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A CERTAIN CLASS OF POINCARÉ SERIES ON Sp_n, II

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Abstract. We compute the Petersson scalar product of certain Poincaré series introduced in our previous paper against a Siegel cusp form and show that it can be written as a certain averaged cycle integral. This generalizes earlier work by Katok, Zagier and the first named author in the case of genus 1.

1. Introduction. In our previous paper [4] we had introduced a certain type of Poincaré series $P_{n,k,Q}(z)$, where n and k are positive integers, Q is an even integral symmetric $(2n, 2n)$-matrix which is a symplectic similitude with integral negative scale l and z is a variable in the Siegel upper half-space $\mathcal{H}_n$ of genus n. The main result of that paper states that if $k > n(n+1)/2$ and $-l$ is not a perfect integral square, then $P_{n,k,Q}(z)$ converges in sufficiently large domains and is a cusp form of weight $2k$ with respect to the full Siegel modular group $\Gamma_n := Sp_n(\mathbf{Z})$ of genus n. If $n = 1$, the $P_{n,k,Q}(z)$ are just the classical hyperbolic Poincaré series studied in [1, 2, 7].

It is natural to try to evaluate the Petersson scalar product of $P_{n,k,Q}$ against a Siegel cusp form f of weight $2k$ and genus n. Optimistically, in analogy with the classical case $n = 1$ [1, 2], one could hope that this scalar product up to a universal constant is equal to a "normalized" integral of f over a fundamental domain $\mathcal{F}_{n,Q}$ for the action of $\Gamma_{n,Q}$ on $C_{n,Q}$. Here $\Gamma_{n,Q} \subset \Gamma_n$ is the stabilizer subgroup of Q and Γ_n acts on matrices Q as described above by $(Q, M) \mapsto Q[M] := M'QM$ with M' the transpose of M. Furthermore

$$C_{n,Q} := \left\{ z \in \mathcal{H}_n \,\middle|\, (\bar{z}\ \ 1_n)Q\begin{pmatrix} z \\ 1_n \end{pmatrix} = 0 \right\},$$

which is a real-analytic submanifold of $\mathcal{H}_n$ of real dimension $n(n+1)/2$ (if $n = 1$, then $C_{n,Q}$ is just a classical "Heegner cycle"). Note that the natural action of Γ_n on $\mathcal{H}_n$ induces an action of $\Gamma_{n,Q}$ on $C_{n,Q}$.

In the present paper we shall prove a result in the above direction. However, instead of obtaining an integral along $\mathcal{F}_{n,Q}$, we only get integrals along certain submanifolds $F_{n,Q;u}$ of $\mathcal{H}_n$ depending on a parameter u which is a real symmetric (n, n)-matrix (one has $F_{n,Q;0} = \mathcal{F}_{n,Q}$), and then these integrals are averaged by integrating with respect to u in an appropriate way. So far we have not been able to reduce the integration over u in a further way. To avoid taking square roots of complex determinants at several places, in stating our Theorem we have also supposed that n is odd, for the sake of simplicity.

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In Section 2 we shall prove two technichal Lemmas needed later. In Section 3 we will state our main result in detail. The proof will be given in Section 4, along with some comments.

We use the same *notation* as in [4]. In particular, A' denotes the transpose of a matrix A and $A[B] := B'AB$ for matrices A and B of appropriate sizes. We write $1 = 1_n$ for the unit matrix of size n if there is no danger of confusion.

2. Two lemmas. Let us write $G = GL_n(\boldsymbol{R})$ and denote by T (resp. B) the subgroups of G consisting of diagonal matrices with positive entries (resp. unipotent upper triangular matrices). Let $K = O(n, \boldsymbol{R})$. According to the Iwasawa decomposition, every $g \in G$ has a unique expression as $\kappa t b$ with $\kappa \in K,\ t \in T,\ b \in B$. Hence G acts on $T \times B$ by

$$(t, b) \circ g = (t_0, b_0)\,, \tag{1}$$

where

$$tbg = \kappa t_0 b_0 \quad (\kappa \in K, t_0 \in T, b_0 \in B)\,. \tag{2}$$

Let $\mathcal{S}_n$ be the space of symmetric real (n, n)-matrices and $\mathcal{P}_n$ the subset of positive definite ones. We have diffeomorphisms

$$T \times B \to \mathcal{P}_n, \quad (t, b) \mapsto 1[tb]$$

(Jacobi decomposition) and

$$\phi : \mathcal{S}_n \times T \times B \to \mathcal{H}_n\,, \quad (u, t, b) \mapsto z = (u + i)[tb]\,.$$

