TSUKUBA J. MATH.
Vol. 42 No. 1 (2018), 97–125

ASSOCIATED BINOMIAL INVERSION FOR UNIFIED STIRLING NUMBERS AND COUNTING SUBSPACES GENERATED BY SUBSETS OF A ROOT SYSTEM

By

Tomohiro KAMIYOSHI, Makoto NAGURA and Shin-ichi OTANI

Abstract. We introduce an associated version of the binomial inversion for unified Stirling numbers defined by Hsu and Shiue. This naturally appears when we count the number of subspaces generated by subsets of a root system. We count such subspaces of any dimension by using associated unified Stirling numbers, and then we will also give a combinatorial interpretation of our inversion formula. In particular, the well-known explicit formula for classical Stirling numbers of the second kind can be understood as a special case of our formula.

Introduction

For a sequence $a = (a_0, a_1, \ldots)$, we define a new sequence $b = (b_0, b_1, \ldots)$ by

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \cdot \binom{n}{k} \cdot a_k = \sum_{j=0}^{n} (-1)^{j} \cdot \binom{n}{j} \cdot a_{n-j}.$$

We say that b is the binomial transform of a, and then we have $a_n = \sum_{j=0}^{n} \binom{n}{j} \cdot b_j$ for any n. This fact is well-known (see, for example, Riordan [7, page 43]), and we call it the binomial inversion formula. In this paper, we will introduce an associated version of binomial inversion, and explain some combinatorial interpretation of it.

Let r be a fixed non-negative integer. For non-negative integers n and k with $n \geq rk$, we define the integer $C_r(n,k)$ by

$$C_r(n,k) = \frac{n! \cdot (r!)^{-k}}{k! \cdot (n-rk)!}.$$

2010 *Mathematics Subject Classification*: Primary 11B73; Secondary 17B22, 05A15, 15A21.
Key words and phrases: associated binomial inversion, unified Stirling number, root system.
Received March 26, 2018.
Revised July 12, 2018.

This appears in various situation of combinatorics; for example, see Wall [8]. We call it the r-associated binomial coefficient $C_r(n,k)$. Indeed, this is a generalization of the ordinary binomial coefficient, since we have $C_1(n,k) = \binom{n}{k}$ in the case of $r = 1$, see Remark 1.5(4). Using this, for a sequence $a = (a_0, a_1, \dots)$, we define the *r-associated binomial transform of degree δ with residue p* (here δ is a non-zero constant and p is a non-negative integer less than r) by

$$b_n = \sum_{j=0}^{n} (-1)^j \cdot \delta^j \cdot C_r(rn + p, j) \cdot a_{n-j}. \tag{0.1}$$

In this paper, we will give an inversion formula (Theorem 1.14) for r-associated unified Stirling numbers $f_r(n,k) = f_r(n,k;\alpha,\beta,\gamma)$ which are also an associated version of the unified Stirling numbers defined by Hsu and Shiue [4]. It follows from our inversion formula that the $(r+1)$-associated unified Stirling numbers f_{r+1} can be obtained from f_r by the r-associated binomial transformation. In particular, the well-known explicit formula for classical Stirling numbers of the second kind can be regarded as the binomial transform of the "0-associated" f_0.

On the other hand, a kind of 2-associated unified Stirling numbers naturally appear in the context of counting the number of subspaces that are generated by some roots in a root system; see Corollaries 3.6, 3.16 and 3.21. Although such subspaces of co-dimension one have been counted in [5], a combinatorial interpretation of our inversion formula concerns with the number of subspaces, not only of co-dimension one but also any co-dimension; so, in this paper, we count such subspaces as distinct sets. To do this, we gave a standard form of the matrix corresponding to such a subspace in the case where the root system is classical type (that is, $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$ or $\mathbf{D}_n$-type), see Theorems 3.7, 3.17 and 3.22. For exceptional type, we use the computer as in [5].

The authors would like to dedicate this paper on the occasion of his seventieth birthday to Professor Tatsuo KIMURA, who gave to the authors much encouragement and valuable suggestions for their study, or life.

1. Associated Unified Stirling Numbers

1.1. Definition and Some Remarks. In this section we will introduce an associated version of the unified Stirling numbers defined by Hsu and Shiue [4]. First we define it and give some remarks.

Definition 1.1. For a positive integer n, we denote by $(\gamma|\alpha)_n$ the falling factorial which is defined by

$$(\gamma|\alpha)_n = \gamma(\gamma-\alpha)(\gamma-2\alpha)\cdots(\gamma-(n-1)\alpha);$$

and put $(\gamma|\alpha)_0 = 1$ for any α and γ if its subscript is equal to zero.

Definition 1.2. Let r be a fixed positive integer. For two real numbers α and β, we define the constant $\delta_r = \delta_r(\alpha, \beta)$ by

$$\delta_r = \delta_r(\alpha,\beta) = \begin{cases} 1 & \text{if } (\beta-\alpha|\alpha)_{r-1} = 0, \\ (\beta-\alpha|\alpha)_{r-1} & \text{otherwise.} \end{cases}$$

Definition 1.3. Let r be a fixed positive integer, and we fix three real parameters α, β and γ. For positive integers n and k, we define the *r-associated unified Stirling number* $f_r(n,k) = f_r(n,k;\alpha,\beta,\gamma)$ by the following recurrence relation:

$$f_r(n,k) = \{-\alpha(n-1)+\beta k+\gamma\}\cdot f_r(n-1,k) + \delta_r\cdot\binom{n-1}{r-1}\cdot f_r(n-r,k-1),$$

where we put $f_r(n,0) = (\gamma|\alpha)_n$ for $n \geq 0$, and $f_r(n,k) = 0$ for $n < 0$.

Note that we have $f_r(n,k) = 0$ if $n < rk$.

Definition 1.4. Let $S_r(n,k)$ be the r-associated Stirling number of the second kind; that is, the number of partitions of the set N with $\#N = n$, into k blocks, all of cardinality grater than or equal to r (see Comtet [1, page 221]).

Remark 1.5. Specializing parameters α, β and γ, we have the following:

(1) In the case of $r = 1$, we have $\delta_r = 1$ for any α and β. Hence $f_1(n,k)$ is nothing but the unified Stirling number defined by Hsu and Shiue [4]: $f_1(n,k;\alpha,\beta,\gamma) = S(n,k;\alpha,\beta,\gamma)$.

(2) If $(\alpha,\beta,\gamma) = (1,0,0)$, then it is the r-associated signed Stirling number of the first kind: $f_r(n,k;1,0,0) = t_r(n,k)$. On the other hand, in the case of $(\alpha,\beta,\gamma) = (-1,0,0)$, it is the r-associated signless Stirling number of the first kind $f_r(n,k;-1,0,0) = T_r(n,k)$ (see Comtet [1, page 256]); that is, $t_r(n,k) = (-1)^{n-k}\cdot T_r(n,k)$.

