

AN INDECOMPOSABLE CONTINUUM AS SUBPOWER HIGSON CORONA

By

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Abstract. In this paper, we study topological properties of the subpower Higson coronas of proper metric spaces and show that the subpower Higson corona of the half open interval with the usual metric is an indecomposable continuum. Continuous surjections from Higson-type coronas onto a Higson-type compactifications of the half open interval are also constructed.

1. Introduction

Let (X, d) be a metric space with a metric d and let $B_d(x, r)$ denote the closed ball of radius r centered at $x \in X$. A metric d on X is called *proper* if all balls $B_d(x, r)$ are compact. The Higson compactification is a compactification defined by the coarse structure of a proper metric space and plays an important role in the large-scale geometry [14]. Also the sublinear Higson compactification is known as a compactification defined by the sublinear coarse structure of a proper metric space [3], [5]. There are several ways to define a Higson type compactification, by a coarse structure [4], [5], [14], by a large scale structure [6], and by a closed subring of the algebra of all continuous bounded real-valued functions [10], [11]. The subpower Higson compactification was introduced in [11] as a compactification defined by a closed subring of the algebra of all continuous bounded real-valued functions. And the asymptotic power dimension was studied in [12] related to the subpower Higson corona.

In this paper, we adopt the definition associated with a closed subrings of the algebra of all continuous bounded real-valued functions and study topological properties of the subpower Higson compactifications and their coronas. It is

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known that a Higson-type corona νX can be realized by a discrete subspace of a proper metric space X . The Higson corona and the subpower Higson corona contain a copy of the Stone-Čech remainder $\beta\mathbf{N}\setminus\mathbf{N}$ of the natural numbers \mathbf{N} [10], [11]. Also it was shown that the Higson corona of the half open interval $[0, \infty)$ with the usual metric is, like the Stone-Čech remainder $\beta[0, \infty)\setminus[0, \infty)$ [2], an indecomposable continuum [9]. We show that the subpower Higson corona of the half open interval with the usual metric is also an indecomposable continuum. Then we construct continuous surjections from Higson-type coronas onto Higson-type compactifications of the half open interval.

2. Basic properties of the subpower Higson compactification

A (not necessarily continuous) function $s : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ between the set of positive real numbers is called *asymptotically subpower* (resp. *asymptotically sublinear*) if for every $\alpha > 0$ there exists $t_0 > 0$ such that $s(t) < t^\alpha$ (resp. $s(t) < \alpha t$) whenever $t > t_0$. The set of all asymptotically subpower functions (resp. all asymptotically sublinear functions, all positive constant functions) is denoted by \mathcal{P} (resp. \mathcal{L} , \mathcal{H}). Since every constant functions are asymptotically subpower and every asymptotically subpower functions are asymptotically sublinear, it follows that $\mathcal{H} \subset \mathcal{P} \subset \mathcal{L}$.

In what follows, a metric space (X, d) is assumed to have a base point x_0 , and the distance $d(x_0, x)$ of $x \in X$ from x_0 is denoted by $|x|$. For a subset A of X , the diameter of A is denoted by $\text{diam}_d A$, that is, $\text{diam}_d A = \sup\{d(x, y) \mid x, y \in A\}$.

Let (X, d) and (Y, ρ) be proper metric spaces. A map $f : X \rightarrow Y$ is called *Higson subpower* (resp. *Higson sublinear*, *Higson*) provided that

$$(*)_s^f \quad \lim_{|x| \rightarrow \infty} \text{diam}_\rho f(B_d(x, s(|x|))) = 0$$

for each $s \in \mathcal{P}$ (resp. $s \in \mathcal{L}$, $s \in \mathcal{H}$). Hence, a map $f : X \rightarrow Y$ is Higson subpower if and only if, given asymptotically subpower function s and $\varepsilon > 0$, there exists a compact subset $K \subset X$ such that $\text{diam}_\rho f(B_d(x, s(|x|))) < \varepsilon$ whenever $x \in X \setminus K$.

Let $C^*(X)$ be the set of all bounded real-valued continuous functions on X . For a subset F of $C^*(X)$, let $e_F : X \rightarrow \prod_{f \in F} I_f$, where $I_f = [\inf f, \sup f]$, $f \in F$, be the *evaluation map* of F defined by $(e_F(x))_f = f(x)$ for every $x \in X$. It is known that if $F \subset C^*(X)$ separates points from closed sets, then e_F is an embedding [15, 8.16]. Identifying X with $e_F(X)$, the closure $\overline{e_F(X)}$ of $e_F(X)$ in $\prod_{f \in F} I_f$ gives a compactification of X .

For a proper metric space X , we consider the following three subsets of $C^*(X)$:

$$\begin{aligned}
 C_H(X) &= \{f \in C^*(X) \mid f \text{ satisfies } (*)_s^f \text{ for every } s \in \mathcal{H}\}, \\
 C_P(X) &= \{f \in C^*(X) \mid f \text{ satisfies } (*)_s^f \text{ for every } s \in \mathcal{P}\}, \\
 C_L(X) &= \{f \in C^*(X) \mid f \text{ satisfies } (*)_s^f \text{ for every } s \in \mathcal{L}\}.
 \end{aligned}$$

Then they are closed subrings of $C^*(X)$ with respect to the sup-metric. Also, they contain all constant maps and separate points from closed sets. Hence the subrings $C_H(X)$, $C_P(X)$ and $C_L(X)$ determine compactifications of X . Since $\mathcal{H} \subset \mathcal{P} \subset \mathcal{L}$, it follows that $C_L(X) \subset C_P(X) \subset C_H(X)$ (cf. [11]).

Let c_1X and c_2X be compactifications of X . We say $c_1X \succeq c_2X$ provided that there is a continuous map $f : c_1X \rightarrow c_2X$ such that $f|_X = \text{id}_X$. We note that a continuous map $f : c_1X \rightarrow c_2X$ with $f|_X = \text{id}_X$ is unique and surjective. If $c_1X \preceq c_2X$ and $c_1X \succeq c_2X$, then we say that c_1X and c_2X are *equivalent*.

The *Higson compactification* $h_H(X)$, the *subpower Higson compactification* $h_P(X)$ and the *sublinear Higson compactification* $h_L(X)$ are compactifications defined by closed subrings $C_H(X)$, $C_P(X)$ and $C_L(X)$ respectively, that is, $h_A(X)$ is equivalent to $\overline{e_{C_A}(X)}$, where $A \in \{H, P, L\}$ and $C_A = C_A(X)$. Their *coronas* are defined by $v_AX = h_A(X) \setminus X$ for each $A \in \{H, P, L\}$. These three compactifications are referred to as the *Higson type compactifications*. We note that $h_L(X) \preceq h_P(X) \preceq h_H(X)$ since $C_L(X) \subset C_P(X) \subset C_H(X)$.

One of the basic property of the Higson type compactifications is the following:

- (‡) A bounded continuous map $f : X \rightarrow \mathbf{R}$ has an extension $\hat{f} : h_A(X) \rightarrow \mathbf{R}$ if and only if $f \in C_A(X)$,

where $A \in \{H, P, L\}$. This condition holds for any closed subring of $C^*(X)$ which contains all constant maps and separates points from closed sets [7, Problem 3.12.22(e)]. See [1] for a comprehensive and detailed description of this property.

