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ON THE UNIQUE EXTREMALITY OF QUASICONFORMAL MAPPINGS WITH DILATATION BOUNDS

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Abstract. Concerning the problem of extremality of quasiconformal mappings with dilatation bounds, we discuss the unique extremality of the problem and prove the if part of a conjecture on the unique extremality ([G1], [R1]). To this end, we need to investigate a new extremal problem in the infinitesimal setting. In particular, we give a complete description of the unique infinitesimal extremality of partially zero Beltrami differentials.

1. Introduction. The problem of extremality of quasiconformal mappings with dilatation bounds has been much investigated in the literature. In this paper, we will discuss the unique extremality of the problem and prove the if part of a conjecture concerning the unique extremality. To make this precise, we state the problem as follows.

Let R' and R be two hyperbolic Riemann surfaces covered by the unit disk $\Delta = \{z : |z| < 1\}$ in the complex plane. Let a compact, possibly empty, subset E' of R' be given in such a way that $R' \setminus E'$ has positive measure, a non-negative measurable function $b(w)$ on E' (known as a dilatation bound function) with $\|b\|_\infty < 1$, and a quasiconformal mapping F of R' onto R such that the complex dilatation $\tilde{\mu}$ of F satisfies $|\tilde{\mu}(w)| \le b(w)$ for a.e. $w \in E'$. We denote by $Q(F, E', b)$ the class of all quasiconformal mappings G of R' onto R such that G is homotopic to F (mod $\partial R'$) and that the complex dilatation $\tilde{\nu}$ of G satisfies $|\tilde{\nu}(w)| \le b(w)$ for a.e. $w \in E'$. Here $\partial R'$ is the ideal boundary of R' in the standard sense (see [G2]). Then F, E' and b determine the extremal maximal dilatation $K(F, E', b) \ge 1$, defined as

$$K(F, E', b) = \inf\{K[G|R' \setminus E'] : G \in Q(F, E', b)\}, \tag{1.1}$$

where $K[G|R' \setminus E']$ is the maximal dilatation of G on $R' \setminus E'$. To avoid triviality, we will always assume that $K(F, E', b) > 1$, that is, $Q(F, E', b)$ contains no mapping which is conformal in $R' \setminus E'$. An element G of $Q(F, E', b)$ is called extremal if $K[G|R' \setminus E'] = K(F, E', b)$, and uniquely extremal if $K[G'|R' \setminus E'] > K[G|R' \setminus E']$ for any other $G' \in Q(F, E', b)$. If F is (uniquely) extremal in $Q(F, E', b)$, we occasionally say simply that F is (uniquely) extremal.

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As stated above, there have been many literatures on the problem of extremality of quasiconformal mappings with dilatation bounds (see, for example, [F], [FS], [G1], [R1], [Sa1-5]). Among others, it is known that there always exists at least one extremal mapping in $Q(F, E', b)$, and a complete characterization of the extremality also appeared in the literature. For our purpose, we recall this precisely as follows.

As usual, we denote by $Q(R)$ the space of all integrable holomorphic quadratic differentials on the surface R, and by $M(R)$ the unit ball of the space $L^\infty(R)$ of all essentially bounded Beltrami differentials on R. Now we let $E'_0 = \{w \in E' : b(w) = 0\}$. For the mapping F, let μ denote the complex dilatation of the inverse mapping $f = F^{-1}$, $k_F = \|\tilde{\mu}|R' \setminus E'\|_\infty$, and set

$$\tau_F(z) = \begin{cases} \mu(z) & z \in R \setminus F(E' \setminus E'_0)\,, \\ k_F \mu(z)/b(f(z)) & z \in F(E' \setminus E'_0)\,. \end{cases} \tag{1.2}$$

Then we have the following result (see, for example, [Sa4]).

PROPOSITION 1.1. *F is extremal if and only if the Beltrami differential τ_F satisfies the condition*

$$\sup\left\{\left|\iint_{R\setminus F(E'_0)} \tau_F \phi\right| ; \phi \in Q(R), \|\phi\|_{R\setminus F(E'_0)} = 1\right\} = \|\tau_F\|_\infty\,.$$

While Proposition 1.1 completely characterizes the extremality of the mapping F, less is known for the unique extremality. In several articles (see, for example, [G1], [R1]) it has been pointed out that the unique extremality of F is closely related to the uniqueness of the Hahn-Banach extension of the linear functional $\Lambda_\tau \in (Q(R)|R \setminus F(E'_0))^*$ induced by $\tau = \tau_F$, $\Lambda_\tau(\phi) = \iint_{R\setminus F(E'_0)} \tau\phi$. In fact, it was conjectured that F is uniquely extremal if and only if Λ_τ has a unique norm-preserving extension to a bounded linear functional from $Q(R)|R \setminus F(E'_0)$ to $L^1(R \setminus F(E'_0))$. Here $Q(R)|R \setminus F(E'_0)$ means the restriction to $R \setminus F(E'_0)$ of $Q(R)$.

When E' is the empty set, the unique extremality has been much discussed recently (see [BLMM], [BMM], [MM], [R4], [Sh1], [Sh2]), and the conjecture was proved affirmatively in [BLMM]. In this paper, we will study the unique extremality for a general set E', proving that the if part of the conjecture is still true in this general case.

THEOREM 1.1. *Let F be extremal (in the class $Q(F, E', b)$). If Λ_τ ($\tau = \tau_F$) has a unique norm-preserving extension to a bounded linear functional from $Q(R)|R \setminus F(E'_0)$ to $L^1(R \setminus F(E'_0))$, then F is uniquely extremal.*

In order to prove Theorem 1.1, we need to investigate a new extremal problem in the infinitesimal setting, namely, the extremal problem for partially zero Beltrami differentials. In Section 2, we will introduce such an extremal problem and explain how these two extremal problems are related to each other. In Sections 3 and 4, we will give a complete description of the unique infinitesimal extremality of partially zero Beltrami differentials. In Section 5, we shall establish a fundamental inequality, which will be used to prove Theorem 1.1 in Section 6.

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2. Partially zero Beltrami differentials. In this section, we will introduce a somewhat new extremal problem in an infinitesimal setting, which, as will be seen, is closely related to the extremal problem of quasiconformal mappings with dilatation bounds. Indeed, in the unit disk case, such an extremal problem has been introduced and dicussed in [SC], where it was used to prove the existence of non-decreasable dilatations in a non-zero infinitesimally equivalent class. For completeness and for generality, we will repeat some discussions from [SC].

Let E_0, which will be fixed through out Sections 2 through 4, be a compact, possibly empty, subset of R such that $R \setminus E_0$ has positive measure, and $\mu_0 \in L^\infty(R)$ be a Beltrami differential which vanishes on the set E_0. Recall that two elements μ and ν in $L^\infty(R)$ are infinitesimally equivalent, denoted by $\mu \approx \nu$, if $\iint_R \mu\phi = \iint_R \nu\phi$ for all $\phi \in Q(R)$. We denote by $\mathrm{Belt}(\mu_0)$ the set of all elements μ in $L^\infty(R)$ infinitesimally equivalent to μ_0 and set

$$\mathrm{Belt}(\mu_0, E_0) = \{\mu \in \mathrm{Belt}(\mu_0)\,;\, \mu(z) = 0 \text{ a.e. } z \in E_0\}\,, \tag{2.1}$$

$$\|\mu_0\|_{E_0} = \inf\{\|\mu\|_\infty\,;\, \mu \in \mathrm{Belt}(\mu_0, E_0)\}\,. \tag{2.2}$$

An element $\mu \in \mathrm{Belt}(\mu_0, E_0)$ is called infinitesimally extremal if $\|\mu\|_\infty = \|\mu_0\|_{E_0}$, and uniquely infinitesimally extremal if for any other $\nu \in \mathrm{Belt}(\mu_0, E_0)$, $\|\nu\|_\infty > \|\mu\|_\infty$. If μ_0 is (uniquely) infinitesimally extremal in $\mathrm{Belt}(\mu_0, E_0)$, we occasionally say simply that μ_0 is (uniquely) infinitesimally extremal.

We then have the following basic result.