(3) If $(\alpha,\beta,\gamma) = (0,1,0)$, then we have the r-associated Stirling number of the second kind: $f_r(n,k;0,1,0) = S_r(n,k)$.

(4) If $(\alpha, \beta, \gamma) = (0, 0, 1)$, then we have $\delta_r = 1$; so that it is a generalization of the binomial coefficient (see Wall [8]):

$$f_r(n, k; 0, 0, 1) = \frac{n! \cdot (r!)^{-k}}{k! \cdot (n - rk)!} =: C_r(n, k).$$

We will call this the r-associated binomial coefficient. Indeed, $C_1(n, k) = \binom{n}{k}$ for the case of $r = 1$ is nothing but the ordinary binomial coefficient.

1.2. Exponential Generating Function. Now we define the exponential generating function (GF) of $f_r(n, k) = f_r(n, k; \alpha, \beta, \gamma)$ by

$$F_{r,k}(x) = \sum_{n=0}^{\infty} \frac{f_r(n, k)}{n!} x^n = \sum_{n=rk}^{\infty} \frac{f_r(n, k)}{n!} x^n.$$

Here we note that we are enough to take the sum from $n = rk$, since we have $f_r(n, k) = 0$ if $n < rk$.

PROPOSITION 1.6. *The GF $F_{r,k}(x)$ of the r-associated unified Stirling numbers $f_r(n, k)$, for fixed k, satisfies the following linear differential equation of rank one*:

$$(1.1) \qquad (1 + \alpha x) \cdot \frac{d}{dx} F_{r,k}(x) = (\beta k + \gamma) \cdot F_{r,k}(x) + \frac{\delta_r}{(r-1)!} x^{r-1} \cdot F_{r,k-1}(x).$$

Moreover $F_{r,k}(x)$ can be expressed as follows:

$$(1.2) \qquad F_{r,k}(x) = \frac{1}{k!} \cdot p(x) \cdot \delta_r^k \cdot q_r(x)^k,$$

where we put

$$(1.3) \qquad p(x) = \sum_{i=0}^{\infty} \frac{(\gamma|\alpha)_i}{i!} x^i, \quad q_r(x) = \sum_{j=0}^{\infty} \frac{(\beta - r\alpha|\alpha)_j}{(r+j)!} x^{r+j}.$$

PROOF. From the recurrence relation, we have

$$\begin{aligned} \frac{d}{dx} F_{r,k}(x) &= \sum_{n=1}^{\infty} \frac{f_r(n, k)}{(n-1)!} x^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{-\alpha(n-1) \cdot f_r(n-1, k)}{(n-1)!} x^{n-1} + (\beta k + \gamma) \cdot \sum_{n=1}^{\infty} \frac{f_r(n-1, k)}{(n-1)!} x^{n-1} \end{aligned}$$

$$+\delta_r \cdot \binom{n-1}{r-1} \cdot \sum_{n=1}^{\infty} \frac{f_r(n-r,k-1)}{(n-1)!} x^{n-1}$$

$$= -\alpha x \cdot \frac{d}{dx} F_{r,k}(x) + (\beta k + \gamma) \cdot F_{r,k}(x) + \frac{\delta_r}{(r-1)!} x^{r-1} \cdot F_{r,k}(x);$$

so that $F_{r,k}(x)$ satisfies the DE (1.1). The power series $p(x)$, $q_r(x)$ defined in (1.3) satisfies

$$(1+\alpha x) \cdot \frac{d}{dx} p(x) = \gamma \cdot p(x),$$

$$(1+\alpha x) \cdot \frac{d}{dx} q_r(x) = \beta \cdot q_r(x) + \frac{1}{(r-1)!} x^{r-1},$$

respectively. Since δ_r is a constant, we see that the GF $F_{r,k}(x)$ defined in (1.2) satisfies the DE (1.1). $\square$

Remark 1.7. Although we give the GF of $f_r(n,k)$ as the product of power serieses (1.3), we can rewrite them as follows: If $\alpha \neq 0$, then we have $p(x) = (1+\alpha x)^{\gamma/\alpha}$. In addition, we see that the r-th derivative of $q_r(x)$ is the function $(1+\alpha x)^{\beta/\alpha - r}$, and that $q_r(x)$ can be expressed as the following hyper geometric series:

$$q_r(x) = \frac{x^r}{r!} \cdot {}_2F_1\left(1, -\frac{\beta}{\alpha} + r, r+1; -\alpha x\right);$$

hence this is a polynomial in the case where $-\beta/\alpha + r$ is a negative integer. On the other hand, if $\alpha = 0$, we have

$$q_r(x) = \frac{x^r}{r!} \cdot {}_1F_1(1, r+1; \beta x).$$

Thus we see that $q_r(x) = x^r/r!$ when $\alpha = 0$ and $\beta = 0$. In the case of $(\beta - \alpha|\alpha)_{r-1} \neq 0$, we see that $\delta_r \cdot q_r(x)$ can be expressed by the following power series:

$$\delta_r \cdot q_r(x) = (\beta - \alpha|\alpha)_{r-1} \cdot q_r(x) = \sum_{j=r}^{\infty} \frac{(\beta - \alpha|\alpha)_{j-1}}{j!} x^j. \tag{1.4}$$

Note that $\delta_1 \cdot q_1(x)$ for $r = 1$ is a primitive function of $(1+\alpha x)^{\beta/\alpha - 1}$ if $\alpha \neq 0$.

REMARK 1.8. Specializing parameters α, β and γ, we have the following:

(1) If $\alpha\beta \neq 0$ and $(\beta-\alpha|\alpha)_{r-1} \neq 0$, then we have

$$F_{r,k}(x) = \frac{1}{k!} \cdot (1+\alpha x)^{\gamma/\alpha} \cdot \frac{1}{\beta^k} \cdot \left((1+\alpha x)^{\beta/\alpha} - \sum_{j=0}^{r-1} \frac{(\beta|\alpha)_j}{j!} x^j \right)^k. \tag{1.5}$$

This is a generalization of [4, Theorem 2] which is given by Hsu and Shiue.

(2) If $\alpha = 0$ and $\beta \neq 0$, we have $(\beta-\alpha|\alpha)_{r-1} \neq 0$; so that $(\beta-\alpha|\alpha)_{j-1} = \beta^{j-1}$ for each $j \geq 1$. Hence the GF can be expressed as

$$\begin{aligned} F_{r,k}(x) &= \frac{1}{k!} \cdot e^{\gamma x} \cdot \left(\frac{1}{\beta} \cdot \sum_{j=r}^{\infty} \frac{1}{j!} (\beta x)^j \right)^k \\ &= \frac{1}{k!} \cdot e^{\gamma x} \cdot \frac{1}{\beta^k} \cdot \left(e^{\beta x} - \sum_{j=0}^{r-1} \frac{1}{j!} (\beta x)^j \right)^k. \end{aligned} \tag{1.6}$$

This can be understood as the limit of (1.5) as $\alpha \to 0$.