In this section, we shall derive some basic properties concerning the Higson subpower compactifications using the basic ideas of [4], [5] and [10].

The following proposition is a fundamental property of the Higson type compactification which is derived from the property (‡) and can be shown by a similar argument to that in [10, Proposition 1].

PROPOSITION 2.1 (cf. [10]). *Let X be a proper metric space. Then the subpower (resp. sublinear) Higson compactification is the unique compactification of*

X such that if Y is any compact metric space and $f : X \rightarrow Y$ is continuous, then f has a continuous extension $\hat{f} : h_P(X) \rightarrow Y$ (resp. $\hat{f} : h_L(X) \rightarrow Y$) if and only if f satisfies $(*)_s^f$ for any $s \in \mathcal{P}$ (resp. for any $s \in \mathcal{L}$).

Let (X, d) be a metric space. We call a finite system E_1, \dots, E_n of closed subsets of X *diverges as a power function* if there exist $\alpha > 0$ and $r_0 > 0$ such that $\max\{d(x, E_i) \mid 1 \leq i \leq n\} \geq |x|^\alpha$ whenever $|x| > r_0$. If a finite system E_1, \dots, E_n diverge as a power function then $(\bigcap_{i=1}^n E_i) \cap (X \setminus B_d(x_0, r)) = \emptyset$ for some $r > 0$. As in the case of Higson compactification [4, Lemma 2.2], we have the following:

LEMMA 2.2 (cf. [4]). *Let (X, d) be a non-compact proper metric space. Let E_1, \dots, E_n be a finite system of closed subsets of X such that $\bigcap_{i=1}^n E_i = \emptyset$. Let $f_i : X \rightarrow \mathbf{R}_+$ be the map defined by $f_i(x) = d(x, E_i)$, $1 \leq i \leq n$ and let $F = \sum_{i=1}^n f_i$. If the system E_1, \dots, E_n diverges as a power function then the map $g_i = f_i/F : X \rightarrow [0, 1]$ is Higson subpower.*

PROOF. First we note that the well-definedness of g_i follows from the assumption $\bigcap_{i=1}^n E_i = \emptyset$. Let $\alpha, r_1 > 0$ be positive numbers such that $\max\{d(x, E_i) \mid 1 \leq i \leq n\} \geq |x|^{2\alpha}$ whenever $|x| > r_1$. Then $F(x) \geq |x|^{2\alpha}$ whenever $|x| > r_1$. Let $s : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be an asymptotically subpower function. Then there exists $r_2 > 0$ such that $s(t) < t^\alpha$ for every $t > r_2$. Given $\varepsilon > 0$, we can take $r_3 > 0$ so that $(n+1)/t^\alpha < \varepsilon/2$ for every $t > r_3$.

Put $r_0 = \max\{r_1, r_2, r_3\}$. Let x be a point with $|x| > r_0$ and let $y \in B_d(x, s(|x|))$. Then we have

$$\begin{aligned} |g_i(x) - g_i(y)| &= \left| \frac{f_i(x)}{F(x)} - \frac{f_i(y)}{F(y)} \right| = \left| \frac{f_i(x) - f_i(y)}{F(x)} + \frac{f_i(y)}{F(y)} \cdot \frac{F(y) - F(x)}{F(x)} \right| \\ &\leq \left| \frac{d(x, y)}{F(x)} \right| + \left| \frac{F(y) - F(x)}{F(x)} \right| \leq \frac{(n+1)s(|x|)}{F(x)} \\ &\leq \frac{(n+1)s(|x|)}{|x|^{2\alpha}} < \frac{(n+1)|x|^\alpha}{|x|^{2\alpha}} = \frac{(n+1)}{|x|^\alpha} < \frac{\varepsilon}{2}. \end{aligned}$$

Hence, $\text{diam } g_i(B_d(x, s(|x|))) \leq \varepsilon$ whenever $|x| > r_0$. Thus g_i is Higson subpower for every $1 \leq i \leq n$. \square

For a subset A of a proper metric space X , \bar{A} denotes the closure of A in the subpower Higson compactification $h_P X$ of X . As in case of the Higson corona

[4, Proposition 2.3] and of the sublinear Higson corona [5, Lemma 2.3], we have the following.

PROPOSITION 2.3 (cf. [4], [5]). *Let (X, d) be a non-compact proper metric space. For a system E_1, \dots, E_n of closed subsets of X , the following conditions are equivalent:*

- (1) $v_P X \cap (\bigcap_{k=1}^n \overline{E_k}) = \emptyset$,
- (2) *the system E_1, \dots, E_n diverges as a power function.*

PROOF. Suppose that the condition (2) does not hold. Then there exists a sequence $(x_i)_{i=1}^\infty$ in X such that $\lim_{i \rightarrow \infty} |x_i| = \infty$, $|x_i| < |x_{i+1}|$ and $d(x_i, E_k) < |x_i|^{1/i}$ for every $i \in \mathbb{N}$ and $1 \leq k \leq n$. Let p be a cluster point of the sequence $(x_i)_{i=1}^\infty$. Then $p \in v_P X$. Put $t_i = |x_i|$ for each $i \in \mathbb{N}$. Let $s : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be the (non-continuous) function defined by $s(t) = |x_i|^{1/i}$ when $t \in [t_i, t_{i+1})$ and $s(t) = 1$ when $t \in [0, t_1)$. It is easy to see that s is an asymptotically subpower function and $d(x_i, E_k) < s(|x_i|)$ for every $i \in \mathbb{N}$ and $1 \leq k \leq n$. Hence, if f is any element of $C_P(X)$, the condition $(*)_s^f$ implies that the distance between $f(x_i)$ and $f(E_k)$ tends to zero as i tends to infinity. Considering the evaluation map $e_{C_P(X)} : X \rightarrow \prod_{f \in C_P(X)} I_f$, it follows from this fact that $p \in \overline{E_k}$ for every $1 \leq k \leq n$. Thus $p \in v_P X \cap (\bigcap_{k=1}^n \overline{E_k}) \neq \emptyset$.

Suppose that the system E_1, \dots, E_n diverges as a power function. Then we have $(\bigcap_{k=1}^n E_k) \cap (X \setminus B_d(x_0, r)) = \emptyset$ for some $r > 0$. Put $F_k = \text{cl}_X(E_k \setminus B_d(x_0, r + 1))$ for each $k = 1, \dots, n$. Then the system F_1, \dots, F_n diverges as a power function and $\bigcap_{k=1}^n F_k = \emptyset$. Note that $\overline{F_k} \cap v_P X = \overline{E_k} \cap v_P X$ for every $1 \leq k \leq n$. By Lemma 2.2 there exist Higson subpower maps $g_k : X \rightarrow [0, 1]$, $1 \leq k \leq n$, such that $F_k \subset g_k^{-1}(0)$ and $\sum_{k=1}^n g_k = 1$. By Proposition 2.1, there exists an extension $G_k : h_P X \rightarrow [0, 1]$ of g_k . Then $\sum_{k=1}^n G_k = 1$ and $\overline{F_k} \subset G_k^{-1}(0)$. However, the condition $\sum_{k=1}^n G_k = 1$ leads that $v_P X \cap (\bigcap_{k=1}^n \overline{E_k}) \subset \bigcap_{k=1}^n \overline{F_k} \subset \bigcap_{k=1}^n G_k^{-1}(0) = \emptyset$. □

We note that the implication (1) \Rightarrow (2) for the case $n = 2$ of the above proposition was in fact shown by Kucab and Zarichnyi in [13].