THEOREM 2.1. *There always exists at least one infinitesimally extremal Beltrami differential in* $\mathrm{Belt}(\mu_0, E_0)$. *Furthermore*, *if* $\mathrm{Belt}(\mu_0, E_0)$ *contains more than one infinitesimally extremal Beltrami differential*, *then it must contain infinitely many.*

PROOF 1. Let $\mu_n \in \mathrm{Belt}(\mu_0, E_0)$ satisfy $\|\mu_n\|_\infty \to \|\mu_0\|_{E_0}$ as $n \to \infty$. When restricted on $R \setminus E_0$, (μ_n) is a bounded sequence in $L^\infty(R \setminus E_0)$. By the *-weak compactness, there exists a subsequence, also denoted by (μ_n), which converges to a limit $\mu \in L^\infty(R \setminus E_0)$ in the *-weak topology, that is, $\iint_{R\setminus E_0} \mu_n\phi \to \iint_{R\setminus E_0} \mu\phi$ for any $\phi \in L^1(R \setminus E_0)$. Now, when $\phi \in Q(R)$, since $\mu_n \in \mathrm{Belt}(\mu_0, E_0)$, $\iint_{R\setminus E_0} \mu_n\phi = \iint_R \mu_n\phi = \iint_R \mu_0\phi$, we obtain that $\iint_{R\setminus E_0} \mu\phi = \iint_R \mu_0\phi$. Extending μ to E_0 be zero, we conclude that $\mu \in \mathrm{Belt}(\mu_0, E_0)$. On the other hand, since (μ_n) converges to μ in the *-weak topology, $\|\mu\|_\infty \le \liminf \|\mu_n\|_\infty = \|\mu_0\|_{E_0}$, which implies that $\mu \in \mathrm{Belt}(\mu_0, E_0)$ is infinitesimally extremal.

Suppose now that μ and ν are two distinct infinitesimally extremal Beltrami differentials in $\mathrm{Belt}(\mu_0, E_0)$. For $0 < t < 1$, set $\mu_t = t\mu + (1 - t)\nu$. It is then easy to see that μ_t is infinitesimally extremal in $\mathrm{Belt}(\mu_0, E_0)$.

LEMMA 2.1. *For any* $\mu \in \mathrm{Belt}(\mu_0, E_0)$, *it holds that*

$$\sup\left\{\left|\iint_{R\setminus E_0}\mu\phi\right|;\ \phi\in Q(R),\ \|\phi\|_{R\setminus E_0}=1\right\}=\|\mu_0\|_{E_0}.$$

PROOF 2. Let $\mu \in \mathrm{Belt}(\mu_0, E_0)$ be given. For any $\nu \in \mathrm{Belt}(\mu_0, E_0)$ and $\phi \in Q(R)$ with $\|\phi\|_{R\setminus E_0} = 1$, since $\iint_{R\setminus E_0}\mu\phi = \iint_{R\setminus E_0}\nu\phi$, it follows that $|\iint_{R\setminus E_0}\mu\phi| \leq \|\nu\|_\infty$, which implies that

$$\begin{aligned}\sup&\left\{\left|\iint_{R\setminus E_0}\mu\phi\right|;\ \phi\in Q(R),\ \|\phi\|_{R\setminus E_0}=1\right\}\\ &\leq \inf\{\|\nu\|_\infty;\ \nu\in\mathrm{Belt}(\mu_0,E_0)\}=\|\mu_0\|_{E_0}.\end{aligned}\tag{2.3}$$

On the other hand, since the set $\{\phi|R\setminus E_0;\ \phi\in Q(R)\}$ is a closed subspace of $L^1(R\setminus E_0)$, by the Hahn-Banach theorem and the Riesz representative theorem, there exists some $\nu \in L^\infty(R\setminus E_0)$ such that

$$\iint_{R\setminus E_0}\mu\phi=\iint_{R\setminus E_0}\nu\phi\quad\text{for all }\phi\in Q(R)\tag{2.4}$$

and that

$$\sup\left\{\left|\iint_{R\setminus E_0}\mu\phi\right|;\ \phi\in Q(R),\ \|\phi\|_{R\setminus E_0}=1\right\}=\|\nu\|_\infty.\tag{2.5}$$

Extending ν to be zero on E_0, we obtain from (2.4) that $\nu \in \mathrm{Belt}(\mu_0, E_0)$. Then (2.3) and (2.5) imply that

$$\sup\left\{\left|\iint_{R\setminus E_0}\mu\phi\right|;\ \phi\in Q(R),\ \|\phi\|_{R\setminus E_0}=1\right\}=\|\mu_0\|_{E_0}.$$

The proof of Lemma 2.1 also shows that $\nu \in \mathrm{Belt}(\mu_0, E_0)$ is infinitesimally extremal, which gives another proof of the existence part of Theorem 2.1. An immediate consequence of Lemma 2.1 is the following theorem.

THEOREM 2.2. $\mu \in \mathrm{Belt}(\mu_0, E_0)$ *is infinitesimally extremal if and only if*

$$\sup\left\{\left|\iint_{R\setminus E_0}\mu\phi\right|;\ \phi\in Q(R),\ \|\phi\|_{R\setminus E_0}=1\right\}=\|\mu\|_\infty.$$

Now, we point out how the two extremal problems are related to each other. Noting Proposition 1.1 and Theorem 2.2, we find that F is extremal in $Q(F, E', b)$ if and only if the Beltrami differential τ_F is infinitesimally extremal in $\mathrm{Belt}(\tau_F, F(E_0'))$. On the other hand, if $\mu \in \mathrm{Belt}(\mu_0, E_0)$ is infinitesimally extremal, then we can conclude by the proof of Lemma 2.1 that μ is uniquely infinitesimally extremal if and only if the linear functional $\Lambda_\mu \in (Q(R)|R\setminus E_0)^*$ induced by μ, $\Lambda_\mu(\phi) = \iint_{R\setminus E_0}\mu\phi$, has a unique norm-preserving extension to a bounded linear functional from $Q(R)|R\setminus E_0$ to $L^1(R\setminus E_0)$. Thus, the conjecture in the Introduction is equivalent to the one that F is uniquely extremal in $Q(F, E', b)$ if and only if τ_F is uniquely infinitesimally extremal in $\mathrm{Belt}(\tau_F, F(E_0'))$. Hence Theorem 1.1 can be restated as follows.

THEOREM 2.3. *Given the class $Q(F, E', b)$, if τ_F is uniquely infinitesimally extremal in* $\mathrm{Belt}(\tau_F, F(E_0'))$, *then F is uniquely extremal.*

In order to prove Theorem 2.3, we need to investigate the unique infinitesimal extremality of partially zero Beltrami differentials, which will be done in Sections 3 and 4. Here we want to discuss a special case. Recall that the boundary dilatation of μ_0 is defined to be

$$b(\mu_0) = \inf\{\|\mu|R \setminus E\|_\infty \,;\ \text{for all } \mu \in \mathrm{Belt}(\mu_0) \text{ and compact subsets } E \text{ in } R\}\,. \tag{2.6}$$

It can be defined equivalently as (see [EGL], [GL])

$$b(\mu_0) = \sup\left\{\limsup_{n\to\infty}\left|\iint_R \mu_0\phi_n\right| ;\ \text{all degenerating sequences } (\phi_n) \text{ with } \|\phi_n\| \to 1\right\}. \tag{2.7}$$

Recall that a sequence (ϕ_n) in $Q(R)$ is said to be degenerating if $\phi_n \to 0$ locally uniformly in R. Clearly, $b(\mu_0) \le \|\mu_0\|_{E_0}$.

THEOREM 2.4. *Let μ be infinitesimally extremal in* $\mathrm{Belt}(\mu_0, E_0)$. *If $b(\mu_0) < \|\mu_0\|_{E_0}$, then μ is uniquely infinitesimally extremal. Furthermore, there exists some element $\phi \in Q(R)$ with $\|\phi\|_{R\setminus E_0} = 1$ such that $\mu = \|\mu\|_\infty |\phi|/\phi \chi_{R\setminus E_0}$, where χ stands for the characteristic function of a set.*

PROOF 3. Since μ is infinitesimally extremal, it follows from Theorem 2.2 that there exists a sequence (ϕ_n) in $Q(R)$ with $\|\phi_n\|_{R\setminus E_0} = 1$ such that

$$\left|\iint_{R\setminus E_0} \mu\phi_n\right| \to \|\mu\|_\infty\,. \tag{2.8}$$

Since E_0 is compact in R, it follows that (ϕ_n) is a bounded sequence in $Q(R)$. Otherwise, there would exist some subsequence, also denoted by (ϕ_n), such that $\|\phi_n\| \to \infty$ as $n \to \infty$. Letting $\tilde{\phi_n} = \phi_n/\|\phi_n\|$, we obtain $\|\tilde{\phi_n}\| = 1$. So there exists a subsequence, still denoted by $(\tilde{\phi_n})$, such that $\tilde{\phi_n}$ tends locally uniformly in R to some $\tilde{\phi}$ in $Q(R)$. By Fatou's Lemma, $\|\tilde{\phi}\| \le 1$. Then, for any subset F of R, noting that $0 \le |\tilde{\phi_n} - \tilde{\phi}| - |\tilde{\phi_n}| + |\tilde{\phi}| \le 2|\tilde{\phi}|$, we conclude by Lebesgue's dominated convergence theorem that $\|\tilde{\phi_n} - \tilde{\phi}\|_F - \|\tilde{\phi_n}\|_F + \|\tilde{\phi}\|_F \to 0$ as $n \to \infty$. If $F = E_0$, then $\|\tilde{\phi_n} - \tilde{\phi}\|_{E_0} \to 0$. On the other hand, $\|\tilde{\phi_n}\|_{E_0} = \|\phi_n\|_{E_0}/\|\phi_n\| = (\|\phi_n\| - 1)/\|\phi_n\| \to 1$, so we have $\|\tilde{\phi}\|_{E_0} = 1$, which contradicts $\|\tilde{\phi}\| \le 1$.