Under the map ϕ^{-1} the action $z \mapsto z[g]$ of G on $\mathcal{H}_n$ corresponds to the action of G on $\mathcal{S}_n \times T \times B$ given by

$$(u, t, b) \circ g = (u[\kappa], (t, b) \circ g)\,,$$

where $(t, b) \circ g$ and κ are determined by (1) and (2).

LEMMA 1. *The functional determinant of the map ϕ is given by*

$$J(\phi) = (-1)^{\frac{n(n-1)}{2}} 2^n (\det t)^{3n+2} \prod_{j=1}^{n} t_j^{-2j}\,.$$

PROOF. We have

$$J(\phi) = \det \begin{pmatrix} \partial x/\partial u & \partial x/\partial t & \partial x/\partial b \\ \partial y/\partial u & \partial y/\partial t & \partial y/\partial b \end{pmatrix} = \det(\partial x/\partial u)\det(\partial y/\partial t\ \ \partial y/\partial b)\,,$$

since $(\partial y/\partial u) = 0$.

One has $x = u[tb]$. Write $x = (x_{ij})_{1 \le i,j \le n}$, $u = (u_{ij})_{1 \le i,j \le n}$, $t = \mathrm{diag}(t_1, \dots, t_n)$ and denote the upper diagonal components of b by b_{ij} $(1 \le i \le j \le n)$; in particular, $b_{ii} = 1$ for all i. Then

$$x_{ij} = \sum_{1 \le \mu \le i, 1 \le \nu \le j} t_\mu t_\nu u_{\mu\nu} b_{\mu i} b_{\nu j} \quad (1 \le i \le j \le n)$$

and hence $(\partial x/\partial u)$ is a lower triangular matrix and

$$\det(\partial x/\partial u) = \prod_{1\le i\le j\le n} \frac{\partial x_{ij}}{\partial u_{ij}} = \prod_{1\le i\le j\le n} t_i t_j$$
$$= \left(\prod_{i=1}^{n} t_i\right)^{n+1} = (\det t)^{n+1} .$$

By interchanging columns, we transform the matrix $\big(\partial y/\partial t \;\; \partial y/\partial b\big)$ to the form

(3) $$\big(\cdots \quad \partial y/\partial t_i \quad \partial y/\partial b_{i,i+1} \quad \partial y/\partial b_{i,i+2} \quad \cdots \quad \partial y/\partial b_{in} \quad \cdots \partial y/\partial t_n\big)_{1\le i\le n-1} .$$

The required number of changes of columns, modulo 2 is $n(n-1)/2 \pmod 2$, and hence we find that $\det\big(\partial y/\partial t \;\; \partial y/\partial b\big)$ is equal to $(-1)^{n(n-1)/2}$ times the determinant of the matrix in (3).

We have $y = 1[tb] = b' t^2 b$ and therefore

$$y_{ij} = \sum_{\lambda=1}^{\min\{i,j\}} t_\lambda^2 b_{\lambda i} b_{\lambda j} .$$

From this we see that (3) is a lower triangular matrix and that

$$\det\big(\partial y/\partial t \;\; \partial y/\partial b\big) = (-1)^{n(n-1)/2} \prod_{i=1}^{n} \left(\frac{\partial y_{ii}}{\partial t_i} \prod_{j=i+1}^{n} \frac{\partial y_{ij}}{\partial b_{ij}} \right)$$

$$= (-1)^{n(n-1)/2} \prod_{i=1}^{n} \left(2t_i \prod_{j=i+1}^{n} t_i^2 \right) = (-1)^{n(n-1)/2} 2^n (\det t)^{2n+1} \prod_{j=1}^{n} t_j^{-2j} .$$

Hence we obtain our assertion.