PROPOSITION 2.4 (cf. [10]). *Let X be a proper metric space and let Y be a closed subset of X with the induced metric from X . Then the closure \overline{Y} of Y in $h_P(X)$ is equivalent to the subpower Higson compactification $h_P(Y)$ of Y .*

PROOF. By [7, Theorem 3.5.5], it suffices to show $\text{cl}_{h_P(X)} A \cap \text{cl}_{h_P(X)} B = \emptyset$ if and only if $\text{cl}_{h_P(Y)} A \cap \text{cl}_{h_P(Y)} B = \emptyset$ for every pair A, B of closed subsets of Y . It follows from Proposition 2.3 that the condition $\text{cl}_{h_P(X)} A \cap \text{cl}_{h_P(X)} B = \emptyset$ if and only if the system A, B diverges as a power function in X and $A \cap B = \emptyset$. This is equivalent to the condition that the system A, B diverges as a power function in Y and $A \cap B = \emptyset$ since the metric on Y is inherited from X . \square

Let (X, d) be a metric space. For any $R > 0$, a subset Y of X is called *R-dense* in X if $B_d(x, R) \cap Y \neq \emptyset$ for every $x \in X$.

COROLLARY 2.5 (cf. [10]). *Let (X, d) be a proper metric space and let Y be a subset of X . If Y is R -dense in X for some $R > 0$ then $\bar{Y} \setminus Y = v_P X$. In particular, $v_P Y$ is homeomorphic to $v_P X$.*

PROOF. Let $x \in v_P X$. Let U and V be open neighborhoods of x in $h_P(X)$ such that $x \in V \subset \bar{V} \subset U$. Put $E = X \cap (h_P(X) \setminus U)$ and $F = X \cap \bar{V}$. By Proposition 2.3, the system E, F diverges as a power function since $v_P X \cap \bar{E} \cap \bar{F} = \emptyset$. Thus, there are $\alpha, r > 0$ such that $d(y, E) > |y|^\alpha$ whenever $y \in F$ and $|y| > r$. Since $V \cap v_P X \neq \emptyset$ and X is a proper metric space, we can take $z \in F$ so that $|z| > r$. We may assume that $R < r^\alpha$. Then $B_d(z, R) \cap E = \emptyset$, i.e., $B_d(z, R) \subset U$. Since Y is R -dense in X , $\emptyset \neq Y \cap B_d(z, R) \subset Y \cap U$. This fact leads that $x \in \bar{Y}$ since we can take U as an arbitrarily small neighborhood of x . Hence $v_P X$ is contained in $\bar{Y} \setminus Y$, i.e., $\bar{Y} \setminus Y = v_P X$. By Proposition 2.4, $\bar{Y} \setminus Y$ is homeomorphic to $v_P Y$. Thus $v_P Y$ is homeomorphic to $v_P X$. \square

3. Continua as subpower Higson coronas of $[0, \infty)$

Throughout this section, 0 is assumed to be the base point of the half-open interval $[0, \infty)$, that is, $x_0 = 0 \in [0, \infty)$. A metric d on $[0, \infty)$ defined by $d(x, y) = \sqrt{(x - y)^2}$ is called *the usual metric* on $[0, \infty)$. For notational simplicity, we identify t with $|t| = d(x_0, t)$ for every $t \in [0, \infty)$ when we consider the usual metric on $[0, \infty)$. A *continuum* is a nonempty, compact and connected Hausdorff space. A *subcontinuum* is a continuum which is a subset of a continuum. A continuum is called *decomposable* if it can be represented as the union of two of its proper subcontinua. A continuum which is not decomposable is said to be *indecomposable*.

Let $X = [0, \infty)$ be the half open interval with the usual metric. The indecomposability of the Stone-Ćech remainder $\beta X \setminus X$ and the Higson corona $v_H X$ was proved in [2] and in [9] respectively. We show here that the subpower Higson

corona $v_P X$ is also an indecomposable continuum. Let K be a proper closed subset of $\beta X \setminus X$ with non-empty interior in $\beta X \setminus X$. To see the indecomposability of $\beta X \setminus X$, Bellamy [2] constructed a continuous map $f : X \rightarrow [0, 1]$ such that $f^*(K) = \{0, 1\}$, where $f^* : \beta X \rightarrow [0, 1]$ is the extension of f . Hence K is disconnected. We note that such the extension f^* to the Stone-Ćech compactification always exists since f is a bounded continuous map. And then he constructed a continuous surjection from a given non-degenerate subcontinuum of $\beta X \setminus X$ onto a given metric continuum. Our strategy is essentially the same as the Bellamy's. To see the indecomposability of $v_P X$, we will construct a map $\psi : X \rightarrow [0, 1]$ such that $\psi^*(K') = \{0, 1\}$ for a given proper closed subset K' of $v_P X$ with non-empty interior in $v_P X$, where $\psi^* : h_P X \rightarrow [0, 1]$ is the extension of ψ . In order to ensure that a map $\psi : X \rightarrow [0, 1]$ has the extension $\psi^* : h_P X \rightarrow [0, 1]$, we will carefully construct ψ to be Higson subpower (cf. Proposition 2.1). Then we will construct a continuous surjection from $v_P X$ onto the Higson type compactification $h_A(X)$ for each $A \in \{H, P, L\}$.

The following lemma plays an essential role to analyze the subpower Higson corona of the half-open interval $[0, \infty)$.

LEMMA 3.1. *Let $X = [0, \infty)$ be the half open interval with the usual metric d . Let U and V be non-empty disjoint open subsets of $h_P(X)$ such that $U \cap v_P X \neq \emptyset$ and $V \cap v_P X \neq \emptyset$. Then there exist a natural number $k \geq 3$ and sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ with $0 = b_0 < a_1 < b_1 < \dots < b_{n-1} < a_n < b_n < \dots$ satisfying the following conditions:*

- (1) $(a_n)^{1/k} > 2^n$,
- (2) $b_{n-1} + (b_{n-1})^{1/k} < a_n$,
- (3) $[b_{n-1}, a_n] \cap U \neq \emptyset$,
- (4) $b_n = a_n + (a_n)^{2/k}$,
- (5) $[a_n, b_n] \subset V \cap X$,
- (6) $a_n - b_{n-1} > 2^{n-1}(a_n)^{1/k}$,
- (7) $b_n - a_n > 2^n(b_n)^{1/k}$.