Now, since (ϕ_n) is a bounded sequence in $Q(R)$, there exists a subsequence, also denote by (ϕ_n), which converges to some function $\phi \in Q(R)$ locally uniformly in R. By the same reasoning as above, we have

$$\lim_{n\to\infty} (\|\phi_n - \phi\|_F - \|\phi_n\|_F + \|\phi\|_F) = 0 \tag{2.9}$$

for any subset F of R. In particular, we have

$$\lim_{n\to\infty} \|\phi_n - \phi\|_{R\setminus E_0} = 1 - \|\phi\|_{R\setminus E_0}\,. \tag{2.10}$$

Under the assumption that $b(\mu_0) < \|\mu_0\|_{E_0}$, we want to show that $\|\phi\|_{R\setminus E_0} = 1$. Suppose to the contrary that $\|\phi\|_{R\setminus E_0} < 1$, and set

$$\psi_n = \frac{\phi_n - \phi}{\|\phi_n - \phi\|_{R\setminus E_0}}.$$

Then (ψ_n) is a sequence in $Q(R)$ which satisfies that $\|\psi_n\|_{R\setminus E_0} = 1$, $\psi_n \to 0$ locally uniformly in R and that

$$\left|\iint_{R\setminus E_0} \mu\psi_n\right| \to \|\mu\|_\infty. \tag{2.11}$$

Since E_0 is compact in R, $\|\psi_n\| \to 1$, so (2.11) implies that $b(\mu_0) = b(\mu) = \|\mu\|_\infty = \|\mu_0\|_{E_0}$. This is a contradiction.

Consequently, $\|\phi\|_{R\setminus E_0} = 1$, which implies from (2.8) and (2.10) that $\iint_{R\setminus E_0} \mu\phi = \|\mu\|_\infty$ and hence that $\mu = \|\mu\|_\infty |\phi|/\phi \chi_{R\setminus E_0}$ as required. Finally, it is easy to see that μ is uniquely infinitesimally extremal.

REMARK 2.1. For simplicity, we say that a Beltrami differential μ_0 which vanishes on the compact set E_0 is a Strebel differential (with respect to E_0) if $b(\mu_0) < \|\mu_0\|_{E_0}$.

3. Characterization of unique infinitesimal extremality. In this section we will characterize the unique infinitesimal extremality of partially zero Beltrami differentials under certain condition. In its proof, we need the following fundamental inequality. Recall that μ_0 is a Beltrami differential on R which vanishes on the compact subset E_0.

LEMMA 3.1. *Let μ and ν be two Beltrami differentials in the class* $\mathrm{Belt}(\mu_0, E_0)$. *If $\|\nu\|_\infty \le \|\mu\|_\infty$, then*

$$\iint_{R\setminus E_0} |\mu-\nu|^2|\phi| \le 8\|\mu\|_\infty\left(\|\mu\|_\infty\|\phi\|_{R\setminus E_0} - \mathrm{Re}\iint_{R\setminus E_0}\mu\phi\right) \tag{3.1}$$

for all $\phi \in Q(R)$.

REMARK 3.1. When $E_0 = \emptyset$, Lemma 3.1 was proved in [BLMM](see also [R2], [R3]), and called the infinitesimal delta inequality. For completeness, we give here a short proof using a discussion from [GL].

PROOF OF LEMMA 3.1. Let $k = \|\mu\|_\infty$. For any $\phi \in Q(R)$, $\iint_{R\setminus E_0}\mu\phi = \iint_{R\setminus E_0}\nu\phi$. Therefore, Lemma 3.1 follows from the following calculation.

$$\begin{aligned}
\iint_{R\setminus E_0} |\mu-\nu|^2|\phi| &= \iint_{R\setminus E_0} \left|\mu - k\frac{|\phi|}{\phi} + k\frac{|\phi|}{\phi} - \nu\right|^2 |\phi| \\
&\le 2\iint_{R\setminus E_0} \left|\mu - k\frac{|\phi|}{\phi}\right|^2 |\phi| + 2\iint_{R\setminus E_0} \left|\nu - k\frac{|\phi|}{\phi}\right|^2 |\phi| \\
&= 2\iint_{R\setminus E_0} \left(|\mu|^2 + k^2 - 2k\mathrm{Re}\,\mu\frac{\phi}{|\phi|}\right)|\phi| + 2\iint_{R\setminus E_0} \left(|\nu|^2 + k^2 - 2k\mathrm{Re}\,\nu\frac{\phi}{|\phi|}\right)|\phi| \\
&\le 2\iint_{R\setminus E_0} (2k^2|\phi| - 2k\mathrm{Re}\,\mu\phi) + 2\iint_{R\setminus E_0} (2k^2|\phi| - 2k\mathrm{Re}\,\nu\phi) \\
&= 8k\left(k\iint_{R\setminus E_0} |\phi| - \mathrm{Re}\iint_{R\setminus E_0} \mu\phi\right).
\end{aligned}$$

For a Beltrami differential μ in the class $\mathrm{Belt}(\mu_0, E_0)$, the set $R(\mu) = \{z \in R\,;\, |\mu(z)| = \|\mu\|_\infty\}$ is called the extremal set for μ. We introduce the Reich's functional δ_μ on $Q(R)|R\setminus E_0$ induced by μ, $\delta_\mu(\phi) = \|\mu\|_\infty\|\phi\|_{R\setminus E_0} - \mathrm{Re}\iint_{R\setminus E_0}\mu\phi$. We say that μ satisfies Reich's condition on a set $E \subset R \setminus E_0$ if there exists a sequence (ϕ_n) in $Q(R)$ such that $\delta_\mu(\phi_n) \to 0$ and $\liminf|\phi_n(z)| > 0$ for almost all z in E. We are in a position to prove the main result of this section.

THEOREM 3.1. *Let μ be a Beltrami differential in the class* $\mathrm{Belt}(\mu_0, E_0)$ *with* $|\mu| = \|\mu\|_\infty$ *almost everywhere on* $R \setminus E_0$. *Then the following conditions are equivalent*:

(a) *μ is uniquely infinitesimally extremal in the class* $\mathrm{Belt}(\mu_0, E_0)$.

(b) *μ is infinitesimally extremal in the class* $\mathrm{Belt}(\mu_0, E_0)$ *and, for every compact subset E of $R \setminus E_0$ with positive measure and every $r > 0$, $\mu\chi_E + (1/(1+r))\mu\chi_{R\setminus E}$ is a Strebel differential* (*with respect to E_0*).

(c) *For every measurable subset E of $R \setminus E_0$ with positive measure, there exists a sequence (ϕ_n) in $Q(R)$ with $\|\phi_n\|_{R\setminus E_0} = 1$ such that*

$$\left(\|\mu\|_\infty - \mathrm{Re}\iint_{R\setminus E_0} \mu\phi_n\right)\bigg/\iint_E |\phi_n| \to 0\,.$$

(d) *μ satisfies Reich's condition on $R \setminus E_0$.*

PROOF 4. Suppose that μ is uniquely infinitesimally extremal. For every compact subset E of $R\setminus E_0$ with positive measure and every $r > 0$, let $\mu(r, E) = \mu\chi_E + (1/(1+r))\mu\chi_{R\setminus E}$. We need to show that $\mu(r, E)$ is a Strebel differential, that is, $b(\mu(r, E)) < \|\mu(r, E)\|_{E_0}$.

It is easy to see that $b(\mu(r, E)) \le \|\mu\|_\infty/(1 + r)$. Suppose to the contrary that $b(\mu(r, E)) = \|\mu(r, E)\|_{E_0}$. Let $\nu(r, E)$ be an infinitesimally extremal Beltrami differential in the class $\mathrm{Belt}(\mu(r, E), E_0)$. Then $\|\nu(r, E)\|_\infty = \|\mu(r, E)\|_{E_0} = b(\mu(r, E)) \le \|\mu\|_\infty/(1 + r)$. Clearly, $\nu = \mu - \mu(r, E) + \nu(r, E) = (r/(1+r))\mu\chi_{R\setminus E} + \nu(r, E) \in \mathrm{Belt}(\mu_0, E_0)$. Since $\|\nu\|_\infty \le (r/(1 + r))\|\mu\|_\infty + \|\nu(r, E)\|_\infty \le \|\mu\|_\infty$, we conclude by the unique infinitesimal extremality of μ that $\nu = \mu$, that is, $\mu(r, E) = \nu(r, E)$ is infinitesimally extremal. Since $|\mu| = \|\mu\|_\infty$ almost everywhere on $R \setminus E_0$, this case cannot occur. So (a) implies (b).