(3) If $\alpha \neq 0$ and $\beta = 0$, we have $(\beta-\alpha|\alpha)_{r-1} \neq 0$; so that $(\beta-\alpha|\alpha)_{j-1} = (-1)^{j-1} \cdot (j-1)! \cdot \alpha^{j-1}$ for each $j \geq 1$. It follows from (1.4) that

$$\begin{aligned} F_{r,k}(x) &= \frac{1}{k!} \cdot (1+\alpha x)^{\gamma/\alpha} \cdot \left(\frac{1}{\alpha} \cdot \sum_{j=r}^{\infty} \frac{(-1)^{j-1}}{j} (\alpha x)^j \right)^k \\ &= \frac{1}{k!} \cdot (1+\alpha x)^{\gamma/\alpha} \cdot \frac{1}{\alpha^k} \cdot \left(\log(1+\alpha x) - \sum_{j=1}^{r-1} \frac{(-1)^{j-1}}{j} (\alpha x)^j \right)^k. \end{aligned}$$

This can be understood as the limit of (1.5) as $\beta \to 0$.

(4) In the case of $\alpha = 0$ and $\beta = 0$, we have $\delta_r = 1$. Then the GF is given by

$$F_{r,k}(x) = \frac{1}{k!} \cdot e^{\gamma x} \cdot \left(\frac{x^r}{r!} \right)^k.$$

Therefore, in this case, the r-associated unified Stirling numbers can be explicitly given by the formula

$$f_r(n,k) = \frac{n! \cdot (r!)^{-k}}{k! \cdot (n-rk)!} \cdot \gamma^{n-rk}.$$

In addition, when $\gamma = 1$, this is the r-associated binomial coefficient $C_r(n,k)$ mentioned in Remark 1.5(4).

In fact, if $\alpha = 0$ or $\beta = 0$, then $f_r(n,k)$ can be expressed by r-associated Stirling numbers of the first or second kind, respectively.

Proposition 1.9. *If* $\alpha = 0$ *and* $\beta \neq 0$, *then* $f_r(n,k) = f_r(n,k;0,\beta,\gamma)$ *can be expressed by r-associated Stirling numbers* $S_r(n,k)$ *of the second kind*:

$$f_r(n,k;0,\beta,\gamma) = \sum_{j=rk}^{n} \binom{n}{j} \cdot \beta^{j-k} \cdot \gamma^{n-j} \cdot S_r(j,k). \tag{1.7}$$

In addition, in the case of $\gamma = 0$, *the terms of the right-hand side of* (1.7) *vanish only except* $j = n$, *we have* $f_r(n,k;0,\beta,0) = \beta^{n-k} \cdot S_r(n,k)$. *Here we should consider* $0^0 = 1$.

Proof. If $\alpha = 0$ and $\beta \neq 0$, then it follows from (1.6) that the exponential GF $F_{r,k}(x)$ of $f_r(n,k) = f_r(n,k;\alpha,\beta,\gamma)$ can be expressed as follows:

$$F_{r,k}(x) = e^{\gamma x} \cdot \beta^{-k} \cdot H_{r,k}(\beta x), \tag{1.8}$$

where $H_{r,k}(x)$ is the GF of $S_r(n,k) = f_r(n,k;0,1,0)$:

$$H_{r,k}(x) = \sum_{n=0}^{\infty} \frac{S_r(n,k)}{n!} x^n = \frac{1}{k!} \cdot \left(\sum_{j=r}^{\infty} \frac{x^j}{j!} \right)^k.$$

Comparing coefficients of x^n in (1.8), we have

$$\frac{f_r(n,k)}{n!} = \beta^{-k} \cdot \sum_{j=0}^{n} \frac{\gamma^{n-j}}{(n-j)!} \cdot \frac{S_r(n,k)}{j!} \cdot \beta^j;$$

thus we obtain our assertion. □

In the case of $\alpha \neq 0$ and $\beta = 0$, similarly we have the following:

Proposition 1.10. *If* $\alpha \neq 0$ *and* $\beta = 0$, *then* $f_r(n,k) = f_r(n,k;\alpha,0,\gamma)$ *can be expressed by signless r-associated Stirling numbers* $T_r(n,k)$ *of the first kind*:

$$f_r(n,k;\alpha,0,\gamma) = \sum_{j=rk}^{n} \binom{n}{j} \cdot (-\alpha)^{j-k} \cdot (\gamma|\alpha)_{n-j} \cdot T_r(j,k). \tag{1.9}$$

In addition, in the case of $\gamma = 0$, *the terms of the right-hand side of* (1.9) *vanish only except* $j = n$, *we have* $f_r(n,k;\alpha,0,0) = (-\alpha)^{n-k} \cdot T_r(n,k)$.

1.3. Binomial Inversion Formula. We will define the r-associated binomial transformation, which is defined by using the r-associated binomial coefficient.

DEFINITION 1.11. Let δ be a non-zero constant and p a non-negative integer less than r. For a sequence $a = (a_0, a_1, \dots)$, we define a new sequence $b = (b_0, b_1, \dots)$ by the equation (0.1). We call b the *r-associated binomial transform of degree δ with residue p*. Then we have the inverse

$$a_n = \sum_{j=0}^{n} \delta^j \cdot C_r(rn + p, j) \cdot b_{n-j}. \tag{1.10}$$

That is to say, we have the following:

PROPOSITION 1.12. *For two sequences $a = (a_0, a_1, \dots)$ and $b = (b_0, b_1, \dots)$, the equation* (0.1) *holds for any n if and only if we have* (1.10) *for any n.*

PROOF. We consider generating functions such as

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{(rn+p)!} x^{rn+p}, \quad B(x) = \sum_{n=0}^{\infty} \frac{b_n}{(rn+p)!} x^{rn+p}.$$

Suppose that (0.1) holds for any n. Then we have

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\delta^{n-k}}{(rn+p)!} \cdot \frac{(rn+p)! \cdot b_k}{(r!)^{n-k} \cdot (n-k)! \cdot (rn+p-r(n-k))!} x^{rn+p} \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{b_k}{(rk+p)!} x^{rk+p} \cdot \left(\frac{\delta}{r!}\right)^{n-k} \cdot \frac{x^{r(n-k)}}{(n-k)!} \\ &= B(x) \cdot \exp\left(\frac{\delta}{r!} x^r\right). \end{aligned}$$

Therefore $B(x) = A(x) \cdot \exp(-\delta \cdot x^r / r!)$ and we see that (1.10) holds for any n; vice versa. □

Now we will explain the binomial inversion (Theorem 1.14) for r-associated unified Stirling numbers. The structure of inversion formula is as follows:

PROPOSITION 1.13. *For each non-negative integer r, we let $A_r(x)$ be a function in x depending only on r, and put $G_{r,k}(x) = B_k \cdot A_r(x)^k$ for a positive integer k,* *where* B_k *is a constant or function depending only on* k. *Let*

$$G_{r,k}(x) = \sum_{n=0}^{\infty} \frac{g_r(n,k)}{n!} x^n$$

be the GF of a sequence $g_r(n,k)$. *Suppose that, for* $A_{r+1}(x)$ *and* $A_r(x)$, *there exists a constant* a_r *satisfying the condition for difference*

$$A_r(x) - A_{r+1}(x) = \frac{a_r}{r!} x^r. \tag{1.11}$$

Then we have the following identities:

$$g_{r+1}(n,k) = k! \cdot B_k \sum_{j=0}^{k} \frac{(-1)^j \cdot a_r^j}{(k-j)! \cdot B_{k-j}} \cdot C_r(n,j) \cdot g_r(n-rj, k-j), \tag{1.12}$$

$$g_r(n,k) = k! \cdot B_k \sum_{j=0}^{k} \frac{a_r^j}{(k-j)! \cdot B_{k-j}} \cdot C_r(n,j) \cdot g_{r+1}(n-rj, k-j), \tag{1.13}$$

where $C_r(n,j)$ *is the r-associated binomial coefficient mentioned in Remark* 1.5 (4).

Proof. From the condition $A_r(x) = a_r \cdot x^r/r! + A_{r+1}(x)$, we have

$$\begin{aligned}
G_{r,k}(x) &= B_k \sum_{i=0}^{k} \binom{k}{i} \cdot \left(\frac{a_r}{r!} x^r\right)^{k-i} \cdot A_{r+1}(x)^i \\
&= B_k \sum_{i=0}^{k} \sum_{l=0}^{\infty} \binom{k}{i} \cdot \left(\frac{a_r}{r!}\right)^{k-i} \cdot \frac{1}{B_i} \cdot \frac{g_{r+1}(l,i)}{l!} x^{l+r(k-i)} \\
&= B_k \sum_{n=0}^{\infty} \sum_{j=0}^{k} \binom{k}{k-j} \cdot \left(\frac{a_r}{r!}\right)^j \cdot \frac{1}{B_{k-j}} \cdot \frac{n!}{(n-rj)!} \cdot \frac{g_{r+1}(n-rj, k-j)}{n!} x^n.
\end{aligned}$$

Here we note that

$$\binom{k}{k-j} \cdot \left(\frac{a_r}{r!}\right)^j \cdot \frac{1}{B_{k-j}} \cdot \frac{n!}{(n-rj)!} = \frac{k! \cdot a_r^j}{(k-j)! \cdot B_{k-j}} \cdot C_r(n,j);$$

thus we obtain (1.13). Similarly we can prove (1.12). □

Theorem 1.14. *For r-associated unified Stirling numbers* $f_r(n,k) = f_r(n,k; \alpha, \beta, \gamma)$, *if* $(\beta - \alpha|\alpha)_r \neq 0$, *then we have the following*:

$$f_{r+1}(n,k) = \sum_{j=0}^{k} (-1)^j \cdot \delta_r^j \cdot C_r(n,j) \cdot f_r(n-rj, k-j),$$

$$f_r(n,k) = \sum_{j=0}^{k} \delta_r^j \cdot C_r(n,j) \cdot f_{r+1}(n-rj, k-j).$$

PROOF. The GF of $f_r(n,k)$ was given by Proposition 1.6. Here we let $B_k = p(x)/k!$ and $A_r(x) = \delta_r \cdot q_r(x)$, where $p(x)$ and $q_r(x)$ are the same as in (1.3). Then $A_r(x)$ and $A_{r+1}(x)$ satisfy the condition for difference with $a_r = \delta_r$. Thus Proposition 1.13 implies our theorem. □

Our theorem is a generalization of the identities (4.5) and (4.6) in Howard [3], or Riordan [6, page 102]. Here we will translate this fact to matrix language. For a non-negative integer p, we define a lower-triangular matrix $M_{r,p}$, $M_{r,p}^+$, $C_{r,p}$ and $C'_{r,p}$ (of any size) as follows:

$$M_{r,p} = (f_r(r(i-1)+p, i-j))_{ij}, \qquad C'_{r,p} = ((-\delta_r)^{i-j} \cdot C_r(r(i-1)+p, i-j))_{ij},$$

$$M_{r,p}^+ = (f_{r+1}(r(i-1)+p, i-j))_{ij}, \quad C_{r,p} = (\delta_r^{i-j} \cdot C_r(r(i-1)+p, i-j))_{ij}.$$

Then Theorem 1.14 implies the following:

COROLLARY 1.15. *We have $M_{r,p} = C_{r,p} \cdot M_{r,p}^+$ and $M_{r,p}^+ = C'_{r,p} \cdot M_{r,p}$. Moreover, the matrix $C_{r,p}$ is invertible, and we have $C_{r,p}^{-1} = C'_{r,p}$.*

Thus we see that $(r+1)$-associated f_{r+1}'s are obtained from r-associated f_r's by the r-associated binomial transformation of degree δ_r. We will give a combinatorial interpretation of the above inversion formula in the next section; see (3.11), (3.20) and (3.28).

1.4. An Interpretation in the Case of $r = 0$. Although we have let r be a positive integer, here we will dare to consider the case of $r = 0$. Here we assume that β is not equal to zero. Then, since the binomial coefficient $\binom{n-1}{r-1}$ should be zero for $r = 0$, we will understand the recurrence relation for $f_r(n,k)$ as follows:

$$f_0(n,k) = \{-\alpha(n-1) + \beta k + \gamma\} \cdot f_0(n-1, k).$$

Thus we have $f_0(n,k) = (\beta k + \gamma | \alpha)_n \cdot f_0(0,k)$, and we see that its initial value should be taken as $f_0(0,k) = (\beta^k \cdot k!)^{-1}$ from the view point of the GF (1.6), because $q_0(x)$ in (1.3) can be considered to be the function $\beta^{-1} \cdot (1+\alpha x)^{\beta/\alpha}$. Therefore our 0-*associated* unified Stirling numbers are given explicitly:

$$f_0(n,k) = \frac{(\beta k + \gamma|\alpha)_n}{\beta^k \cdot k!}.$$

Hence, putting $B_k = (1+\alpha x)^{\gamma/\alpha}/k!$ and $A_r(x) = \beta^{-1} \cdot \sum_{j=r}^{\infty} (\beta|\alpha)_j \cdot x^j/j!$ for $r = 0, 1$, we see that Proposition 1.13 (also Theorem 1.14) is still valid for $G_{r,k}(x) = B_k \cdot A_r(x)^k$ with $r = 0, 1$. Therefore we have the inversion formula between $r = 0$ and $r = 1$.