PROOF. Let z be a point in $V \cap v_P X$ and let W be an open neighborhood of z in $h_P X$ such that $z \in W \subset \overline{W} \subset V$. Then

$$v_P X \cap \overline{(X \setminus (V \cap X))} \cap \overline{(\overline{W} \cap X)} \subset v_P X \cap (h_P(X) \setminus V) \cap \overline{W} = \emptyset.$$

By Proposition 2.3, the system $X \setminus (V \cap X)$, $\overline{W} \cap X$ diverges as a power function. Hence, there exist $\alpha > 0$ and $r_0 > 0$ such that

$$d(t, X \setminus (V \cap X)) \geq t^\alpha \quad \text{whenever } t \in \overline{W} \cap X \text{ and } t \geq r_0.$$

Take a natural number $k \geq 3$ so that $\alpha > 2/k$. Then the condition above implies that

$$(A) \quad B_d(t, t^{2/k}) \subset V \cap X \quad \text{whenever } t \in \overline{W} \cap X \text{ and } t \geq r_0.$$

Then we consider the following two inequalities:

$$(B) \quad \begin{cases} x^k > x, \\ x^2 > 2^k(x + x^{2/k}). \end{cases}$$

Since $k \geq 3$, there is $t_1 > 2$ such that x satisfies (B) whenever $x > t_1$. Then we take $a_1 \in W \cap X$ so that $a_1 > \max\{r_0, (t_1)^k\}$. This is possible since the set $W \cap X$ is cofinal in X by the condition $W \cap v_P X \neq \emptyset$. Hence, a_1 satisfies the conditions (1) and (2). Also, the condition $U \cap v_P X \neq \emptyset$ allows us to make a_1 satisfy the condition (3). Then we define b_1 by (4). Since $a_1 \in \overline{W} \cap X$ and $d(a_1, b_1) = (a_1)^{2/k}$, we have $b_1 \in B_d(a_1, (a_1)^{2/k}) \subset V \cap X$ by (A). Then the condition (5) follows from the fact that the metric d is geodesic. To see that a_1 and b_1 satisfy the conditions (6) and (7), put $a_1 = x$. Then $b_1 = x + x^{2/k}$ by (4). Since $a_1 > t_1$, we have $(a_1 - b_0)^k = x^k > x = ((a_1)^{1/k})^k$ and $(b_1 - a_1)^k = x^2 > 2^k(x + x^{2/k}) = (2(b_1)^{1/k})^k$ by (B). Thus the conditions (6) and (7) are satisfied.

Suppose that $b_{i-1} < a_i < b_i$ have been constructed for $i < n$. Then we consider the following two inequalities:

$$(C) \quad \begin{cases} (x - b_{n-1})^k > 2^{(n-1)k}x, \\ x^2 > 2^{nk}(x + x^{2/k}). \end{cases}$$

Since $k \geq 3$, there exists $t_n > 2^n$ such that x satisfies (C) whenever $x > t_n$. As in the first step, we can take $a_n \in W \cap X$ with $a_n > \max\{r_0, (t_n)^k\}$ so that the conditions (1)–(3) are satisfied. Then we define b_n by (4). The condition (5) follows from (A) and the fact that the metric d is geodesic. Finally, to see the conditions (6) and (7), we put $a_n = x$. Then $b_n = x + x^{2/k}$ by (4). Since $a_n > t_n$, it follows from (C) that $(a_n - b_{n-1})^k = (x - b_{n-1})^k > 2^{(n-1)k}x = (2^{n-1}(a_n)^{1/k})^k$ and $(b_n - a_n)^k = x^2 > 2^{nk}(x + x^{2/k}) = (2^n(b_n)^{1/k})^k$. The conditions (6) and (7) are satisfied. \square

PROPOSITION 3.2. *Let $X = [0, \infty)$ be the half open interval with the usual metric d . If K is a proper closed subset of $v_P X$ with non-empty interior in $v_P X$ then K is disconnected.*

PROOF. Let $x \in \text{Int}_{v_P X} K$ and $y \in v_P X \setminus K$. Let U be an open neighborhood of x in $h_P X$ such that $\overline{U} \cap v_P X \subset K$. Since $y \notin \overline{U}$, we can take an

open neighborhood V of y in $h_P X$ such that $y \in V \subset \bar{V} \subset h_P X \setminus (K \cup \bar{U})$. By Lemma 3.1, there exist a natural number $k \geq 3$ and sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ with $0 = b_0 < a_1 < b_1 < \dots < b_{n-1} < a_n < b_n < \dots$ satisfying the following conditions:

- (1) $(a_n)^{1/k} > 2^n$,
- (2) $b_{n-1} + (b_{n-1})^{1/k} < a_n$,
- (3) $[b_{n-1}, a_n] \cap U \neq \emptyset$,
- (4) $b_n = a_n + (a_n)^{2/k}$,
- (5) $[a_n, b_n] \subset V \cap X$,
- (6) $a_n - b_{n-1} > 2^{n-1}(a_n)^{1/k}$,
- (7) $b_n - a_n > 2^n(b_n)^{1/k}$.

CLAIM 3.2.1. *Let $u \in B_d(t, t^{1/k})$. Then we have the following:*

- (i) *If $t \in [a_n, b_n]$ then $u \in (b_{n-1}, a_{n+1})$.*
- (ii) *If $t \in [b_n, a_{n+1}]$ then $u \in (a_n, b_{n+1})$.*

PROOF OF CLAIM 3.2.1. Suppose that $t \in [a_n, b_n]$. If $u < t$ then $u \geq t - t^{1/k} \geq a_n - (a_n)^{1/k} \geq a_n - 2^{n-1}(a_n)^{1/k} > b_{n-1}$ by the monotone increasing-property of the correspondence $s \mapsto s - s^{1/k}$ on $(1, \infty)$ and (6). If $t < u$ then $u \leq t + t^{1/k} < b_n + (b_n)^{1/k} < a_{n+1}$ by (2). Thus the condition (i) is satisfied.

Next suppose that $t \in [b_n, a_{n+1}]$. If $u < t$ then, as above, using (7), $u \geq t - t^{1/k} \geq b_n - (b_n)^{1/k} > b_n - 2^n(b_n)^{1/k} > a_n$. If $u > t$ then $u \leq t + t^{1/k} < a_{n+1} + (a_{n+1})^{1/k} < a_{n+1} + (a_{n+1})^{2/k} = b_{n+1}$ by (4). Thus the condition (ii) is satisfied. ◇

We define $\psi : X \rightarrow [0, 1]$ by

$$\psi(t) = \begin{cases} 0, & \text{if } b_{2n-2} \leq t < a_{2n-1}, \\ \frac{t - a_{2n-1}}{b_{2n-1} - a_{2n-1}}, & \text{if } a_{2n-1} \leq t < b_{2n-1}, \\ 1, & \text{if } b_{2n-1} \leq t < a_{2n}, \\ \frac{b_{2n} - t}{b_{2n} - a_{2n}}, & \text{if } a_{2n} \leq t < b_{2n}. \end{cases}$$

Then it is easy to see that the function ψ is well-defined and continuous. Now we shall evaluate the diameter of $\psi(B_d(t, t^{1/k}))$.

CLAIM 3.2.2. *If $a_n \leq t < a_{n+1}$ then $|\psi(t) - \psi(u)| < 1/2^n$ for every $u \in B_d(t, t^{1/k})$.*

PROOF OF CLAIM 3.2.2. Let $u \in B_d(t, t^{1/k})$. By the definition of ψ , we check the claim by dividing it into four cases, $t \in [a_{2n-1}, b_{2n-1})$, $t \in [b_{2n-1}, a_{2n})$, $t \in [a_{2n}, b_{2n})$ and $t \in [b_{2n}, a_{2n+1})$. We show here the first two cases. The other two cases can be shown in a similar fashion.