Now, let μ satisfies the condition (b). By Theorem 2.4, there exists an element $\phi(r, E)$ in $Q(R)$ with $\|\phi(r, E)\|_{R\setminus E_0} = 1$ such that the infinitesimally extremal Beltrami differential $\nu(r, E)$ in $\mathrm{Belt}(\mu(r, E), E_0)$ has the form $\|\nu(r, E)\|_\infty |\phi(r, E)|/\phi(r, E)\chi_{R\setminus E_0}$. On the other hand, since μ is infinitesimally extremal in $\mathrm{Belt}(\mu_0, E_0)$, and $\nu = \mu - \mu(r, E) + \nu(r, E) = (r/(1+r))\mu\chi_{R\setminus E} + \nu(r, E) \in \mathrm{Belt}(\mu_0, E_0)$, we have $\|\mu\|_\infty \leq \|\nu\|_\infty \leq (r/(1+r))\|\mu\|_\infty + \|\nu(r, E)\|_\infty$, so $\|\nu(r, E)\|_\infty \geq \|\mu\|_\infty/(1+r)$. Consequently,

$$\frac{\|\mu\|_\infty}{1+r} \leq \|\nu(r, E)\|_\infty = \mathrm{Re} \iint_{R\setminus E_0} \nu(r, E)\phi(r, E)$$

$$= \mathrm{Re} \iint_{R\setminus E_0} \mu(r, E)\phi(r, E) = \mathrm{Re} \iint_E \mu\phi(r, E) + \mathrm{Re}\frac{1}{1+r} \iint_{R\setminus E_0\setminus E} \mu\phi(r, E)\,.$$

Thus,

$$\|\mu\|_\infty - \mathrm{Re} \iint_{R\setminus E_0} \mu\phi(r, E) = \mathrm{Re} \iint_E r\mu\phi(r, E) \leq \|\mu\|_\infty r \iint_E |\phi(r, E)|\,. \tag{3.2}$$

For each measurable subset E of R with positive measure, choose a compact subset $\tilde{E}$ of E with positive measure. Then for any $r > 0$, there exists an element $\phi(r, \tilde{E})$ in $Q(R)$ with $\|\phi(r, \tilde{E})\|_{R\setminus E_0} = 1$ such that

$$\|\mu\|_\infty - \mathrm{Re} \iint_{R\setminus E_0} \mu\phi(r, \tilde{E}) \leq \|\mu\|_\infty r \iint_{\tilde{E}} |\phi(r, \tilde{E})| \leq \|\mu\|_\infty r \iint_E |\phi(r, \tilde{E})|\,. \tag{3.3}$$

For $n \geq 1$, set $r = 1/n$ and $\phi_n = \phi(r, \tilde{E})$. Then we conclude by (3.3) that

$$0 \leq \left(\|\mu\|_\infty - \mathrm{Re} \iint_{R\setminus E_0} \mu\phi_n\right) \Big/ \iint_E |\phi_n| \leq \frac{\|\mu\|_\infty}{n} \to 0\,.$$

So (b) implies (c).

Finally, let the condition (c) be satisfied. Suppose that μ is not uniquely infinitesimally extremal. Then there would exist some ν in the class $\mathrm{Belt}(\mu_0, E_0)$ such that $\|\nu\|_\infty \leq \|\mu\|_\infty$ and that $|\nu - \mu| \geq \varepsilon_0 > 0$ on some positive measure subset E of $R \setminus E_0$. Note that for this set E, there exists a sequence (ϕ_n) in $Q(R)$ with $\|\phi_n\|_{R\setminus E_0} = 1$ such that

$$\left(\|\mu\|_\infty - \mathrm{Re} \iint_{R\setminus E_0} \mu\phi_n\right) \Big/ \iint_E |\phi_n| \to 0\,. \tag{3.4}$$

On the other hand, by Lemma 3.1 we have that

$$\varepsilon_0^2 \iint_E |\phi_n| \leq \iint_{R\setminus E_0} |\nu - \mu|^2 |\phi_n| \leq C\|\mu\|_\infty \left(\|\mu\|_\infty - \mathrm{Re} \iint_{R\setminus E_0} \mu\phi_n\right),$$

which contradicts (3.4). So (c) implies (a).

We will prove the equivalence of (a) and (d) in the next section (Theorem 4.1).

REMARK 3.2. From the proof we see that it holds that (b)⇒(c)⇒(a) for any Beltrami differential μ in the class $\mathrm{Belt}(\mu_0, E_0)$. The condition $|\mu| = \|\mu\|_\infty$ almost everywhere in $R \setminus E_0$ is only used in the proof of (a)⇒(b). Indeed, when μ satisfies the condition (a) (without the condition that $|\mu| = \|\mu\|_\infty$ almost everywhere in $R \setminus E_0$), (b) still holds for those compact subsets E of $R \setminus E_0$ with positive measure and $\|\mu|E\|_\infty = \|\mu\|_\infty$. We will use this fact in the next section.

REMARK 3.3. We say that μ is uniquely infinitesimally extremal with respect to $S \subset R \setminus E_0$ with positive measure if for any other $\nu \in \mathrm{Belt}(\mu_0, E_0)$ with $\|\nu\|_\infty \le \|\mu\|_\infty$, $\mu = \nu$ almost everywhere on S. Examining the proof of $(c) \Rightarrow (a)$, we find that if the condition (c) is satisfied for every compact subset E of $S \subset R \setminus E_0$ with positive measure, then μ is uniquely infinitesimally extremal with respect to S.

4. Characterization of unique infinitesimal extremality (continued). We continue to discuss the unique infinitesimal extremality of partially zero Beltrami differentials. We will modify the discussion in [BLMM].

In general, for a bounded linear functional Λ with real part λ on a subspace Y of a normed space X, we may define

$$\overline{\lambda}(x_0) = \inf_{y \in Y} \{\lambda(y) + \|\lambda\| \|y - x_0\|\}$$

and

$$\underline{\lambda}(x_0) = \sup_{y \in Y} \{\lambda(y) - \|\lambda\| \|y - x_0\|\}\,.$$

The analysis in the proof of the Hahn-Banach theorem leads to the following lemma.

LEMMA 4.1. *Λ has a unique norm-preserving extension from Y to X if and only if $\overline{\lambda}(x_0) = \underline{\lambda}(x_0)$ for all $x_0 \in X \setminus Y$.*

We say that λ satisfies the unique approximation property at $x_0 \in X \setminus Y$ if there exists sequences (y_{n1}) and (y_{n2}) in Y such that

$$\lambda(y_{n1} - y_{n2}) = \|\lambda\|(\|y_{n1} - x_0\| + \|y_{n2} - x_0\|) + o(1)\,.$$

Then we have

LEMMA 4.2. *Λ has a unique norm-preserving extension from Y to X if and only if λ satisfies the unique approximation property at each $x_0 \in X \setminus Y$.*

We now proceed to discuss the unique infinitesimal extremality of a Beltrami differential. Let, as before, μ_0 be a Beltrami differential on R which is zero on the compact subset E_0. Then we have

LEMMA 4.3. *If $\mu \in \mathrm{Belt}(\mu_0, E_0)$ satisfies Reich's condition on a set $E \subset R \setminus E_0$, then μ is uniquely infinitesimally extremal in the class $\mathrm{Belt}(\mu_0, E_0)$ with respect to E.*

PROOF 5. Suppose that μ is not uniquely infinitesimally extremal with respect to E. Then there would exist some ν in the class $\mathrm{Belt}(\mu_0, E_0)$ such that $\|\nu\|_\infty \le \|\mu\|_\infty$ and that $|\nu - \mu| \ge \varepsilon_0 > 0$ on some positive measure subset $\tilde{E}$ of E. It follows from Lemma 3.1 that

$$\varepsilon_0^2 \iint_{\tilde{E}} |\phi| \le \iint_{R \setminus E_0} |\nu - \mu|^2 |\phi| \le C\|\mu\|_\infty \delta_\mu(\phi) \tag{4.1}$$

for all $\phi \in Q(R)$.

On the other hand, since μ satisfies Reich's condition on E, there exists a sequence (ϕ_n) in $Q(R)$ such that $\liminf |\phi_n| > 0$ almost everywhere in E and that $\delta_\mu(\phi_n) \to 0$. Applying Fatou's Lemma, we then obtain a contradiction from (4.1).