$$f_1(n,k) = \sum_{j=0}^{k} (-1)^j \cdot a_0^j \cdot C_0(n,j) \cdot f_0(n,k-j), \tag{1.14}$$

where we note that the "0-associated" binomial coefficient

$$C_0(n,j) = \frac{n! \cdot (0!)^{-j}}{j! \cdot (n-0)!} = \frac{1}{j!}$$

does depend only on j. Since we have $a_0 = \beta^{-1}$, the inversion formula (1.14) can be rewritten as follows:

$$\begin{aligned} f_1(n,k) &= \sum_{j=0}^{k} (-1)^j \cdot (\beta^{-1})^j \cdot \frac{1}{j!} \cdot \frac{(\beta(k-j)+\gamma|\alpha)_n}{\beta^{k-j} \cdot (k-j)!} \\ &= \frac{1}{\beta^k \cdot k!} \cdot \sum_{j=0}^{k} (-1)^j \cdot \frac{k!}{j! \cdot (k-j)!} \cdot (\beta(k-j)+\gamma|\alpha)_n \\ &= \frac{1}{\beta^k \cdot k!} \cdot \sum_{l=0}^{k} (-1)^{k+l} \cdot \binom{k}{l} \cdot (\beta l + \gamma|\alpha)_n. \qquad (1.15) \end{aligned}$$

Thus we gain the explicit formula for 1-associated unified Stirling numbers with $\beta \neq 0$. In particular, putting $(\alpha, \beta, \gamma) = (0, 1, 0)$, we conclude that our inversion formula between f_1 and f_0 gives the famous formula for Stirling numbers of the second kind. Note that the explicit formula (1.15) has been given by He [2, Corollary 2.3].

2. Counting Subspaces Generated by Subsets of a Root System

2.1. Preliminaries for Counting Subspaces. In [5], we have counted the number of subspaces, which are generated by subsets of a root system, of only co-dimension one. Here we will generalize the result. In this subsection, we give preliminaries for counting such subspaces of any co-dimension.

NOTATION 2.1. For k column vectors $w_1, \dots, w_k \in \mathbf{R}^n$, we denote by $\mathbf{M}(w_1, \dots, w_k)$ the $n \times k$ matrix aligning them, and by $\mathbf{L}(w_1, \dots, w_k)$ the subspace of $\mathbf{R}^n$ generated by them:

$$\mathbf{M}(w_1, \dots, w_k) = [w_1 \,|\, w_2 \,|\, \cdots \,|\, w_k], \quad \mathbf{L}(w_1, \dots, w_k) = \langle w_1, w_2, \dots, w_k \rangle_{\mathbf{R}}.$$

On the other hand, we sometimes write by $\mathbf{L}(M)$ the subspace of $\mathbf{R}^n$ generated by the column vectors $w_1, \dots, w_k$ of an $n \times k$ matrix $M = \mathbf{M}(w_1, \dots, w_k)$.

NOTATION 2.2. We write $A \sim B$ if a matrix B can be obtained from a matrix A by right-elementary transformation (i.e., elementary transformation for columns) and/or permutation of rows. On the other hand, we write $A \simeq B$ if B can be obtained from A only by right-elementary transformation; thus we have $\mathbf{L}(A) = \mathbf{L}(B)$ if $A \simeq B$.

NOTATION 2.3. For a non-zero vector $x = (x_i)_{i=1}^n \in \mathbf{R}^n$, we put $\operatorname{supp} x = \{i;\, x_i \neq 0\}$. Then we will call $\mathrm{s}(x) = \min \operatorname{supp} x$ the *starting of* x and $\mathrm{e}(x) = \max \operatorname{supp} x$ its *ending*.

DEFINITION 2.4. If an $n \times k$ matrix M can be, by right-elementary transformation and/or permutation of rows, expressed as

$$M \sim \left[\begin{array}{c|c} X & O \\ \hline O & Y \end{array}\right], \tag{2.1}$$

we call it to be *reducible*. Here X is an $l \times m$ matrix and Y is $(n-l) \times (k-m)$ $(1 \le l \le n-1,\ 1 \le m \le k-1)$. Then we say that M is the *direct sum* of X and Y, and denote it by $X \oplus Y$. We will similarly define the direct sum of some (more than two) matrices. If M is not reducible, we call it to be *irreducible*.

DEFINITION 2.5. We say that a matrix $M = \mathbf{M}(x_1, x_2, \dots, x_k)$ is *sincere* if M does not have a row consisting of zeros; that is, for any number i $(1 \le i \le n)$, there exists j $(1 \le j \le k)$ such that $i \in \operatorname{supp} x_j$. A subspace $L = \mathbf{L}(M)$ is also called *sincere* if the corresponding matrix M is sincere.

DEFINITION 2.6. An $n \times k$ matrix M is called to be *full-rank* if we have $\operatorname{rank} M = \min\{n, k\}$.

REMARK 2.7. For the terminology defined in the above, the following facts are fundamental:

(1) For an $n \times k$ matrix M with $n \geq 2$ and $k \geq 2$, if there exists a column u of M satisfying $\mathrm{s}(u) = \mathrm{e}(u)$, then M is reducible. In particular, each identity matrix of degree more than one is reducible.

(2) If $M \sim M'$ and M is irreducible (resp. sincere, full-rank), then M' is also irreducible (resp. sincere, full-rank); these properties are kept under any permutation of rows and/or right-elementary transformation.

LEMMA 2.8. *For a matrix M of size $n \times k$ with $k \geq 2$, we let $M \sim M_1 \oplus \cdots \oplus M_q$ be a direct sum of irreducible matrices, and put*

$$(2.2) \qquad PMQ = M_1 \oplus M_2 \oplus \cdots \oplus M_q \quad \textit{and} \quad {}^t(PM) = [{}^tY_1 \,|\, {}^tY_2 \,|\, \cdots \,|\, {}^tY_q],$$

where each M_i, Y_i is of size $n_i \times m_i$, $n_i \times k$, respectively. Then each Y_i is irreducible. Moreover, if $n_i \geq 2$, then the ending of each column of Y_i is different from its starting unless it is the zero vector.

PROOF. We make an $n_i \times k$ matrix $\widetilde{M_i} = [O \,|\, M_i \,|\, O]$ by adding some zero vectors to M_i. Then $\widetilde{M_i}$ is irreducible. Thus so is Y_i, because we have $Y_i \simeq \widetilde{M_i}$ by the assumption. Since $n_i \geq 2$, our assertion follows from Remark 2.7(1) immediately. □

NOTATION 2.9. We denote the standard basis of $\mathbf{R}^n$ by $\boldsymbol{e}_1^{(n)}, \boldsymbol{e}_2^{(n)}, \ldots, \boldsymbol{e}_n^{(n)}$. We will sometimes omit the superscripts (write as $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ for simply) if they are clearly in $\mathbf{R}^n$.

DEFINITION 2.10. Let k be a positive integer, and $k = m_1 + \cdots + m_q$ a representation as a sum of q positive integers. Then we call $(m_1, \ldots, m_q)$ a *partition* of k into q summands.