Suppose that $t \in [a_{2n-1}, b_{2n-1})$. By Claim 3.2.1, it suffices to consider the three cases that $u \in (b_{2n-2}, a_{2n-1})$, $u \in [a_{2n-1}, b_{2n-1})$ and $u \in [b_{2n-1}, a_{2n})$. If $u \in (b_{2n-2}, a_{2n-1})$ then, using (7),

$$\begin{aligned} |\psi(t) - \psi(u)| &= \psi(t) = \frac{t - a_{2n-1}}{b_{2n-1} - a_{2n-1}} \leq \frac{t - u}{b_{2n-1} - a_{2n-1}} \\ &\leq \frac{t^{1/k}}{b_{2n-1} - a_{2n-1}} < \frac{(b_{2n-1})^{1/k}}{b_{2n-1} - a_{2n-1}} \\ &< \frac{(b_{2n-1})^{1/k}}{2^{2n-1}(b_{2n-1})^{1/k}} < \frac{1}{2^{2n-1}}. \end{aligned}$$

If $u \in [a_{2n-1}, b_{2n-1})$ then we have

$$|\psi(t) - \psi(u)| = \frac{|t - u|}{b_{2n-1} - a_{2n-1}} \leq \frac{t^{1/k}}{b_{2n-1} - a_{2n-1}} < \frac{1}{2^{2n-1}}.$$

If $u \in [b_{2n-1}, a_{2n})$ then we have

$$\begin{aligned} |\psi(t) - \psi(u)| &= 1 - \psi(t) = 1 - \frac{t - a_{2n-1}}{b_{2n-1} - a_{2n-1}} = \frac{b_{2n-1} - t}{b_{2n-1} - a_{2n-1}} \\ &\leq \frac{u - t}{b_{2n-1} - a_{2n-1}} \leq \frac{t^{1/k}}{b_{2n-1} - a_{2n-1}} < \frac{1}{2^{2n-1}}. \end{aligned}$$

Thus $|\psi(t) - \psi(u)| < 1/2^{2n-1}$ whenever $t \in [a_{2n-1}, b_{2n-1})$.

Next suppose $t \in [b_{2n-1}, a_{2n})$. By Claim 3.2.1, it suffices to consider the three cases that $u \in (a_{2n-1}, b_{2n-1})$, $u \in [b_{2n-1}, a_{2n})$ and $u \in [a_{2n}, b_{2n})$. Let $u \in (a_{2n-1}, b_{2n-1})$. Since $u \geq t - t^{1/k} \geq b_{2n-1} - (b_{2n-1})^{1/k}$, it follows that

$$(\dagger) \quad b_{2n-1} - u \leq (b_{2n-1})^{1/k}.$$

So, using (7) and (\dagger), we have

$$\begin{aligned} |\psi(t) - \psi(u)| &= 1 - \psi(u) = 1 - \frac{u - a_{2n-1}}{b_{2n-1} - a_{2n-1}} = \frac{b_{2n-1} - u}{b_{2n-1} - a_{2n-1}} \\ &\leq \frac{(b_{2n-1})^{1/k}}{b_{2n-1} - a_{2n-1}} < \frac{(b_{2n-1})^{1/k}}{2^{2n-1}(b_{2n-1})^{1/k}} < \frac{1}{2^{2n-1}}. \end{aligned}$$

If $u \in [b_{2n-1}, a_{2n})$ then $|\psi(t) - \psi(u)| = 1 - 1 = 0$. Finally, let $u \in [a_{2n}, b_{2n})$. Then, using (4) and (1), we have

$$\begin{aligned} |\psi(t) - \psi(u)| &= 1 - \psi(u) = 1 - \frac{b_{2n} - u}{b_{2n} - a_{2n}} = \frac{u - a_{2n}}{b_{2n} - a_{2n}} \\ &< \frac{u - t}{b_{2n} - a_{2n}} \leq \frac{t^{1/k}}{b_{2n} - a_{2n}} = \frac{t^{1/k}}{(a_{2n})^{2/k}} \\ &< \frac{(a_{2n})^{1/k}}{(a_{2n})^{2/k}} = \frac{1}{(a_{2n})^{1/k}} < \frac{1}{2^{2n}} < \frac{1}{2^{2n-1}}. \end{aligned}$$

Thus $|\psi(t) - \psi(u)| < 1/2^{2n-1}$ whenever $t \in [b_{2n-1}, a_{2n})$. ◇

CLAIM 3.2.3. *The map $\psi : X \rightarrow [0, 1]$ is Higson subpower.*

PROOF OF CLAIM 3.2.3. Let $\varepsilon > 0$ and let $s : X \rightarrow \mathbf{R}_+$ be an asymptotically subpower function. Then there exists $r > 0$ such that $s(t) < t^{1/k}$ for every $t > r$. Let m be a natural number such that $1/2^m < \varepsilon/2$. Put $t_0 = \max\{r, a_m\}$. Then, for every $t > t_0$ and $u \in B_d(t, t^{1/k})$, we have $B_d(t, s(t)) \subset B_d(t, t^{1/k})$ and $|\psi(t) - \psi(u)| < 1/2^m$ by Claim 3.2.2. Thus $\text{diam } \psi(B_d(t, s(t))) \leq \text{diam } \psi(B_d(t, t^{1/k})) \leq 2/2^m < \varepsilon$ for every $t > t_0$. Hence ψ is Higson subpower. ◇

By Claim 3.2.3 and Proposition 2.1, there exists an extension $\Psi : h_p X \rightarrow [0, 1]$ of ψ . By (3), we obtain a sequence $(c_n)_{n=1}^\infty$ such that $b_n < c_n < a_{n+1}$ and $c_n \in U$. Then $\psi(c_{2n}) = 0$ and $\psi(c_{2n-1}) = 1$ for every n . Let z_0 and z_1 be cluster points of the sequences $(c_{2n})_{n=1}^\infty$ and $(c_{2n-1})_{n=1}^\infty$ respectively. Then both z_0 and z_1 are contained in $\bar{U} \cap v_p X \subset K$. It follows from the continuity of Ψ that $\Psi(z_0) = 0$ and $\Psi(z_1) = 1$. Hence $z_0 \in \Psi^{-1}(0) \cap K \neq \emptyset$ and $z_1 \in \Psi^{-1}(1) \cap K \neq \emptyset$. However, $\Psi^{-1}((0, 1)) \cap K = \emptyset$. Indeed, if there exists $z \in K$ such that $\Psi(z) = t \in (0, 1)$ then z has a neighborhood $W \subset h_p X$ such that $\Psi(W) \subset (0, 1)$. Since $K \cap \bar{V} = \emptyset$, we may assume $W \cap V = \emptyset$. Then we have $\Psi(W \cap X) = \psi(W \cap X) \subset (0, 1)$, i.e., $W \cap X \subset \psi^{-1}((0, 1))$. On the other hand, $\psi^{-1}((0, 1)) \subset V \cap X$ by (5) and our construction of ψ . So we have $\emptyset \neq W \cap X \subset V \cap X$ which implies that $W \cap V \neq \emptyset$, a contradiction. Thus $\Psi(K) = \{0, 1\}$, i.e., K is disconnected. □