LEMMA 2. *Let u be a fixed complex symmetric matrix of size n. Put*

(4) $$z = u[tb] \quad (t \in T,\ b \in B) .$$

Then the complex functional determinant of the map (4) *is given by*

(5) $$\det\big(\partial z/\partial t \quad \partial z/\partial b\big) = (-1)^{n(n-1)/2} 2^n \left(\prod_{j=1}^{n} \det u^{(j)} \right) \cdot (\det t)^{2n+1} \prod_{j=1}^{n} t_j^{-2j} ,$$

where $u^{(1)} = u_{11},\ u^{(2)}, \ldots, u^{(n-1)},\ u^{(n)} = u$ are the principal submatrices of u.

PROOF. Denote the determinant on the left-hand side of (5) by $\alpha(u^{(n)}, t^{(n)}, b^{(n)})$. We shall proceed by induction on n, the case $n = 1$ being clear.

Suppose $n \ge 2$. We have

(6) $$z_{ij} = \sum_{1\le\mu\le i,\, 1\le\nu\le j} t_\mu t_\nu u_{\mu\nu} b_{\mu i} b_{\nu j} \quad (1 \le i \le j \le n) .$$

Observe that z_{ij} for $1 \le i \le j \le n-1$ does not depend on t_n, b_{en} $(1 \le e \le n-1)$ and u_{fn} $(1 \le f \le n)$.

Therefore one easily sees that

$$\alpha(u^{(n)}, t^{(n)}, b^{(n)}) = \alpha(u^{(n-1)}, t^{(n-1)}, b^{(n-1)}) \cdot \beta(u^{(n)}, t^{(n)}, b^{(n)}) ,$$

where

$$\beta = \beta(u^{(n)}, t^{(n)}, b^{(n)}) = \det \big(\partial z_{in}/\partial t_n \quad \partial z_{in}/\partial b_{en}\big)_{1\le i\le n,\, 1\le e\le n-1}$$

(to transform $\det \big(\partial z/\partial t \quad \partial z/\partial b\big)$ into a lower triangular block matrix, one has to perform an equal number of changes of columns and rows).

Hence by the induction hypothesis it is sufficient to show that

$$\beta = (-1)^{n-1} \cdot 2 \cdot \det u \cdot \prod_{j=1}^{n-1} t_j \cdot \prod_{j=1}^{n} t_j .$$

From (6) we find that

$$\frac{\partial z_{in}}{\partial t_n} = (1+\delta_{ni}) \sum_{1\le\mu\le i} b_{\mu i} u_{\mu n} t_\mu$$

and

$$\frac{\partial z_{in}}{\partial b_{en}} = (1+\delta_{ni}) t_e \Big(\sum_{1\le\mu\le i} b_{\mu i} u_{\mu e} t_\mu \Big) \quad (1 \le e \le n-1) ,$$

where δ_{ni} denotes the Kronecker delta. Therefore we obtain after a simple computation

$$\beta = (-1)^{n-1} \cdot 2 \, \cdot t_n^{-1} \, \cdot \det(b'c) ,$$

where

$$c = u[t] .$$

This proves our assertion.

COROLLARY. *Let $u \in \mathcal{S}_n$ and $g \in G$ be fixed. For $t \in T, b \in B$ write*

$$tbg = \kappa_{g,t,b}\, t_0 b_0 \quad (\kappa_{g,t,b} \in K, t_0 \in T, b_0 \in B)$$

(*compare* (2)). *Then the complex functional determinants of the maps*

$$z = \big((u[\kappa_{g,t,b}^{-1}] + i\big)[tb] \quad (t \in T, b \in B) \tag{7}$$

and

$$z = (u+i)[tb] \quad (t \in T, b \in B) \tag{8}$$

differ by the constant factor $(\operatorname{sign} \det g)^{n+1}$.