LEMMA 4.4. *If μ is uniquely infinitesimally extremal in the class* Belt(μ_0, E_0), *then μ satisfies Reich's condition on its extremal set.*

PROOF 6. Suppose that μ is uniquely infinitesimally extremal in the class Belt(μ_0, E_0), and let $E = R(\mu)$ be its extremal set. Without loss of generality, we may assume that $\|\mu\|_\infty = 1$. Take $\phi \in Q(R)$ such that $\|\phi\|_{R\setminus E_0} = 1$, and let $\psi = |\phi|\overline{\mu}\chi_E$. Clearly, $\psi \in L^1(R \setminus E_0)$ and

$$\iint_{R\setminus E_0} \mu\psi = \iint_E |\phi| = \|\psi\|_{R\setminus E_0}\,.$$

Since μ is uniquely infinitesimally extremal in Belt(μ_0, E_0), $\Lambda_\mu(\psi) = \iint_{R\setminus E_0} \mu\psi$ is the unique norm-preserving extension from $Q(R)|R \setminus E_0$ to $L^1(R \setminus E_0)$. By Lemma 4.1, for the real part $\lambda_\mu = \operatorname{Re} \Lambda_\mu$, noting that $\|\lambda_\mu\| = \|\Lambda_\mu\| = \|\mu\|_\infty = 1$, there exists a sequence (ϕ_n) in $Q(R)$ such that $\lambda_\mu(\phi_n) - \|\phi_n - \psi\|_{R\setminus E_0} \to \lambda_\mu(\psi)$, that is,

$$\operatorname{Re} \iint_{R\setminus E_0} \mu\psi + \|\phi_n - \psi\|_{R\setminus E_0} - \operatorname{Re} \iint_{R\setminus E_0} \mu\phi_n \to 0\,.$$

Consequently,

$$\begin{aligned}
0 \le \delta_\mu(\phi_n) &= \|\phi_n\|_{R\setminus E_0} - \operatorname{Re} \iint_{R\setminus E_0} \mu\phi_n \\
&\le \|\psi\|_{R\setminus E_0} + \|\phi_n - \psi\|_{R\setminus E_0} - \operatorname{Re} \iint_{R\setminus E_0} \mu\phi_n \\
&= \operatorname{Re} \iint_{R\setminus E_0} \mu\psi + \|\phi_n - \psi\|_{R\setminus E_0} - \operatorname{Re} \iint_{R\setminus E_0} \mu\phi_n \to 0\,.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
0 &\le \iint_E (|\psi| + |\phi_n - \psi| - |\phi_n|) \\
&\le \|\psi\|_{R\setminus E_0} + \|\phi_n - \psi\|_{R\setminus E_0} - \|\phi_n\|_{R\setminus E_0} \\
&\le \|\psi\|_{R\setminus E_0} + \|\phi_n - \psi\|_{R\setminus E_0} - \operatorname{Re} \iint_{R\setminus E_0} \mu\phi_n \to 0\,,
\end{aligned}$$

we may assume without loss of generality that $|\psi| + |\phi_n - \psi| - |\phi_n| \to 0$ for almost all $z \in E$. Hence $\liminf |\phi_n(z)| \ge |\psi(z)| = |\phi(z)| > 0$ for almost all $z \in E$.

An immediate consequence of Lemmas 4.3 and 4.4 is the following theorem, which gives another characterization of the unique infinitesimal extremality of Beltrami differentials with constant absolute value on $R \setminus E_0$ (see Theorem 3.1).

THEOREM 4.1. *If* $\mu \in \mathrm{Belt}(\mu_0, E_0)$ *satisfies the condition that* $|\mu| = \|\mu\|_\infty$ *almost everywhere in* $R \setminus E_0$, *then* μ *is uniquely infinitesimally extremal in* $\mathrm{Belt}(\mu_0, E_0)$ *if and only if* μ *satisfies Reich's condition on* $R \setminus E_0$.

We are now in a position to characterize the unique infinitesimal extremality for an arbitrary Beltrami differential. Let $\mu \in \mathrm{Belt}(\mu_0, E_0)$. We say that a Beltrami differential η is an admissible variation of μ if η equals zero on E_0, $\|\eta\|_\infty \le \|\mu\|_\infty$, and if there exists some, possibly empty, subset E of $R \setminus E_0$ such that $|\mu| \le k_0 < k = \|\mu\|_\infty$ almost everywhere in E and $\mu = \eta$ in $R \setminus E$.

LEMMA 4.5. *If* μ *is (uniquely) infinitesimally extremal in the class* $\mathrm{Belt}(\mu_0, E_0)$, *then every admissible variation* η *of* μ *is (uniquely) infinitesimally extremal.*

PROOF 7. Suppose μ is infinitesimally extremal in the class $\mathrm{Belt}(\mu_0, E_0)$, and η is any admissible variation of μ. Then there exists a subset E of $R \setminus E_0$ such that $|\mu| \le k_0 < k = \|\mu\|_\infty$ almost everywhere in E and $\mu = \eta$ in $R \setminus E$. Take any real number $t > 2k/(k - k_0)$. Then, for any Beltrami differential η' in the class $\mathrm{Belt}(\eta, E_0)$, $t\mu + \eta' - \eta \in \mathrm{Belt}(t\mu, E_0)$. Since $t\mu$ is infinitesimally extremal, we have

$$\begin{aligned} tk = \|t\mu\|_\infty &\le \|\eta'\|_\infty + \|t\mu - \eta\|_\infty \\ &\le \|\eta'\|_\infty + \max\{(t-1)k, tk_0 + k\} = \|\eta'\|_\infty + (t-1)k\,, \end{aligned}$$

so $\|\eta'\|_\infty \ge k \ge \|\eta\|_\infty$, that is, η is infinitesimally extremal.

If μ is uniquely infinitesimally extremal in the class $\mathrm{Belt}(\mu_0, E_0)$, the above reasoning also shows that η is uniquely infinitesimally extremal.

Now we can prove the following general characterization theorem for the unique infinitesimal extremality of Beltrami differentials.

THEOREM 4.2. *Let* $\mu \in \mathrm{Belt}(\mu_0, E_0)$. *The following conditions are equivalent*:

(1) μ *is uniquely infinitesimally extremal in* $\mathrm{Belt}(\mu_0, E_0)$.

(2) μ *is infinitesimally extremal in* $\mathrm{Belt}(\mu_0, E_0)$ *and, for every* $r > 0$, *every admissible variation* η *of* μ, *and every compact subset* E *of* $R(\eta)$ *with positive measure*, $\eta\chi_E + (1/(1+r))\eta\chi_{R\setminus E}$ *is a Strebel differential.*

(3) *For every admissible variation* η *of* μ *and every compact subset* E *of* $R(\eta)$ *with positive measure, there exists a sequence* (ϕ_n) *in* $Q(R)$ *with* $\|\phi_n\|_{R\setminus E_0} = 1$ *such that*

$$\left(\|\eta\|_\infty - Re \iint_{R\setminus E_0} \eta\phi_n\right) \Big/ \iint_E |\phi_n| \to 0\,.$$

(4) *Every admissible variation* η *of* μ *is uniquely infinitesimally extremal with respect to* $R(\eta)$.

(5) *Every admissible variation* η *of* μ *is uniquely infinitesimally extremal.*

(6) *Every admissible variation* η *of* μ *satisfies reich's condition on* $R(\eta)$.

(7) λ_μ *satisfies the unique approximation property at each* $\psi \in L^1(R \setminus E_0) \setminus Q(R)|R \setminus E_0$.

PROOF 8. (1) is equivalent to (7) by Lemma 4.2.

(1) $\Rightarrow$ (2). Since μ is uniquely infinitesimally extremal, it is definitely infinitesimally extremal, and every admissible variation η of μ is also uniquely infinitesimally extremal by Lemma 4.5. Hence (2) can be proved by the same method as in the proof of $(a) \Rightarrow (b)$ of Theorem 3.1 (see remark 3.2).

(2) $\Rightarrow$ (3). Since μ is infinitesimally extremal, every admissible variation η of μ is also infinitesimally extremal by Lemma 4.5. Therefore (3) can be proved by the same method as in the proof of $(b) \Rightarrow (c)$ of Theorem 3.1.

(3) $\Rightarrow$ (4). This can be proved by the same method as in the proof of $(c) \Rightarrow (a)$ of Theorem 3.1(see remark 3.3).

(4) $\Rightarrow$ (1). Suppose that μ is not uniquely infinitesimally extremal. Then there would exist some ν in the class Belt(μ_0, E_0) such that $\|\nu\|_\infty \leq \|\mu\|_\infty$ and that $|\nu - \mu| \geq \varepsilon_0 > 0$ on some positive measure compact subset E of $R \setminus E_0$. Noting that μ itself is an admissible variation of μ, we conclude that the set $\{z \in E\,;\ |\mu(z)| = \|\mu\|_\infty\}$ must be a set of measure zero. So we may assume that for some $k_0 < k = \|\mu\|_\infty$, the set $\tilde{E} = \{z \in E : |\mu(z)| \leq k_0\}$ is compact and has positive measure.