As we have seen, the function $\psi : X \rightarrow [0, 1]$ constructed above is Higson subpower. However it is not Higson sublinear. Indeed, take sequences $(a_n)_{n=1}^\infty$,

$(b_n)_{n=1}^\infty$ and $k \geq 3$ as in Proposition 3.2. Let $\xi : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be the function defined by $\xi(t) = t^{2/k}$. Since $k \geq 3$, it follows that ξ is an asymptotically sublinear function. Since $b_n = a_n + (a_n)^{2/k}$, the closed ball $B_d(a_n, \xi(a_n))$ contains at least two points a_n and b_n . By our construction of ψ , we have $(\psi(a_n), \psi(b_n)) \in \{(0, 1), (1, 0)\}$, i.e., $\text{diam } \psi(B_d(a_n, \xi(a_n))) = 1$ for every n . Hence, ψ is not Higson sublinear.

THEOREM 3.3. *Let $X = [0, \infty)$ be the half open interval with the usual metric. Then the subpower Higson corona $v_P X$ is a non-metrizable indecomposable continuum.*

PROOF. Let $A_n = [n, \infty) \subset X$. Then $\overline{A_n}$ is a continuum and $\overline{A_n} \supset \overline{A_{n+1}}$ for every $n \in \mathbf{N}$. Note that $v_P X = \bigcap_{n=1}^\infty \overline{A_n}$. Thus $v_P X$ is a continuum as the intersection of the decreasing sequence $(\overline{A_n})_{n=1}^\infty$ of continua. Non-metrizability of $v_P X$ follows from the fact that $v_P X$ contains a copy of $\beta\mathbf{N} \setminus \mathbf{N}$ [11] which has the cardinality 2^c [8]. Finally, if $v_P X$ is expressed as the union of two non-degenerate proper closed subsets K and L of $v_P X$ then both of K and L must have non-empty interiors in $v_P X$. Thus both K and L are disconnected by Proposition 3.2. \square

The following is an example, given in [9], of a proper metric on $[0, \infty)$ which derives a decomposable Higson corona. We show that the same example also derives a decomposable subpower Higson corona.

EXAMPLE 3.4. There exists a proper metric ρ on the half open interval $[0, \infty)$ such that the Higson subpower corona is a decomposable continuum. Indeed, let $f : [0, \infty) \rightarrow \mathbf{R}^2$ be the embedding given by $f(t) = (t, t \sin t)$ and let $X = f([0, \infty))$. Let ρ be the metric on X inherited from \mathbf{R}^2 . We shall show that the Higson subpower corona $v_P X$ of (X, ρ) is a decomposable continuum. Put

$$Y = \{(x, y) \in X \mid y \geq 0\},$$

$$Z = \{(x, y) \in X \mid y \leq 0\},$$

$$A = \{(x, y) \in \mathbf{R}^2 \mid x \geq 0, |y| \leq x\},$$

$$B = \{(x, y) \in A \mid y \geq 0\},$$

$$C = \{(x, y) \in A \mid y \leq 0\}.$$

Then $X = Y \cup Z$, $A = B \cup C$, $Y \subset B$ and $Z \subset C$. For each $A \subset X$, \bar{A} denotes the closure of A in the subpower Higson compactification $h_P(X)$ of X .

First we note that

$$v_P X = \bar{X} \setminus X = (\bar{Y} \setminus Y) \cup (\bar{Z} \setminus Z).$$

Indeed, $\bar{X} \setminus X \supset (\bar{Y} \cup \bar{Z}) \setminus (Y \cup Z) = (\bar{Y} \setminus Y) \cup (\bar{Z} \setminus Z)$. If $z \in \bar{X} \setminus X$ has an open neighborhood U with $U \cap Y = \emptyset$ then every neighborhood of z must intersect with Z by the density of X in \bar{X} , i.e., $z \in \bar{Z} \setminus Z$.

Next we shall show that both $\bar{Y} \setminus Y$ and $\bar{Z} \setminus Z$ are subcontinua of $v_P X$. By Proposition 2.4, $\bar{Y} \setminus Y$ is homeomorphic to $v_P Y$ (notation: $\bar{Y} \setminus Y \approx v_P Y$) and $\bar{Z} \setminus Z \approx v_P Z$. It is easy to see that Y is R -dense in B for some $R > 0$. Hence $v_P Y$ is homeomorphic to $v_P B$ by Corollary 2.5. Thus, we have $\bar{Y} \setminus Y \approx v_P B$. Similarly, we have $\bar{Z} \setminus Z \approx v_P C$. For each $n \in \mathbf{N}$, let $B_n = \{(x, y) \in B \mid x \geq n\}$. Then $v_P B = \bigcap_{i=1}^{\infty} \text{cl}_{v_P B}(B_n)$. Hence $v_P B$ is a continuum as the intersection of the decreasing sequence $(\text{cl}_{v_P B}(B_n))_{n=1}^{\infty}$ of continua. Similarly, it follows that $v_P C$ is a continuum. Thus, both $\bar{Y} \setminus Y$ and $\bar{Z} \setminus Z$ are subcontinua of $v_P X$.

Finally, we shall show that both $\bar{Y} \setminus Y$ and $\bar{Z} \setminus Z$ are proper subcontinua of $v_P X$. Let $D = \{f(t) \mid t = \pi/2 + 2n\pi, n \in \mathbf{N}\}$ and $E = \{f(t) \mid t = 3\pi/2 + 2n\pi, n \in \mathbf{N}\}$. Then $\bar{Y} \setminus Y \supset \bar{D} \setminus D \neq \emptyset$. It is easy to see that the systems D, Z diverges as a power function. Hence we have $\bar{D} \cap \bar{Z} \cap v_P X = \emptyset$ by Proposition 2.3, i.e., $(\bar{Y} \setminus Y) \setminus (\bar{Z} \setminus Z) \supset \bar{D} \setminus D \neq \emptyset$. Similarly, we have $(\bar{Z} \setminus Z) \setminus (\bar{Y} \setminus Y) \supset \bar{E} \setminus E \neq \emptyset$. Thus, both $\bar{Y} \setminus Y$ and $\bar{Z} \setminus Z$ are proper subcontinua of $v_P X$ with $v_P X = (\bar{Y} \setminus Y) \cup (\bar{Z} \setminus Z)$. Hence $v_P X$ is a decomposable continuum.

Let $K \subset X$ be a continuum and let $a, b \in K$ be distinct two points. If there is no proper subcontinuum of K containing both a and b then K is said to be *irreducible between a and b* .

The following lemma was proved by Bellamy [2, Lemma 1] for the Stone-Ćech compactification of $[0, \infty)$. The proof is valid for any compactification of $[0, \infty)$. Here we give a proof for the reader's sake.