PROOF. Write $J_0(t,b)$ and $J(t,b)$ for the functional determinants of the maps (7) and (8), respectively. By definition, we have

$$((u[\kappa_{g,t,b}^{-1}] + i)[tb] = (u+i)[t_0 b_0]\,[g^{-1}] .$$

Since $z \mapsto z[g^{-1}]$ has determinant $(\det g)^{-n-1}$, we obtain

$$J_0(t,b) = (\det g)^{-n-1}\, J(t_0, b_0) \det \left(\frac{\partial((t,b)\circ g)}{\partial t \partial b} \right) . \tag{9}$$

The measure $(\det y)^{-(n+1)/2}dy$ $(y \in \mathcal{P}_n)$ is G-invariant with respect to the action $y \mapsto y[g]$. Since the functional determinant of the map $y = 1[tb]$ $(t \in T, b \in B)$ is equal to $(\det t)^{2n+1} \prod_{j=1}^n t_j^{-2j}$ (argue in the same way as in Lemma 2 or use [6, Chap. 4.1, ex. 21a)]), we see that $(\det t)^n \prod_{j=1}^n t_j^{-2j}\, dt db$ is a G-invariant measure on $T \times B$. Hence we have

$$\det\left(\frac{\partial((t,b)\circ g)}{\partial t \partial b}\right) = (\det t)^n \prod_{j=1}^n t_j^{-2j} \cdot (\det t_0)^{-n} \prod_{j=1}^n (t_0)_j^{2j} .$$

Using (9), Lemma 2 and the fact that $\det t_0 = \det t \cdot \det g \cdot \det \kappa_{g,t,b}^{-1}$, $\det \kappa_{g,t,b} = \pm 1$, we obtain our assertion.

3. The Petersson scalar product. To state our result we have to recall several definitions and results from [3, 4].

Let Q be as in Section 1 and assume that $-l$ is not a perfect integral square. Let k be an integer with $k > n(n+1)/2$ and define

$$P_{n,k,Q}(z) := \sum_{M \in \Gamma_{n,Q}\backslash\Gamma_n} \left(\det(Q[M]\,[\begin{pmatrix} z \\ 1 \end{pmatrix}])\right)^{-k} \quad (z \in \mathcal{H}_n)\,. \tag{10}$$

It was proved in [4] that the series (10) is absolutely uniformly convergent on subsets of the form

$$V_n(\delta) := \{z = x + iy \in \mathcal{H}_n \,|\, \mathrm{tr}(x'x) \le 1/\delta,\ y \ge \delta 1_n\} \quad (\delta > 0)$$

and is a cusp form of weight $2k$ on Γ_n.

Write

$$Q = \begin{pmatrix} a & b \\ b' & c \end{pmatrix}$$

with a, b and c integral (n,n)-matrices, a and c symmetric and

$$ab' = ba\,, \quad b'c = cb'\,, \quad ac - b^2 = l1_n\,.$$

Replacing Q by a Γ_n-equivalent form if necessary, we can and will assume that a is invertible [3, Lemma 5]. Let σ be the non-trivial automorphism of the field extension $\boldsymbol{Q}(\sqrt{-l})/\boldsymbol{Q}$, and set

$$w_1 := a^{-1}(-b + \sqrt{-l})\,, \quad w_2 := w_1^\sigma = a^{-1}(-b - \sqrt{-l})\,.$$

Then w_1 and w_2 are symmetric and satisfy

$$Q[\begin{pmatrix} w_1 \\ 1 \end{pmatrix}] = 0\,, \quad Q[\begin{pmatrix} w_2 \\ 1 \end{pmatrix}] = 0\,.$$

Put

$$R := \begin{pmatrix} -w_2 & w_1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\,.$$

Then R is a symplectic similitude with scale $2\sqrt{-l}$ and

$$Q[R] = (-2l) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[4, p. 429]. In the sequel, $R \circ z$ denotes the usual action of R on $z \in \mathcal{H}_n$.

Write $-l = df^2$ with $d, f \in \boldsymbol{N}$ and d squarefree. Let $\mathcal{O}$ be the ring of integers of $\boldsymbol{Q}(\sqrt{-l})$. Put

$$q := |\det a|$$

and

$$H_{a^{-1}} := \{g \in GL_n(\mathcal{O}) |\, a^{-1} g'^{\sigma} a = g^{-1},\ g \equiv 1 \pmod{2qf}\}\,.$$

Then $H_{a^{-1}}$ is a discrete subgroup of $GL_n(\boldsymbol{R})$, and

$$H_{a^{-1}}^{\uparrow} := \left\{ \begin{pmatrix} g & 0 \\ 0 & g'^{-1} \end{pmatrix} \middle|\, g \in H_{a^{-1}} \right\}$$

is contained in the conjugate subgroup $R^{-1}\Gamma_{n,Q}R$ with finite index [4, p. 430].