3. Standard Form Attached to Subspaces

3.1. $\mathbf{A}_n$-type. In this subsection, we denote by $\Phi = \Phi(\mathbf{A}_n)$ the root system of type $\mathbf{A}_n$, which can be regarded as a finite subset of $E^{(n+1)}$:

$$\Phi = \Phi(\mathbf{A}_n) = \{\pm(\boldsymbol{e}_i^{(n+1)} - \boldsymbol{e}_j^{(n+1)}) \in E^{(n+1)};\ 1 \leq i < j \leq n+1\},$$

where $E^{(n+1)}$ is the hyperplane of vectors orthogonal to ${}^t(1, 1, \ldots, 1) \in \mathbf{R}^{n+1}$; that is, $E^{(n+1)} = \{\boldsymbol{x} = (x_i)_{i=1}^{n+1} \in \mathbf{R}^{n+1};\ x_1 + x_2 + \cdots + x_{n+1} = 0\}$.

Notation 3.1. We denote by $R(m)$ the $(m+1)\times m$ matrix such that

$$R(m) = \mathbf{M}(v_1, v_2, \dots, v_m) \quad \text{with } v_i = \boldsymbol{e}_i^{(m+1)} - \boldsymbol{e}_{i+1}^{(m+1)} \in \mathbf{R}^{m+1} \quad \text{for } i = 1, 2, \dots, m.$$

For such a matrix $R(m)$, the following fact is fundamental:

Lemma 3.2. *Let $\boldsymbol{m} = (m_1, \dots, m_q)$ and $\boldsymbol{m}' = (m_1', \dots, m_q')$ be partitions of a positive integer k into q summands with the condition*

$$m_1 \le \dots \le m_q \quad \text{and} \quad m_1' \le \dots \le m_q'. \tag{3.1}$$

Put $M = R(m_1) \oplus \dots \oplus R(m_q)$ and $M' = R(m_1') \oplus \dots \oplus R(m_q')$. If $M \sim M'$, then we have $\boldsymbol{m} = \boldsymbol{m}'$.

Proof. For a positive integer m, we note that the reduced row echelon form (rref) of ${}^tR(m)$ is given by $[I_m \mid \boldsymbol{n}(m)]$ with the vector $\boldsymbol{n}(m) = {}^t(-1, \dots, -1) \in \mathbf{R}^m$, where I_m denotes the identity matrix of degree m. Hence we have

$$M \sim {}^t[I_k \mid \boldsymbol{n}(m_1) \oplus \dots \oplus \boldsymbol{n}(m_q)] \quad \text{and} \quad M' \sim {}^t[I_k \mid \boldsymbol{n}(m_1') \oplus \dots \oplus \boldsymbol{n}(m_q')].$$

Suppose that $\boldsymbol{m} \ne \boldsymbol{m}'$. Since, for each matrix, its rref is uniquely determined by the theory of Linear Algebra, we see that ${}^t(\boldsymbol{n}(m_1) \oplus \dots \oplus \boldsymbol{n}(m_q)) \not\sim {}^t(\boldsymbol{n}(m_1') \oplus \dots \oplus \boldsymbol{n}(m_q'))$; thus we obtain $M \not\sim M'$. $\square$

Proposition 3.3. *Let $L = \mathbf{L}(w_1, w_2, \dots, w_k)$ be the subspace generated by k roots $w_1, w_2, \dots, w_k \in \Phi$, and $M = \mathbf{M}(w_1, w_2, \dots, w_k)$ the corresponding matrix. If M is full-rank, sincere, and irreducible, then we can take positive roots $v_1, v_2, \dots, v_k$ as a basis of L, where each v_i is of the form $v_i = \boldsymbol{e}_i^{(n+1)} - \boldsymbol{e}_{i+1}^{(n+1)} \in \mathbf{R}^{(n+1)}$. That is to say, for such M, we have $k = n$, and $M \simeq R(n)$.*

Proof. Since our assertion for $k = 1$ is trivial, we consider the case of $k \ge 2$. By appropriate right-elementary transformation, we may assume that $1 = \mathrm{s}(w_1) \le \mathrm{s}(w_2) \le \dots \le \mathrm{s}(w_k)$. Here, if $\mathrm{s}(w_i) = \mathrm{s}(w_{i+1})$, replacing the $(i+1)$-th column of M by $w_{i+1}' = w_{i+1} + w_i$ or $w_{i+1} - w_i$, we have $\mathrm{s}(w_i) < \mathrm{s}(w_{i+1}')$. Thus we may assume that

$$1 = \mathrm{s}(w_1) < \mathrm{s}(w_2) < \dots < \mathrm{s}(w_k). \tag{3.2}$$

Moreover, we may assume that M satisfies the following condition:

$$\mathrm{e}(w_i) \ne \mathrm{s}(w_j) \quad \text{for arbitrary indices } i \text{ and } j. \tag{3.3}$$

Indeed, if there exist indices i and j satisfying $\mathrm{e}(w_i) = \mathrm{s}(w_j)$, then we have $1 \le i < j \le k$ by the condition (3.2), and we can change w_i for $w_i' = w_i + w_j$ or $w_i - w_j$ if necessary, so that $\mathrm{e}(w_i') > \mathrm{s}(w_j)$. Now, we again let $M_1 = \mathbf{M}(w_1, w_2, \ldots, w_k)$ be the matrix satisfying conditions (3.2) and (3.3), which is obtained from M by appropriate right-elementary transformation. Then, since M is irreducible, we see that M_1 satisfies, in addition, the following condition

$$\mathrm{s}(w_k) < \mathrm{e}(w_1) = \mathrm{e}(w_2) = \cdots = \mathrm{e}(w_k). \tag{3.4}$$

Since M_1 is also sincere, we have $\mathrm{s}(w_i) = i$ and $\mathrm{e}(w_i) = k + 1$ for each $i = 1, 2, \ldots, k$; that is,

$$\mathrm{s}(w_k) = k < \mathrm{e}(w_1) = \mathrm{e}(w_2) = \cdots = \mathrm{e}(w_k) = k + 1 \ \ (= n + 1).$$

Thus we conclude that $k = n$; so that, applying [5, Lemma 3.1], we obtain our assertion. □

Theorem 3.4. *Let $L = \mathbf{L}(w_1, w_2, \ldots, w_k)$ be the subspace generated by k roots $w_1, w_2, \ldots, w_k \in \Phi$ of type $\mathbf{A}_n$ $(n \ge 1)$, and $M = \mathbf{M}(w_1, w_2, \ldots, w_k)$ the corresponding matrix. Assume that M is full-rank and sincere, then there exists a partition $(m_1, \ldots, m_q)$ of k into q $(= n + 1 - k)$ summands satisfying $M \sim R(m_1) \oplus \cdots \oplus R(m_q)$. These $R(m_1), \ldots, R(m_q)$ are uniquely determined from L, up to numbering.*