Define $\eta = k((\mu - \nu)/|\mu - \nu|)\chi_{\tilde{E}} + \mu\chi_{R\setminus\tilde{E}}$. Then η is an admissible variation of μ. Now set $\eta' = \eta + \nu - \mu$. Then $\eta' \in$ Belt(η, E_0). Noting that

$$\eta' = k\frac{\mu - \nu}{|\mu - \nu|}\chi_{\tilde{E}} + \mu\chi_{R\setminus\tilde{E}} + \nu - \mu = \left(\frac{k}{|\mu - \nu|} - 1\right)(\mu - \nu)\chi_{\tilde{E}} + \nu\chi_{R\setminus\tilde{E}}\,,$$

we get $|\eta'| \leq \max\{|k - |\mu - \nu||, k\} = k$, which implies that $\|\eta'\|_\infty \leq k$. Since $|\eta| = k$ on $\tilde{E}$, and η is uniquely infinitesimally extremal with respect to its extremal subset $R(\eta)$, we conclude that $\eta' = \eta$ and consequently that $\mu = \nu$ on $\tilde{E} \subset E$, which is a contradiction.

(1) $\Rightarrow$ (5). This follows directly from Lemma 4.5.

(5) $\Rightarrow$ (6). This follows directly from Lemma 4.4.

(6) $\Rightarrow$ (4). This follows directly from Lemma 4.3.

5. A fundamental inequality. In this section, we establish a fundamental inequality parallel to the delta inequality in [BLMM], which will be used to prove Theorem 2.3 in the next section. We shall repeat some discussion from [BLMM] for completeness.

LEMMA 5.1. *Let G be a quasiconformal mapping in the class $Q(F, E', b)$. Let, as before, $\tilde{\mu}$, $\tilde{\nu}$, μ and ν denote the complex dilatations of the mappings F, G, $f = F^{-1}$ and $g = G^{-1}$, respectively. If $k_G \leq k_F$, then*

$$\iint_{R\setminus F(E_0')} \left|\frac{\tilde{\mu}(f) - \tilde{\nu}(f)}{1 - \overline{\tilde{\mu}(f)}\tilde{\nu}(f)}\right|^2 |\phi| \leq C\left(k_F\|\phi\|_{R\setminus F(E_0')} - \mathrm{Re}\iint_{R\setminus F(E_0')} \tau_F\phi\right),$$

for all $\phi \in Q(R)$. The constant C depends only on k_F and $\|b\|_\infty$.

PROOF 9. Set $\tau = \tau_F$, $k = k_F$, $\alpha = \tilde{\mu}(f)$, $\beta = \tilde{\nu}(f)$. Since $G \in Q(F, E', b)$, it follows from the main inequality of Reich-Strebel (see [G2]) that

$$\iint_R |\phi| \leq \iint_R |\phi| \frac{|1 - \mu\phi/|\phi||^2}{1 - |\mu|^2} \frac{|1 + \beta(\mu/\alpha)(\phi/|\phi|)(1 - \overline{\mu}\overline{\phi}/|\phi|)/(1 - \mu\phi/|\phi|)|^2}{1 - |\beta|^2},$$

for all $\phi \in Q(R)$, or equivalently (see [R2], [R3]),

$$\text{(5.1)} \qquad \operatorname{Re} \iint_R \frac{(\alpha - \beta)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{\mu}{\alpha}\phi \leq \iint_R \frac{|\alpha - \beta|^2}{(1 - |\alpha|^2)(1 - |\beta|^2)} |\phi| .$$

Noting that $\mu/\alpha = -\overline{\partial f}/\partial f$ and $\alpha = \beta = 0$ on $R \setminus F(E_0')$, we obtain from (5.1) that

$$\text{(5.2)} \qquad \operatorname{Re} \iint_{R \setminus F(E_0')} \frac{(\alpha - \beta)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{\mu}{\alpha}\phi \leq \iint_{R \setminus F(E_0')} \frac{|\alpha - \beta|^2}{(1 - |\alpha|^2)(1 - |\beta|^2)} |\phi| .$$

Adding $\operatorname{Re} \iint_{R \setminus F(E_0')} ((\beta - \alpha)(1 - \alpha\bar{\beta})/(1 - |\alpha|^2)(1 - |\beta|^2))(|\alpha|/\alpha)|\phi|$ to both sides of the above inequality, we conclude that

$$\iint_{R \setminus F(E_0')} \frac{(1 - |\alpha|)^2|\alpha - \beta|^2 + (1 - |\alpha|^2)(|\alpha|^2 - |\beta|^2)}{2|\alpha|(1 - |\alpha|^2)(1 - |\beta|^2)} |\phi|$$
$$\leq \operatorname{Re} \iint_{R \setminus F(E_0')} \frac{(\alpha - \beta)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{1}{\alpha}(|\mu||\phi| - \mu\phi) ,$$

or equivalently,

$$\text{(5.3)} \qquad \begin{aligned} &\iint_{R \setminus F(E_0')} \frac{(1 - |\alpha|)|\alpha - \beta|^2}{2|\alpha|(1 + |\alpha|)(1 - |\beta|^2)} |\phi| \\ &\leq \operatorname{Re} \iint_{R \setminus F(E_0')} \frac{(\alpha - \beta)(1 - \alpha\bar{\beta})}{(1 - |\alpha|^2)(1 - |\beta|^2)} \frac{1}{\alpha}(|\mu||\phi| - \mu\phi) \\ &\quad + \iint_{R \setminus F(E_0')} \frac{|\beta|^2 - |\alpha|^2}{2|\alpha|(1 - |\beta|^2)} |\phi| . \end{aligned}$$

We first assume that $|\tau| \geq k/2$ on $R \setminus F(E_0')$, that is, $|\alpha| = |\mu| \geq k/2$ on $R \setminus F(E')$ and $|\alpha| = |\mu| \geq b(f)/2$ on $F(E' \setminus E_0')$. Note that on $R \setminus F(E')$

$$\text{(5.4)} \qquad \frac{(1 - |\alpha|)|\alpha - \beta|^2}{2|\alpha|(1 + |\alpha|)(1 - |\beta|^2)} |\phi| \geq \frac{(1 - \max\{k, \|b\|_\infty\})|\alpha - \beta|^2}{4} |\phi| .$$

Using the identity

$$||w| - w|^2 = 2|w|(|w| - \operatorname{Re} w) ,$$

we get

$$
\begin{aligned}
&\operatorname{Re}\frac{(\alpha-\beta)(1-\alpha\bar{\beta})}{(1-|\alpha|^2)(1-|\beta|^2)}\frac{1}{\alpha}(|\mu||\phi|-\mu\phi)\\
&\quad\le \frac{|\alpha-\beta||1-\alpha\bar{\beta}|}{(1-|\alpha|^2)(1-|\beta|^2)}\frac{1}{|\alpha|}\{2|\mu||\phi|(|\mu||\phi|-\operatorname{Re}\mu\phi)\}^{1/2}\\
&\quad\le \begin{cases} 4/(1-\|b\|_\infty^2)^2\sqrt{1/k}\sqrt{|\alpha-\beta|^2|\phi|}\sqrt{k|\phi|-\operatorname{Re}\tau\phi} & \text{in } F(E'\setminus E_0')\\ 4/(1-k^2)^2\sqrt{1/k}\sqrt{|\alpha-\beta|^2|\phi|}\sqrt{k|\phi|-\operatorname{Re}\tau\phi} & \text{in } R\setminus F(E')\end{cases}\\
&\quad\le \frac{4}{\sqrt{k}(1-(\max\{k,\|b\|_\infty\})^2)^2}\sqrt{|\alpha-\beta|^2|\phi|}\sqrt{k|\phi|-\operatorname{Re}\tau\phi}\,.
\end{aligned}\tag{5.5}
$$

We also have

$$
\begin{aligned}
\frac{|\beta|^2-|\alpha|^2}{2|\alpha|(1-|\beta|^2)}|\phi| &\le \begin{cases} 2/(k(1-\|b\|_\infty^2))(k|\phi|-\operatorname{Re}\tau\phi) & \text{in } F(E'\setminus E_0')\\ 2/(k(1-k^2))(k|\phi|-\operatorname{Re}\tau\phi) & \text{in } R\setminus F(E')\end{cases}\\
&\le \frac{2}{k(1-(\max\{k,\|b\|_\infty\})^2)}(k|\phi|-\operatorname{Re}\tau\phi)\,.
\end{aligned}\tag{5.6}
$$

It follows from (5.3) through (5.6) that

$$
\begin{aligned}
&\iint_{R\setminus F(E_0')}|\alpha-\beta|^2|\phi|\\
&\le C_1(k,\|b\|_\infty)\left(\iint_{R\setminus F(E_0')}\sqrt{|\alpha-\beta|^2|\phi|}\sqrt{k|\phi|-\operatorname{Re}\tau\phi}+\iint_{R\setminus F(E_0')}(k|\phi|-\operatorname{Re}\tau\phi)\right),
\end{aligned}\tag{5.7}
$$

where

$$
C_1(k,\|b\|_\infty)=\frac{32}{k(1-(\max\{k,\|b\|_\infty\})^2)^3}\,.
$$

Using the Cauchy-Schwartz inequality, we obtain from (5.7) that

$$
\begin{aligned}
\iint_{R\setminus F(E_0')}|\alpha-\beta|^2|\phi| \le\ & C_1(k,\|b\|_\infty)\\
&\times\left(\sqrt{\iint_{R\setminus F(E_0')}|\alpha-\beta|^2|\phi|\iint_{R\setminus F(E_0')}(k|\phi|-\operatorname{Re}\tau\phi)}\right.\\
&\left.+\iint_{R\setminus F(E_0')}(k|\phi|-\operatorname{Re}\tau\phi)\right).
\end{aligned}\tag{5.8}
$$