Put

$$\iota_{a^{-1}} := [R^{-1}\Gamma_{n,Q}R : H_{a^{-1}}^{\uparrow}]\,.$$

We will denote by F_a a fundamental domain for the action of H_a on $T \times B$ (cf. Section 2). We also put

$$F_{n,Q;u} := \{R \circ (u+i)[tb] \,|\, (t,b) \in F_a\}\,.$$

For $k \in \boldsymbol{N}$ finally write $S_{2k}(\Gamma_n)$ for the space of cusp forms of weight $2k$ with respect to Γ_n. If $f, g \in S_{2k}(\Gamma_n)$, we denote by

$$\langle f, g\rangle = \int_{\Gamma_n \backslash \mathcal{H}_n} f(z)\bar{g}(z)(\det y)^{2k} \frac{dxdy}{(\det y)^{n+1}} \quad (z = x+iy)\,,$$

the Petersson scalar product of f and g.

We can now state our main result.

THEOREM. *Suppose that n is odd. Let $f \in S_{2k}(\Gamma_n)$. Then with the above assumption and notation we have*

$$\text{(11)} \qquad \langle f, P_{n,k,Q}\rangle = (-1)^{\frac{n(n-1)}{2}} (2\sqrt{-l})^{\frac{n(n+1)}{2}-2nk} \iota_{a^{-1}} \int_{\mathcal{S}_n} \det(u^2+1)^{-k} (\det(u+i))^{\frac{n+1}{2}}$$

$$\cdot \prod_{j=1}^{n} \det(u^{(j)} + i1_j)^{-1} \left(\int_{F_{n,Q;u}} f(z) \left(\det(Q[\begin{pmatrix} z \\ 1 \end{pmatrix}]) \right)^{k-\frac{n+1}{2}} dz \right) du\,.$$

Here $u^{(1)} = u_{11}, u^{(2)}, \ldots, u^{(n-1)}, u^{(n)} = u$ denote the principal submatrices of u and the orientable connected submanifolds $F_{n,Q;u} \subset \mathcal{H}_n \subset \boldsymbol{R}^{n(n+1)/2}$ are endowed with the positive orientation. Furthermore, we have put $dz = \prod_{1 \le i \le j \le n} dz_{ij}$ where the variables are taken in lexicographical order.

REMARKS. i) Clearly the differential form appearing in the inner integral on the right-hand side of (11) is invariant under $\Gamma_{n,Q}$. Also

$$d(R \circ z) = (2\sqrt{-l})^{n(n+1)/2} \frac{dz}{\det(-az+1)^{n+1}}$$

and

$$f(R\circ z)\left(\det(Q[\begin{pmatrix} R\circ z\\ 1\end{pmatrix}])\right)^{k-\frac{n+1}{2}} d(R\circ z)$$
$$= (2\sqrt{-l})^{2nk-\frac{n(n+1)}{2}}\cdot f(R\circ z)(\det z)^{k-\frac{n+1}{2}}\det(-az+1)^{-2k}\,dz\,,$$

as easily follows from the definitions. Hence, using the Corollary to Lemma 2, we see that the expression on the right of (11) is independent of the choice of the fundamental domain F_a.

ii) By definition we have $F_{n,Q;0} = \mathcal{F}_{n,Q}$ (cf. Section 1). Therefore the expression on the right-hand side of (11) could be interpreted as an "appropriately averaged" integral.

4. Proof of Theorem. We will first discuss the convergence of the integrals on the right-hand side of (11). In the inner integral on the right-hand side of (11) we transform $z \mapsto R\circ z$. The formula

$$\det \mathrm{Im}(R\circ z) = (2\sqrt{-l})^n\,\frac{\det \mathrm{Im} z}{|\det(-az+1)|^2}$$

implies that

$$f(R\circ z)\left(\det(Q[\begin{pmatrix} R\circ z\\ 1\end{pmatrix}])\right)^{k-\frac{n+1}{2}} d(R\circ z)$$
$$= (2\sqrt{-l})^{nk-\frac{n(n+1)}{2}}\cdot\big(\det\mathrm{Im}(R\circ z)\big)^k f(R\circ z)\cdot\frac{|\det(-az+1)|^{2k}}{\det(-az+1)^{2k}}\cdot\frac{(\det z)^{k-\frac{n+1}{2}}}{(\det \mathrm{Im}\, z)^k}\quad dz\,.$$