Proof. If M is irreducible, our theorem follows from Proposition 3.3. Suppose that M is reducible; then we have an irreducible decomposition as in (2.2) with a permutation P and an invertible Q. In (2.2), each M_i, Y_i is an $n_i \times m_i$, $n_i \times k$ matrix, respectively, and they are full-rank, sincere, and irreducible. Then, by Lemma 2.8 each column of Y_i is either the zero vector or a root of type $\mathbf{A}_{n_i - 1}$. Thus, applying Proposition 3.3 to each M_i, we see that

$$M \sim M_1 \oplus \cdots \oplus M_q \simeq R(m_1) \oplus \cdots \oplus R(m_q). \tag{3.5}$$

Since each $R(m_i)$ has $n_i = m_i + 1$ rows, the number of rows of the right-hand side of (3.5) is equal to $n + 1 = m_1 + \cdots + m_q + q$. Thus we have $q = n + 1 - k$, because M has $k = m_1 + \cdots + m_q$ columns. The uniqueness of decomposition in the right-hand side of (3.5) follows from Lemma 3.2. Thus we obtain our theorem. □

Remark 3.5. We call the right-hand side of (3.5) a standard form of L. For each subspace generated by some roots of type $\mathbf{A}_n$, its standard form is uniquely determined under the condition $m_1 \leq \cdots \leq m_q$; so that, to classify such subspaces, we are enough to discriminate the standard forms.

COROLLARY 3.6. *The number* $a^0(n,k)$ *of distinct sincere* k*-dimensional subspaces generated by some roots in* $\Phi \subset E^{(n+1)} \subset \mathbf{R}^{n+1}$ *of type* $\mathbf{A}_n$ *is given as follows*:

$$a^0(n,k) = S_2(n+1,q) = S_2(n+1,n+1-k). \tag{3.6}$$

PROOF. We consider the decomposition as in the right-hand side of (3.5). From $n+1$ rows, we select some rows to place q blocks $R(m_1), R(m_2), \dots, R(m_q)$. Since each block has more than one rows, the number of such ways are presented by using the 2-associated Stirling number $S_2(n+1,q)$. □

THEOREM 3.7. *Let* $L = \mathbf{L}(w_1, w_2, \dots, w_k)$ *be the subspace generated by* k *roots* $w_1, w_2, \dots, w_k \in \Phi$ *of type* $\mathbf{A}_n$ $(n \geq 1)$, *and* $M = \mathbf{M}(w_1, w_2, \dots, w_k)$ *the corresponding matrix. Assume that* M *is full-rank and has* s *zero rows. Then there exists a partition* $(m_1, \dots, m_q)$ *of* k *into* q $(= n+1-k-s)$ *summands satisfying*

$$M \sim \left[\frac{R(m_1) \oplus \cdots \oplus R(m_q)}{O} \right],$$

where O *denotes the zero matrix of size* $s \times k$. *These* $R(m_1), \dots, R(m_q)$ *are uniquely determined from* L, *up to numbering.*

PROOF. By appropriate permutation of rows, we may assume that the last s rows of M are zeros. The matrix consisting of the remaining rows is sincere; so that our assertion follows from Theorem 3.4. □

COROLLARY 3.8. *The number* $a(n,k)$ *of distinct* k*-dimensional subspaces* (*which may not be sincere*) *generated by some roots in* $\Phi \subset E^{(n+1)} \subset \mathbf{R}^{n+1}$ *of type* $\mathbf{A}_n$ *is given as follows*:

$$a(n,k) = \sum_{s=0}^{n-k} \binom{n+1}{s} \cdot a^0(n-s,k) \tag{3.7}$$

$$= S_1(n+1,q+s) = S_1(n+1,n+1-k). \tag{3.8}$$

PROOF. The idea of the proof for the first identity is similar to that of Theorem 3.7. The second can be proved by using a similar way to the proof of Corollary 3.6. □

Here we will put $\tilde{a}(n,k) = a(n,n-k)$ and $\tilde{a}^0(n,k) = a^0(n,n-k)$, and rewrite (3.8) and (3.6) respectively. Thus we have the following:

$$\tilde{a}(n,k) = S_1(n+1,k+1) = f_1(n+1,k+1;0,1,0), \tag{3.9}$$

$$\tilde{a}^0(n,k) = S_2(n+1,k+1) = f_2(n+1,k+1;0,1,0). \tag{3.10}$$

By using the explicit formula (1.15) for classical Stirling numbers of the second kind, we can rewrite (3.9), (3.8) as follows; this is the explicit formula for the number of subspaces of type $\mathbf{A}_n$:

$$\tilde{a}(n,k) = \frac{1}{(k+1)!} \cdot \sum_{l=0}^{k+1} (-1)^{k+l+1} \cdot \binom{k+1}{l} \cdot l^{n+1},$$

$$a(n,k) = \frac{1}{(n-k+1)!} \cdot \sum_{l=0}^{n-k+1} (-1)^{n-k+l+1} \cdot \binom{n-k+1}{l} \cdot l^{n+1}.$$

On the other hand, from (3.9), (3.10), we obtain the exponential GF

$$\sum_{n=0}^{\infty} \frac{\tilde{a}(n,k)}{n!} x^n = \frac{d}{dx} F_{1,k+1}(x) = \frac{1}{k!} \cdot (e^x - 1)^k \cdot e^x,$$

$$\sum_{n=0}^{\infty} \frac{\tilde{a}^0(n,k)}{n!} x^n = \frac{d}{dx} F_{2,k+1}(x) = \frac{1}{k!} \cdot (e^x - 1 - x)^k \cdot (e^x - 1),$$

respectively. In addition, from (3.7), we obtain the following formula

$$\tilde{a}(n,k) = \sum_{s=0}^{k} \binom{n+1}{s} \cdot \tilde{a}^0(n-s,k-s). \tag{3.11}$$

This is the formula for Stirling numbers of the second kind (see (4.5) of Howard [3], or Riordan [6, p. 102]); that is obtained from our Theorem 1.14, specializing parameters $r = 1$, $\alpha = 0$, $\beta = 1$ and $\gamma = 0$ (hence $\delta_1 = 1$ and $C_1(n+1,s) = \binom{n+1}{s}$):

$$S_1(n+1,k+1) = \sum_{s=0}^{k} \delta_1^s \cdot C_1(n+1,s) \cdot S_2(n+1-s,k+1-s).$$

3.2. $\mathbf{B}_n$ or $\mathbf{C}_n$-type. In this subsection, we classify subspaces generated by some roots of type $\mathbf{B}_n$ or $\mathbf{C}_n$. Here we denote by Φ the root system either $\Phi(\mathbf{B}_n)$ or $\Phi(\mathbf{C}_n)$, which can be respectively regarded as a finite subset of $\mathbf{R}^n$:

$$\Phi(\mathbf{B}_n) = \{\pm(\boldsymbol{e}_i^{(n)} \pm \boldsymbol{e}_j^{(n)});\ 1 \le i < j \le n\} \cup \{\pm\boldsymbol{e}_1^{(n)}, \pm\boldsymbol{e}_2^{(n)}, \dots, \pm\boldsymbol{e}_n^{(n)}\},$$

$$\Phi(\mathbf{C}_n) = \{\pm(\boldsymbol{e}_i^{(n)} \pm \boldsymbol{e}_j^{(n)});\ 1 \le i < j \le n\} \cup \{\pm 2\boldsymbol{e}_1^{(n)}, \pm 2\boldsymbol{e}_2^{(n)}, \dots, \pm 2\boldsymbol{e}_n^{(n)}\}.$$

So the classification for $\mathbf{C}_n$-type is parallel to that of $\mathbf{B}_n$-type.