LEMMA 3.5. *Every compactification $\alpha([0, \infty))$ of the half-open interval $[0, \infty)$ is an irreducible continuum between 0 and every point z of $\alpha([0, \infty)) \setminus [0, \infty)$.*

PROOF. The connectivity of $\alpha([0, \infty))$ is obvious. Let $z \in \alpha([0, \infty)) \setminus [0, \infty)$. Suppose that there exists a proper closed subset K of $\alpha([0, \infty))$ such that $0, z \in K$. Since $\alpha([0, \infty)) \setminus K$ is a non-empty open subset of $\alpha([0, \infty))$, there exists a point $t \in [0, \infty) \cap (\alpha([0, \infty)) \setminus K)$. Then $U = K \cap [0, t) = K \cap [0, t]$ is a non-empty closed

and open subset of K since $0 \in U$. On the other hand, $z \in K \setminus U \neq \emptyset$, that is, K is not connected. \square

THEOREM 3.6. *Let $X = [0, \infty)$ be the half open interval with the usual metric d . For each $A \in \{H, P, L\}$, there exists a continuous surjection $\xi_A : v_P X \rightarrow h_A(X)$ from the subpower Higson corona $v_P X$ onto the Higson type compactification $h_A(X)$.*

PROOF. We shall construct a continuous surjection $\xi_H : v_P X \rightarrow h_H(X)$ from the subpower Higson corona $v_P X$ onto the Higson compactification $h_H(X)$. Then, for each $A \in \{P, L\}$, a required surjection is obtained by the composition $\alpha_A^H \circ \xi_H : v_P X \rightarrow h_A(X)$ where $\alpha_A^H : h_H(X) \rightarrow h_A(X)$ is the canonical surjection assured by the relation $h_H(X) \succeq h_A(X)$.

Let x and y be distinct two points in $v_P X$. Let U and V be open subsets of $h_P(X)$ such that $x \in U$, $y \in V$ and $\bar{U} \cap \bar{V} = \emptyset$. Then, by Lemma 3.1, there exist a natural number $k \geq 3$ and sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ with $0 = b_0 < a_1 < b_1 < \dots < b_{n-1} < a_n < b_n < \dots$ satisfying the following conditions:

- (1) $(a_n)^{1/k} > 2^n$,
- (2) $b_{n-1} + (b_{n-1})^{1/k} < a_n$,
- (3) $[b_{n-1}, a_n] \cap U \neq \emptyset$,
- (4) $b_n = a_n + (a_n)^{2/k}$,
- (5) $[a_n, b_n] \subset V \cap X$,
- (6) $a_n - b_{n-1} > 2^{n-1}(a_n)^{1/k}$,
- (7) $b_n - a_n > 2^n(b_n)^{1/k}$.

Then we define $\theta : X \rightarrow X$ by

$$\theta(t) = \begin{cases} 0, & \text{if } b_{2n-2} \leq t < a_{2n-1}, \\ \frac{(2n-1)(t - a_{2n-1})}{b_{2n-1} - a_{2n-1}}, & \text{if } a_{2n-1} \leq t < b_{2n-1}, \\ 2n-1, & \text{if } b_{2n-1} \leq t < a_{2n}, \\ \frac{(2n-1)(b_{2n} - t)}{b_{2n} - a_{2n}}, & \text{if } a_{2n} \leq t < b_{2n}. \end{cases}$$

Let $\psi : X \rightarrow [0, 1]$ be the map constructed in Proposition 3.2, regarded as a map of X into X , i.e., $\psi : X \rightarrow X$. Then it follows from our construction that $\theta(t) = (2n-1)\psi(t)$ whenever $b_{2n-2} \leq t < b_{2n}$.

Let $\ell : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be a function defined by $\ell(t) = t/2^t$. One should note that ℓ is monotonically decreasing on $(1/\log 2, \infty)$. In particular, $\lim_{t \rightarrow \infty} \ell(t) = 0$. Then we have the following.

CLAIM 3.6.1. *If $n \geq 3$ and $t \in [a_{2n-1}, a_{2n+1})$ then $|\theta(t) - \theta(u)| < \ell(2n - 1)$ for every $u \in B_d(t, t^{1/k})$.*

PROOF OF CLAIM 3.6.1. Let $u \in B_d(t, t^{1/k})$. If $t \in [a_{2n-1}, a_{2n})$ then $u \in (b_{2n-2}, b_{2n})$ by Claim 3.2.1. Then $\theta(t) = (2n - 1)\psi(t)$ and $\theta(u) = (2n - 1)\psi(u)$. By Claim 3.2.2, we have $|\theta(t) - \theta(u)| = (2n - 1)|\psi(t) - \psi(u)| < (2n - 1)/2^{2n-1} = \ell(2n - 1)$.

Suppose that $t \in [a_{2n}, b_{2n})$. Then $\theta(t) = (2n - 1)\psi(t)$. By Claim 3.2.1, $u \in (b_{2n-1}, a_{2n+1})$. If $u \in (b_{2n-1}, b_{2n})$ then $\theta(u) = (2n - 1)\psi(u)$. If $u \in [b_{2n}, a_{2n+1})$ then $\theta(u) = (2n - 1)\psi(u)$ since $\theta(u) = \psi(u) = 0$. Hence, we obtain $|\theta(t) - \theta(u)| < \ell(2n - 1)$ as above.

Now suppose that $t \in [b_{2n}, a_{2n+1})$. In this case, $\theta(t) = \psi(t) = 0$. So we have $\theta(t) = (2n - 1)\psi(t)$. By Claim 3.2.1, $u \in (a_{2n}, b_{2n+1})$. If $u \in (a_{2n}, b_{2n})$ then $\theta(u) = (2n - 1)\psi(u)$. If $u \in [b_{2n}, a_{2n+1})$ then $\theta(u) = (2n - 1)\psi(u)$ since $\theta(u) = \psi(u) = 0$. Hence, we have $|\theta(t) - \theta(u)| < \ell(2n - 1)$ as above. Finally, if $u \in [a_{2n+1}, b_{2n+1})$ then, by the conditions (4) and (1), we have

$$\begin{aligned} |\theta(t) - \theta(u)| &= \theta(u) = \frac{(2n + 1)(u - a_{2n+1})}{b_{2n+1} - a_{2n+1}} = \frac{(2n + 1)(u - a_{2n+1})}{(a_{2n+1})^{2/k}} \\ &< \frac{(2n + 1)t^{1/k}}{(a_{2n+1})^{2/k}} < \frac{(2n + 1)(a_{2n+1})^{1/k}}{(a_{2n+1})^{2/k}} = \frac{2n + 1}{(a_{2n+1})^{1/k}} \\ &< \frac{2n + 1}{2^{2n+1}} = \ell(2n + 1). \end{aligned}$$

We note that $\ell(2n + 1) < \ell(2n - 1)$ since ℓ is decreasing on $(1/\log 2, \infty)$ and $2n - 1 \geq 5 > 1/\log 2 \doteq 3.3$. Thus we have $|\theta(t) - \theta(u)| < \ell(2n - 1)$. ◇

CLAIM 3.6.2. *If $n \geq 3$ and $t \geq a_{2n-1}$ then $|\theta(t) - \theta(u)| < \ell(2n - 1)$ for every $u \in B_d(t, t^{1/k})$.*

