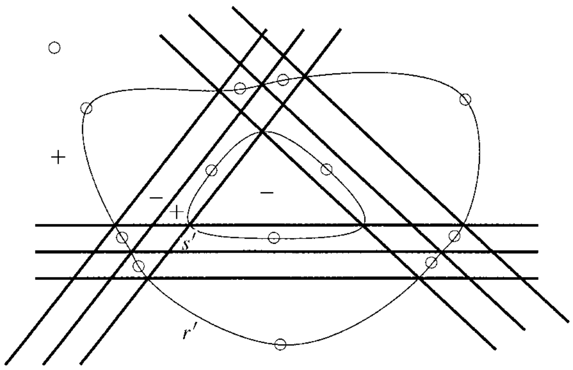

FIGURE 15. Vanishing cycle s on F_Q from a minimum of f_P and cycle r on $F_{P,Q}$.

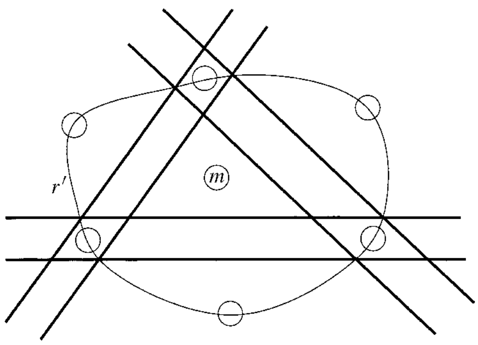

FIGURE 16. Vanishing cycle $\delta_{Q,m}$ on F_Q from a minimum of f_P and cycle r on $F_{P,Q}$.

If p is even, then a minimum m of f_P will be a maximum of f_Q. Let $\delta_{Q,M}$ be the vanishing cycle on F_Q corresponding to maximum $M := m$, see Figure 16. Its projection into D^2 is the point $M := m$. Let r be a simply closed cycle on $F_{P,Q}$ that projects into D^2 upon the curve r' which now surrounds the $-$ region of $M := m$ through the P_+-regions of Q, see Figure 16. In the Manhattan grids r' is just a diagonal, again r' runs through the maxima of the regions. On r we only allow pairs (x, u), where u points inwards to m. It is clear that the cycle r on $F_{P,Q}$ intersects the cycles of the previous construction as the vanishing cycle to the minimum of f_P intersects the vanishing cycles of the critical points of f_P. By applying to $\delta_{Q,M} \subset F_Q$ the Dehn twist corresponding to the critical points of f_Q that are in between the curve r' and the point $M := m$, one can transform the isotopy class of the curve $\delta_{Q,M}$ to the class of the curve r, and proves that r is indeed a quadratic vanishing cycle of the singularity of g. As explained, this terminates the proof.

PROOF OF THEOREM 3. The proof of Theorem 2 constructs a copy $\Gamma_{P,Q}$ of the monodromy group Γ_f of the singulatity f as subgroup in the monodromy group Γ_g of the singularity g. This copy acts with support in a copy $F_{P,Q}$ of the fiber F_P. The the first $a_n - 1$ iterates of the monodromy T_Q of the singularity g constructs a_n copies of F_P in F_Q. By conjugation with T_Q one gets a_n copies from $\Gamma_{P,Q}$. We end the proof by repeating this argument. One gets $a_n a_{n-1} \cdots a_2$ commuting copies of the geometric monodromy group of the singularity $x^{b_1} - y^{a_1}$ in Γ_g.

PROBLEM. We like to state the problem of presenting the geometric monodromy group of plane curve singularities with generators and relations. It would be particulary nice to express the presentation in terms of a divide of the singularity. The same problem can also be stated for the homological monodromy group of plane curve singularities, but I think that the problem for the geometric monodromy group is more tractable, since all reduction curves can be taken into account. The theorems 2 and 3 are possibly first steps towards a solution of this problem. However, an important missing piece in this program is a presentation with generators and relations of the geometric monodromy group of the singularities $y^p - x^q = 0$ for $3 \leq p \leq q$, $7 \leq p + q$. The fundamental group of the complement of the discriminant in the unfolding of the singularity $y^2 - x^q = 0$ is the braid group B_{q-1}. Bernard Perron and Jean-Pierre Vannier have proved for the singularities $y^2 - x^q = 0$ that the geometric monodromy group is a faithful image of the braid group B_{q-1} and that a similar result holds for the singularities $x(y^2 - x^q) = 0$ [P-V]. The fundamental group of the complement of the discriminant in the unfolding of the singularity $y^3 - x^6 = 0$ is the Artin AE_6 group of the Dynkin diagramm E_6. Bronek Wajnryb has proved that the geometric monodromy representation of AE_6 into the mapping class group of the Milnor fiber of the singularity $y^3 - x^6 = 0$ is not faithful. Using the corresponding relation in the mapping class group, Makoto Matsumoto has given a presentation of the mapping class group with local relations [Ma].

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MATHEMATISCHES INSTITUT
UNIVERSITÄT BASEL
RHEINSPRUNG 21, CH-4051 BASEL
SWITZERLAND