Since f is a cusp form of weight $2k$, the function $(\det \mathrm{Im}\, z)^k f(z)$ is bounded on $\mathcal{H}_n$. Substituting $z = (u+i)[tb]$ $(t\in T, b\in B)$ and using Lemma 2, the absolute value of the inner integral in (11) then is estimated from above (up to a non-zero factor depending only on f) by

$$(12)\qquad |(\det(u+i))^{k-\frac{n+1}{2}}|\prod_{j=1}^{n}|\det(u^{(j)}+i1_j)^{-1}|\int_{F_a}(\det t)^n\prod_{j=1}^{n}t_j^{-2j}\,dtdb\,,$$

and the integral occurring in (12) is equal to

$$\int_{\mathcal{P}_n/H_a}\frac{dy}{(\det y)^{(n+1)/2}}$$

(cf. the proof of the Corollary to Lemma 2, sect. 2).

The latter, however, is finite as was shown in [4, p. 431].

To prove the absolute convergence of the integral over u on the right of (11), it is therefore sufficient to show that

$$\int_{\mathcal{S}_n}|\det(u+i)|^{-k}\,du < \infty\,.$$

This also was proved in [4] (under the assumption $k > n(n+1)/2$, cf. p. 430).

Let us now prove equality (11). The validity of interchanging sums and integrals and of any other integral transforms below will follow from stated convergence properties of the series $P_{n,k,Q}$ and from arguments similar as used above, respectively.

We shall write

$$I = \langle f, P_{n,k,Q} \rangle .$$

The usual unfolding argument shows that

$$\begin{aligned} I &= \int_{\Gamma_{n,Q}\backslash\mathcal{H}_n} f(z)\overline{\left(\det(Q[\begin{pmatrix} z \\ 1 \end{pmatrix}])\right)^{-k}} (\det y)^{2k} \frac{dxdy}{(\det y)^{n+1}} \\ &= \int_{R^{-1}\Gamma_{n,Q}R\backslash\mathcal{H}_n} f(R \circ z) \det(-az+1)^{-2k} (\det \overline{z})^{-k} (\det y)^{2k} \frac{dxdy}{(\det y)^{n+1}} , \end{aligned}$$

where to get the last line we have substituted $z \mapsto R \circ z$ and used the formulas given above. Hence

$$\begin{aligned} I = \iota_{a^{-1}} \int_{H^{\uparrow}_{a^{-1}}\backslash\mathcal{H}_n} & f(R \circ z)(\det z)^{k-\frac{n+1}{2}} \det(-az+1)^{-2k} \\ & \cdot (\det z\overline{z})^{-k} (\det z)^{(n+1)/2} (\det y)^{2k} \frac{dxdy}{(\det y)^{n+1}} . \end{aligned}$$

We now substitute $z = (u+i)[tb]$ and use Lemma 1. This gives

$$I = \iota_{a^{-1}} 2^n \int_{\mathcal{S}_n \times T \times B/H_a} f(R \circ ((u+i)[tb]))(\det((u+i)[tb]))^{k-\frac{n+1}{2}}$$

$$\cdot (\det(-a(u+i)[tb]+1))^{-2k} \cdot \det(u^2+1)^{-k} (\det(u+i))^{(n+1)/2} (\det t)^{2n+1} \prod_{j=1}^{n} t_j^{-2j} \, dudtdb .$$

A fundamental domain for the action of H_a on $\mathcal{S}_n \times T \times B$ is given by the points $(u,t,b) \in \mathcal{S}_n \times T \times B$ with u unrestricted and (t,b) running over $T \times B/H_a$. Let us fix a fundamental domain F_a for the action of H_a on $T \times B$. Then we can write

$$\begin{aligned} (13) \qquad I = \iota_{a^{-1}} 2^n \int_{\mathcal{S}_n} & \det(u^2+1)^{-k} \big(\det(u+i)\big)^{(n+1)/2} \Big(\int_{F_a} f(R \circ ((u+i)[tb])) \\ & \cdot \big(\det((u+i)[tb])\big)^{k-\frac{n+1}{2}} \cdot \big(\det(-a(u+i)[tb]+1)\big)^{-2k} (\det t)^{2n+1} \prod_{j=1}^{n} t_j^{-2j} dtdb \Big) du . \end{aligned}$$