PROOF OF CLAIM 3.6.2. Since ℓ is monotonically decreasing on $(1/\log 2, \infty)$ and $2n - 1 \geq 5 > 1/\log 2 \doteq 3.3$, this is an easy consequence of Claim 3.6.1. ◇

CLAIM 3.6.3. *For every $f \in C_H(X)$, the composition $f \circ \theta : X \rightarrow \mathbf{R}$ is Higson subpower.*

PROOF OF CLAIM 3.6.3. Let $f : X \rightarrow \mathbf{R}$ be a continuous Higson map and let $\varepsilon > 0$ be given. If $t \geq a_5$ then $|\theta(t) - \theta(u)| < \ell(5)$ for every $u \in B_d(t, t^{1/k})$ by

Claim 3.6.2. Since f is Higson, there exists $K > 0$ such that

$$(a) \quad \text{diam } f(B_d(t, \ell(5))) < \varepsilon \quad \text{for every } t > K.$$

By the compactness of the interval $[0, K]$ and the continuity of f , we can take $\delta > 0$ so that

$$(b) \quad |f(t) - f(u)| < \varepsilon/3 \quad \text{whenever } t \in [0, K] \text{ and } |t - u| < 3\delta.$$

Without loss of generality, we may assume that $\delta < \ell(5)$. Since $\lim_{t \rightarrow \infty} \ell(t) = 0$, we can take an integer $n_0 \geq 3$ so that $\ell(2n_0 - 1) < \delta/2$.

Let $s : X \rightarrow \mathbf{R}_+$ be an asymptotically subpower function. We take $t_1 > 0$ so that $s(t) < t^{1/k}$ for every $t > t_1$. Then we put $t_0 = \max\{t_1, a_{2n_0-1}\}$.

Let $t > t_0$. It follows from $t > t_1$ that $\theta(B_d(t, s(t))) \subset \theta(B_d(t, t^{1/k}))$. Since $t > a_{2n_0-1}$, it follows from Claim 3.6.2 that $|\theta(t) - \theta(u)| < \ell(2n_0 - 1)$ for every $u \in B_d(t, t^{1/k})$, i.e., $\theta(B_d(t, t^{1/k})) \subset B_d(\theta(t), 2\ell(2n_0 - 1))$. Since $\ell(2n_0 - 1) < \delta/2$, we have $B_d(\theta(t), 2\ell(2n_0 - 1)) \subset B_d(\theta(t), \delta)$. As a summary, we have

$$\theta(B_d(t, s(t))) \subset B_d(\theta(t), \delta).$$

Suppose that $\theta(t) \leq K$. It follows from the condition (b) that

$$\text{diam } f \circ \theta(B_d(t, s(t))) \leq \text{diam } f(B_d(\theta(t), \delta)) < \varepsilon.$$

Now suppose that $\theta(t) > K$. Since $\delta < \ell(5)$, we have $B_d(\theta(t), \delta) \subset B_d(\theta(t), \ell(5))$. By the condition (a), we have

$$\text{diam } f \circ \theta(B_d(t, s(t))) \leq \text{diam } f(B_d(\theta(t), \ell(5))) < \varepsilon.$$

Hence $\text{diam } f \circ \theta(B_d(t, s(t))) < \varepsilon$ for every $t > t_0$. \diamond

CLAIM 3.6.4. *The map $\theta : X \rightarrow X$ has an extension $\Theta : h_P(X) \rightarrow h_H(X)$.*

PROOF OF CLAIM 3.6.4. Let $e : X \rightarrow \prod_{f \in C_H(X)} I_f$ be the evaluation map. Since $h_H(X)$ is equivalent to $\overline{e(X)}$, it suffices to see that $e \circ \theta : X \rightarrow \prod_{f \in C_H(X)} I_f$ can be extended to $h_P(X) \rightarrow \overline{e(X)}$. For every $f \in C_H(X)$, the composition $f \circ \theta : X \rightarrow I_f$ is Higson subpower by Claim 3.6.3, so it can be extended to the map $\widehat{f \circ \theta} : h_P(X) \rightarrow I_f$ by Proposition 2.1. Hence $e \circ \theta$ can be extended to the map $\widehat{e \circ \theta} : h_P(X) \rightarrow \prod_{f \in C_P(X)} I_f$ so that $\widehat{e \circ \theta}(h_P(X)) \subset \overline{e(X)}$. \diamond

Let $(c_n)_{n=1}^\infty$ and $(d_n)_{n=1}^\infty$ be sequences such that $c_n \in [b_{2n-2}, a_{2n-1}]$ and $d_n \in [b_{2n-1}, a_{2n}]$ and let p and q be cluster points of $(c_n)_{n=1}^\infty$ and $(d_n)_{n=1}^\infty$ respectively. Put $C = \{c_n \mid n \in \mathbf{N}\}$ and $D = \{d_n \mid n \in \mathbf{N}\}$. Then the system C, D diverges as a

power function by (1) and (7). Hence $p \in \bar{C} \cap v_P X$ and $q \in \bar{D} \cap v_P X$ are distinct two points in $v_P X$. By Claim 3.6.4, there exists an extension $\Theta : h_P(X) \rightarrow h_H(X)$ of θ . Then $\Theta(p) = 0$ and $\Theta(q) \in v_H X$. Thus $\Theta(v_P X)$ is a continuum containing 0 and $\Theta(q) \in v_H X$. So, $\Theta(v_P X)$ must coincide with $h_H(X)$ by Lemma 3.5. Hence the map $\xi_H = \Theta|_{v_P X} : v_P X \rightarrow h_H(X)$ is a required surjection. □

COROLLARY 3.7. *Let $X = [0, \infty)$ be the half open interval with the usual metric d . For each $A \in \{H, P, L\}$, there exists a continuous surjection $\eta_A : v_H X \rightarrow h_A(X)$ from the Higson corona $v_H X$ onto the Higson type compactification $h_A(X)$.*

PROOF. Let $A \in \{H, P, L\}$ and let $\xi_A : v_P X \rightarrow h_A(X)$ be a continuous surjection given by Theorem 3.6. By the relation $h_H(X) \succeq h_P(X)$, there is the canonical surjection $\alpha : h_H(X) \rightarrow h_P(X)$, such that $\alpha|_X = \text{id}_X$. Since X is dense in $h_H(X)$ and $\alpha|_X = \text{id}_X$, it follows that $\alpha(v_H X) = v_P X$. Thus the composition $\eta_A = \xi_A \circ \alpha|_{v_H X} : v_H X \rightarrow h_A(X)$ is a required surjection. □

QUESTION 1. *Does Proposition 3.2 hold for sublinear Higson corona? In particular, is the sublinear Higson corona of the half open interval with the usual metric an indecomposable continuum?*

Let W be a non-degenerate subcontinuum of $\beta[0, \infty) \setminus [0, \infty)$ and let M be a metric continuum. It is known [2] that there exist continuous surjections $f : W \rightarrow \beta[0, \infty)$ and $g : W \rightarrow M$.

QUESTION 2. *Let $X = [0, \infty)$ be the half open interval with the usual metric d . Let W be a non-degenerate subcontinuum of $v_P(X)$ and let M be a metric continuum. Do there exist continuous surjections $f : W \rightarrow h_P(X)$ and $g : W \rightarrow M$?*

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