By (5.8) it sufficies to show the lemma assuming that $\iint_{R\setminus F(E_0')}(k|\phi|-\operatorname{Re}\tau\phi)\neq 0$. Letting

$$
t^2=\iint_{R\setminus F(E_0')}|\alpha-\beta|^2|\phi|\Big/\iint_{R\setminus F(E_0')}(k|\phi|-\operatorname{Re}\tau\phi)\,,
$$

(5.8) then implies that $t \le C_1(k, \|b\|_\infty)(1 + 1/t)$, which implies that t is bounded, so

$$\iint_{R\setminus F(E_0')} |\alpha-\beta|^2|\phi| \le C_2(k, \|b\|_\infty) \iint_{R\setminus F(E_0')} (k|\phi| - \operatorname{Re}\tau\phi)\,. \tag{5.9}$$

Consequently,

$$\iint_{R\setminus F(E_0')} \left|\frac{\alpha-\beta}{1-\overline{\alpha}\beta}\right|^2 |\phi| \le C_3(k, \|b\|_\infty) \iint_{R\setminus F(E_0')} (k|\phi| - \operatorname{Re}\tau\phi)\,. \tag{5.10}$$

Now we suppose that the set $E = \{z \in R \setminus F(E_0')\,;\ |\tau| < k/2\}$ has positive measure. Choose some non-zero element $\psi \in Q(R')$. We define a Beltrami differential $\tilde{\eta}$ on R' as follows: When $w \in R' \setminus f(E)$, $\tilde{\eta}(w) = 0$. In the case when $w \in f(E) \cap (R' \setminus E')$, if $\tilde{\mu}(w) \ne \tilde{\nu}(w)$, then $\tilde{\eta}(w)$ is the unique point ζ on the hyperbolic circle $\rho(\zeta, \tilde{\mu}(w)) = \rho(0, k/2)$ whose hyperbolic distance to $\tilde{\nu}(w)$ is a minimal; if $\tilde{\mu}(w) = \tilde{\nu}(w) \ne 0$, then $\tilde{\eta}(w) = a\tilde{\mu}(w)$, where a is a positive constant, such that $\tilde{\eta}(w)$ is on the hyperbolic circle $\rho(\zeta, \tilde{\mu}(w)) = \rho(0, k/2)$; if $\tilde{\mu}(w) = \tilde{\nu}(w) = 0$, then $\tilde{\eta}(w) = k|\psi(w)|/(2\psi(w))$. In the case when $w \in f(E) \cap E'$, if $\tilde{\mu}(w) \ne \tilde{\nu}(w)$, then $\tilde{\eta}(w)$ is the unique point ζ on the hyperbolic circle $\rho(\zeta, \tilde{\mu}(w)) = \rho(0, b(w)/2)$ whose hyperbolic distance to $\tilde{\nu}(w)$ is a minimal; if $\tilde{\mu}(w) = \tilde{\nu}(w) \ne 0$, $\tilde{\eta}(w) = a\tilde{\mu}(w)$, where a is a positive constant, such that $\tilde{\eta}(w)$ is on the hyperbolic circle $\rho(\zeta, \tilde{\mu}(w)) = \rho(0, b(w)/2)$; if $\tilde{\mu}(w) = \tilde{\nu}(w) = 0$, then $\tilde{\eta}(w) = b(w)|\psi(w)|/(2\psi(w))$. Let H be a quasiconformal mapping on R' with complex dilatation $\tilde{\eta}$, and set $F_1 = F \circ H^{-1}$, $G_1 = G \circ H^{-1}$. We also denote by $\tilde{\mu}_1$, μ_1, $\tilde{\nu}_1$ and ν_1 the complex dilatations of F_1, $F_1^{-1} = H \circ f$, G_1 and $G_1^{-1} = H \circ g$, respectively.

Noting that

$$\mu_1 = \frac{\tilde{\eta}-\tilde{\mu}}{1-\overline{\tilde{\mu}}\tilde{\eta}} \frac{\partial F}{\overline{\partial F}} \circ f = \mu\chi_{R\setminus E} + \left(\frac{\tilde{\eta}-\tilde{\mu}}{1-\overline{\tilde{\mu}}\tilde{\eta}} \frac{\partial F}{\overline{\partial F}} \circ f\right)\chi_E\,, \tag{5.11}$$

we conclude that

$$\begin{aligned} |\mu_1| &= |\mu|\chi_{R\setminus E} + \left|\frac{\tilde{\eta}-\tilde{\mu}}{1-\overline{\tilde{\mu}}\tilde{\eta}} \circ f\right| \chi_E \\ &= |\mu|\chi_{R\setminus E} + \frac{k}{2}\chi_{(R\setminus F(E'))\cap E} + \frac{b(f)}{2}\chi_{F(E')\cap E}\,. \end{aligned} \tag{5.12}$$

So $|\mu_1| \ge k/2$ on $R \setminus F(E')$ and $|\mu_1| \ge b(f)/2$ on $F(E' \setminus E_0')$. On the other hand, since $|\tilde{\mu}_1(H \circ f)| = |\mu_1|$, we conclude that

$$k_1 = k_{F_1} = \|\tilde{\mu}_1|H(R') \setminus H(E')\|_\infty = \|\mu_1|R \setminus F(E')\|_\infty = k\,,$$

and

$$|\tilde{\mu}_1| = |\tilde{\mu}_1(H \circ f \circ F \circ H^{-1})| \le b(f \circ F \circ H^{-1}) = b(H^{-1}) \quad \text{on } H(E')\,.$$

Similarly,

$$\nu_1 = \frac{\tilde{\eta}-\tilde{\nu}}{1-\overline{\tilde{\nu}}\tilde{\eta}} \frac{\partial G}{\overline{\partial G}} \circ g = \nu\chi_{G\circ f(R\setminus E)} + \left(\frac{\tilde{\eta}-\tilde{\nu}}{1-\overline{\tilde{\nu}}\tilde{\eta}} \frac{\partial G}{\overline{\partial G}} \circ g\right)\chi_{G\circ f(E)}\,, \tag{5.13}$$

so we obtain by the assumption $k_G \le k_F = k$ and the definition of $\tilde{\eta}$ that $k_{G_1} \le k_G \le k$, $|\tilde{\nu}_1| \le b(H^{-1})$ on $H(E')$ and hence $G_1 \in Q(F_1, H(E'), b(H^{-1}))$.

Let $f_1 = F_1^{-1}$. Then by definition,

$$\begin{aligned}\tau_1 = \tau_{F_1} &= \mu_1 \chi_{R\setminus F_1(H(E'\setminus E_0'))} + k_1\mu_1/b(H^{-1}\circ f_1)\chi_{F_1(H(E'\setminus E_0'))}\\ &= \mu_1\chi_{R\setminus F(E'\setminus E_0')} + k\mu_1/b(f)\chi_{F(E'\setminus E_0')}\,,\end{aligned}$$

so $|\tau_1| \ge k/2$ on $R\setminus F(E_0') = R\setminus F_1(H(E_0'))$. Noting that $G_1\circ F_1^{-1} = G\circ F^{-1}$, we conclude by (5.10) that

$$\iint_{R\setminus F(E_0')} \left|\frac{\alpha-\beta}{1-\overline{\alpha}\beta}\right|^2 |\phi| \le C_3(k, \|b\|_\infty) \iint_{R\setminus F(E_0')} (k|\phi| - \operatorname{Re}\tau_1\phi) \tag{5.14}$$

for all $\phi \in Q(R)$.