Consider the integral over t and b on the right-hand side of (13). Applying Lemma 2 (with u replaced by $u+i$, u real), we see that this integral is equal to

$$\begin{aligned} (14) \qquad & (-1)^{n(n-1)/2} 2^{-n} \left(\textstyle\prod_{j=1}^{n} \det(u^{(j)} + i1_j)^{-1} \right) \\ & \cdot \int_{(u+i)[F_a]} f(R \circ z)(\det z)^{k-\frac{n+1}{2}} \det(-az+1)^{-2k} dz . \end{aligned}$$

Hence transforming back $z \mapsto R^{-1} \circ z$ we obtain (11).

REMARK. To reduce further the integration over u in (11) in order to eventually get a single integral over $\mathcal{F}_{n,Q}$ in (11), it would be very suggestive (at least in the case where F_a is compact) to imitate the procedure of the case $n = 1$ and to introduce

$$M_{a,u} := \{(su+i)[tb] \,|\, 0 \le s \le 1, \ (t,b) \in F_a\},$$

which (for $u \neq 0$) is an orientable compact submanifold of $\mathcal{H}_n$ with boundary and of dimension $n(n+1)/2+1$. Its boundary is given by

$$\partial M_{a,u} = i[F_a] \cup (u+i)[F_a] \cup M_{a,u;\partial}\,,$$

where

$$M_{a,u;\partial} := \{(su+i)[tb] \,|\, 0 \leq s \leq 1,\ (t,b) \in \partial F_a\}$$

and ∂F_a is the boundary of F_a.

Using Stokes' theorem, one can then rewrite the integral in (14) as this integral for $u = 0$ plus the term

$$\int_{M^o_{a,u;\partial}} f(R \circ z)(\det z)^{k-\frac{n+1}{2}} \det(-az+1)^{-2k} dz\,, \tag{15}$$

where o is an appropriately chosen orientation, and then one had to show that the latter term is zero.

Again it is suggestive to try to do so by choosing for F_a a fundamental domain defined in terms of the geodesic distance of the Riemann manifold $T \times B$ (for the construction of such a fundamental domain in the symplectic case cf. [5, Sections 19–21]; one eventually has to replace H_a by a torsionfree subgroup of finite index). This fundamental domain is star shaped with respect to geodesic arcs through some point p_0, its boundary consists of finitely many pieces

$$R_{a,h_\nu} \cup R_{a,h_\nu^{-1}}$$

with h_ν running through finitely many elements of $H_a \setminus \{1\}$ and the orientation induced on $R_{a,h_\nu^{-1}}$ is opposite to that of R_{a,h_ν}. We are kindly endebted to A. Deitmar for the above suggestion. However, so far we have not been able to show that (15) is zero in that way.

REFERENCES

[1] S. KATOK, Closed geodesics, periods and arithmetic of modular forms, Invent. Math. 80 (1985), 469–480.

[2] W. KOHNEN AND D. ZAGIER, Modular forms with rational periods, Modular Forms (ed. R. A. Rankin), 197–249, Ellis Horwood, Chichester, 1984.

[3] W. KOHNEN, Special quadratic forms, Siegel modular groups and Siegel modular varieties, Internat. J. Math. 1 (1990), 397–429.

[4] W. KOHNEN AND J. SENGUPTA, A certain class of Poincaré series on Sp_n, Internat. J. Math. 10 (1999), 425–433.

[5] C.-L. SIEGEL, Symplectic Geometry, Collected Works II (eds. K. Chandrasekharan, H. Maaß), 274–359, Springer, Berlin-Heidelberg-New York, 1966.

[6] A. TERRAS, Harmonic Analysis on Symmetric spaces and Applications II, Springer, Berlin-Heidelberg-New York, 1988.

[7] D. ZAGIER, Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields, Modular functions of one variable VI (eds. J.-P. Serre, D. Zagier), 106–169, Lecture Notes in Math. 627, Springer, Berlin-Heidelberg-New York, 1977.

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