Now

$$3\operatorname{Re}\iint_E \tau\phi - \operatorname{Re}\iint_E \tau_1\phi \le (3k/2 + k/2)\iint_E |\phi| = 2k\iint_E |\phi|\,.$$

Hence

$$\begin{aligned}3\operatorname{Re}&\iint_{R\setminus F(E_0')} \tau\phi\\ &= 2\operatorname{Re}\iint_{R\setminus F(E_0')\setminus E}\tau\phi + \operatorname{Re}\iint_{R\setminus F(E_0')\setminus E}\tau\phi + 3\operatorname{Re}\iint_E \tau\phi\\ &\le 2k\iint_{R\setminus F(E_0')\setminus E}|\phi| + \operatorname{Re}\iint_{R\setminus F(E_0')\setminus E}\tau_1\phi + \operatorname{Re}\iint_E\tau_1\phi + 2k\iint_E|\phi|\\ &= 2k\|\phi\|_{R\setminus F(E_0')} + \operatorname{Re}\iint_{R\setminus F(E_0')}\tau_1\phi\,,\end{aligned}$$

that is,

$$k\|\phi\|_{R\setminus F(E_0')} - \operatorname{Re}\iint_{R\setminus F(E_0')}\tau_1\phi \le 3\left(k\|\phi\|_{R\setminus F(E_0')} - \operatorname{Re}\iint_{R\setminus F(E_0')}\tau\phi\right). \tag{5.15}$$

Finally, we obtain from (5.14) and (5.15) that

$$\iint_{R\setminus F(E_0')} \left|\frac{\alpha-\beta}{1-\overline{\alpha}\beta}\right|^2 |\phi| \le 3C_3(k, \|b\|_\infty) \iint_{R\setminus F(E_0')} (k|\phi| - \operatorname{Re}\tau\phi)\,.$$

This completes the proof of Lemma 5.1.

6. Proof of Theorem 2.3. In this section we will prove Theorem 2.3, an equivalent form of Theorem 1.1. We first note the following

LEMMA 6.1. *Given the class $Q(F, E', b)$, if τ_F is uniquely infinitesimally extremal in* Belt$(\tau_F, F(E_0'))$, *then F is uniquely extremal with respect to the extremal set $R(\tau_F) = \{z \in R\,;\ |\tau_F(z)| = k_F\}$ in following sense: If G is any other mapping in the class $Q(F, E', b)$ with $k_G \le k_F$, then the complex dilatations of F and G must coincide on the set $f(R(\tau_F))$.*

PROOF 10. Suppose the contrary. Then there would exist some mapping G in the class $Q(F, E', b)$ with $k_G \le k_F$ such that the complex dilatations $\tilde{\mu}$ and $\tilde{\nu}$ of the mappings F and G satisfy the condition that $|(\tilde{\mu}(f) - \tilde{\nu}(f))/(1 - \overline{\tilde{\mu}(f)}\tilde{\nu}(f))| > \varepsilon_0 > 0$ on some compact subset E of $R(\tau_F)$ with positive measure. By Lemma 5.1, for all $\phi \in Q(R)$, it holds that

$$\varepsilon_0^2 \iint_E |\phi| \le C\left(k_F\|\phi\|_{R-F(E_0')} - \operatorname{Re} \iint_{R\setminus F(E_0')} \tau_F \phi\right). \tag{6.1}$$

On the other hand, since τ_F is uniquely infinitesimally extremal in $\operatorname{Belt}(\tau_F, F(E_0'))$, by our Theorem 4.2, (6.1) implies that the set E has measure zero. This is a contradiction.

Now we prove Theorem 2.3. Let the class $Q(F, E', b)$ be given, and $\tilde{\mu}$ and μ be the complex dilatations of the mappings F and $f = F^{-1}$, respectively. Suppose that $\tau = \tau_F$ is uniquely infinitesimally extremal in $\operatorname{Belt}(\tau_F, F(E_0'))$. We want to show that F is uniquely extremal.

Suppose the contrary. Then there would exist some mapping $G \ne F$ in the class $Q(F, E', b)$ with $k_G \le k_F$. Let $\tilde{\nu}$ and ν denote the complex dilatations of the mappings G and $g = G^{-1}$, respectively. Then the set $E = \{z \in R : \tilde{\nu}(f) \ne \tilde{\mu}(f)\}$ has positive measure. On the other hand, Lemma 6.1 implies that F is uniquely extremal with respect to $R(\tau)$, so the set $\{z \in E\,;\ |\tau| = k = k_F\}$ has measure zero. Hence there exists a constant $k_0 < k$ such that the set $\tilde{E} = \{z \in E\,;\ |\tau(z)| \le k_0\}$ has positive measure. We may assume that $\tilde{E}$ is compact.

Now we define a Beltrami differential $\tilde{\eta}$ on R' as follows: When $w \in R' \setminus f(\tilde{E})$, $\tilde{\eta}(w) = 0$; when $w \in f(\tilde{E}) \cap (R' \setminus E')$, $\tilde{\eta}(w)$ is the unique point ζ on the hyperbolic circle $\rho(\zeta, \tilde{\mu}(w)) = \rho(0, k)$ whose hyperbolic distance to $\tilde{\nu}(w)$ is a minimal; when $w \in f(\tilde{E}) \cap E'$, $\tilde{\eta}(w)$ is the unique point ζ on the hyperbolic circle $\rho(\zeta, \tilde{\mu}(w)) = \rho(0, b(w))$ whose hyperbolic distance to $\tilde{\nu}(w)$ is a minimal. Let H be a quasiconformal mapping on R' with complex dilatation $\tilde{\eta}$, and set $F_1 = F \circ H^{-1}$, $G_1 = G \circ H^{-1}$. We also denote by $\tilde{\mu}_1$, μ_1, $\tilde{\nu}_1$ and ν_1 the complex dilatations of F_1, $F_1^{-1} = H \circ f$, G_1 and $G_1^{-1} = H \circ g$, respectively.

Noting that

$$\mu_1 = \frac{\tilde{\eta} - \tilde{\mu}}{1 - \overline{\tilde{\mu}}\tilde{\eta}} \frac{\partial F}{\overline{\partial F}} \circ f = \mu \chi_{R\setminus\tilde{E}} + \left(\frac{\tilde{\eta} - \tilde{\mu}}{1 - \overline{\tilde{\mu}}\tilde{\eta}} \frac{\partial F}{\overline{\partial F}} \circ f\right)\chi_{\tilde{E}}\,, \tag{6.2}$$

we conclude that

$$\begin{aligned} |\mu_1| &= |\mu|\chi_{R\setminus\tilde{E}} + \left|\frac{\tilde{\eta} - \tilde{\mu}}{1 - \overline{\tilde{\mu}}\tilde{\eta}} \circ f\right| \chi_{\tilde{E}} \\ &= |\mu|\chi_{R\setminus\tilde{E}} + k\chi_{\tilde{E}\cap(R\setminus F(E'))} + b(f)\chi_{\tilde{E}\cap F(E')}\,. \end{aligned} \tag{6.3}$$

Since $|\tilde{\mu}_1(H \circ f)| = |\mu_1|$, we conclude that $k_1 = k_{F_1} = k$, and $|\tilde{\mu}_1| \le b(H^{-1})$ on $H(E')$.

Similarly,

$$\nu_1 = \frac{\tilde{\eta} - \tilde{\nu}}{1 - \overline{\tilde{\nu}}\tilde{\eta}} \frac{\partial G}{\overline{\partial G}} \circ g = \nu \chi_{G\circ f(R\setminus\tilde{E})} + \left(\frac{\tilde{\eta} - \tilde{\nu}}{1 - \overline{\tilde{\nu}}\tilde{\eta}} \frac{\partial G}{\overline{\partial G}} \circ g\right)\chi_{G\circ f(\tilde{E})}\,, \tag{6.4}$$

so we have by the assumption $k_G \leq k_F = k$ and the definition of $\tilde{\eta}$ that $k_{G_1} \leq k_G \leq k$, $|\tilde{\nu}_1| \leq b(H^{-1})$ on $H(E')$ and hence $G_1 \in Q(F_1, H(E'), b(H^{-1}))$.

Now by definition,

$$\tag{6.5} \begin{aligned} \tau_1 = \tau_{F_1} &= \mu_1 \chi_{R \setminus F_1(H(E') \setminus H(E'_0))} + k_1 \mu_1 / b(H^{-1} \circ f_1) \chi_{F_1(H(E') \setminus H(E'_0))} \\ &= \mu_1 \chi_{R \setminus F(E' \setminus E'_0)} + k \mu_1 / b(f) \chi_{F(E' \setminus E'_0)} \\ &= \tau \chi_{R \setminus \tilde{E}} + \mu_1 \chi_{\tilde{E} \cap (R \setminus F(E'))} + k \mu_1 / b(f) \chi_{\tilde{E} \cap F(E')}. \end{aligned}$$

Therefore, $\tau_1 = \tau$ on $R \setminus \tilde{E}$, $\|\tau_1\|_\infty = \|\tau\|_\infty$, and $|\tau_1| = \|\tau_1\|_\infty$ on $\tilde{E}$. In particular, τ_1 is an admissible variation of τ and hence is uniquely infinitesimally extremal. Then Lemma 6.1 implies that F_1 is uniquely extremal with respect to the extremal set $R(\tau_1)$, that is, $\tilde{\mu}_1(f_1) = \tilde{\nu}_1(f_1)$ on $R(\tau_1)$. So $G_1 \circ F_1^{-1} = G \circ F^{-1}$ is conformal in $\tilde{E}$ and $\tilde{\nu}(f) = \tilde{\mu}(f)$ on $\tilde{E}$, which is a contradiction. This completes the proof of Theorem 2.3.

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