NOTATION 3.9. We denote by $\tilde{R}(m)$ the set of $(m+1) \times m$ matrices as follows:

$$\tilde{R}(m) = \{\mathbf{M}(v_1, v_2, \dots, v_m);\ v_i = \boldsymbol{e}_i + \boldsymbol{e}_{i+1} \text{ or } \boldsymbol{e}_i - \boldsymbol{e}_{i+1} \text{ for } i = 1, 2, \dots, m\}.$$

We note that the cardinality of $\tilde{R}(m)$ is equal to 2^m.

REMARK 3.10. For two matrices $X, Y \in \tilde{R}(m)$, we have $X \not\simeq Y$ if $X \neq Y$. Indeed, $X \simeq Y$ implies $\mathbf{L}(X) = \mathbf{L}(Y)$; and then $\mathbf{L}(X)^{\perp} = \mathbf{L}(Y)^{\perp}$. Here $\mathbf{L}(X)^{\perp}$ and $\mathbf{L}(Y)^{\perp}$ are one-dimensional, it follows from [5, Theorem 5.4(2)], that $X = Y$.

LEMMA 3.11. *Let $\boldsymbol{m} = (m_1, \dots, m_q)$ and $\boldsymbol{m}' = (m_1', \dots, m_q')$ be partitions of a positive integer k into q summands with the condition* (3.1). *For each $i = 1, \dots, q$, we choose $X_i \in \tilde{R}(m_i)$ and $X_i' \in \tilde{R}(m_i')$, and put $M = X_1 \oplus \cdots \oplus X_q$ and $M' = X_1' \oplus \cdots \oplus X_q'$. If $M \sim M'$, then we have $\boldsymbol{m} = \boldsymbol{m}'$; in particular, $X_i = X_i'$ for each $i = 1, \dots, q$.*

PROOF. This can be proved by a similar way to the proof of Lemma 3.2. □

LEMMA 3.12. *For k roots $w_1, w_2, \dots, w_k \in \Phi$ $(2 \le k \le n)$, assume that $M = \mathbf{M}(w_1, w_2, \dots, w_k)$ is full-rank and sincere. If $\mathrm{s}(w_1) = \mathrm{s}(w_2)$ and $\mathrm{e}(w_1) = \mathrm{e}(w_2)$, then M is reducible.*

PROOF. If $k = n = 2$, then M is a square matrix of full-rank. Hence it can be transformed to the identity matrix of degree two; that is to say, M is reducible. Assume that $2 \le k < n$. Then the condition $\mathrm{s}(w_1) = \mathrm{s}(w_2)$ and $\mathrm{e}(w_1) = \mathrm{e}(w_2)$ implies that

$$M \sim \left[\begin{array}{c|c} X & Z \\ \hline O & Y \end{array}\right],$$

where X is a square matrix of degree two. In addition, since X is of full-rank, by appropriate right-elementary transformation, we can transform Z to a zero matrix; hence M is reducible. □

Proposition 3.13. *Let* $L = \mathbf{L}(w_1, w_2, \dots, w_n)$ *be the subspace generated by* n *roots* $w_1, w_2, \dots, w_n \in \Phi$, *and* $M = \mathbf{M}(w_1, w_2, \dots, w_n)$ *the corresponding matrix. If* M *is full-rank, then we can take the standard basis* $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ *of* $\mathbf{R}^n$ *as a basis of* L; *in particular, we have* $M \simeq I_n$.

Proof. Since M is full-rank, it is a non-singular matrix of degree n. Hence, multiplying M^{-1} from the right-hand side, we have our assertion. □

Proposition 3.14. *Let* $L = \mathbf{L}(w_1, w_2, \dots, w_k)$ *be the subspace generated by* k $(< n)$ *roots* $w_1, w_2, \dots, w_k \in \Phi$, *and* $M = \mathbf{M}(w_1, w_2, \dots, w_k)$ *the corresponding matrix. If* M *is full-rank, sincere, and irreducible, then we can take positive roots* $v_1, v_2, \dots, v_k$ *as a basis of* L, *where each* v_i *is of the form* $v_i = \boldsymbol{e}_i - \boldsymbol{e}_{i+1}$ *or* $\boldsymbol{e}_i + \boldsymbol{e}_{i+1}$. *That is to say, for such* M, *we can choose a unique* $X \in \tilde{R}(k)$ *satisfying* $M \simeq X$. *In particular, we have* $k = n - 1$.

Proof. First we consider the case of $k = 1$. Then M is not a square matrix and we have $w_1 \neq \pm \boldsymbol{e}_i^{(n)}$, because $k < n$ and M is sincere. Thus the assumption that M is sincere implies that $w_1 = \boldsymbol{e}_1^{(n)} \pm \boldsymbol{e}_2^{(n)}$; hence we have $n = 2$. Next let k be grater than 1. Since M is irreducible, by Remark 2.7(1), we may assume that $\mathrm{s}(w_i) \neq \mathrm{e}(w_i)$ for each column w_i of M; that is,

$$w_i \in \{\pm(\boldsymbol{e}_i^{(n)} \pm \boldsymbol{e}_j^{(n)});\ 1 \leq i < j \leq n\}.$$

In addition, if there exists an index i such that $\mathrm{s}(w_i) = \mathrm{s}(w_{i+1})$, then it follows from Lemma 3.12 that $\mathrm{e}(w_i) \neq \mathrm{e}(w_{i+1})$, since M is irreducible. Here, in addition, we may assume that M satisfies the conditions (3.2), (3.3) and (3.4). Since M is sincere, we have $\mathrm{s}(w_i) = i$ and $\mathrm{e}(w_i) = k + 1\ (= n)$ for each $i = 1, 2, \dots, k$; that is,

$$\mathrm{s}(w_k) = k < \mathrm{e}(w_1) = \mathrm{e}(w_2) = \cdots = \mathrm{e}(w_k) = k + 1\ \ (= n).$$

Thus, M should be one-codimensional; so that our assertion follows from [5, Proposition 5.2]. The uniqueness follows from Remark 3.10. □

Theorem 3.15. *Let* $L = \mathbf{L}(w_1, w_2, \dots, w_k)$ *be the subspace generated by* k *roots* $w_1, \dots, w_k \in \Phi$ *of type* $\mathbf{B}_n$ *or* $\mathbf{C}_n$ $(n \geq 2)$, *and* $M = \mathbf{M}(w_1, \dots, w_k)$ *the corresponding matrix. Assume that* M *is full-rank and sincere.*

(1) *If* $k = n$, *then we have* $M \simeq I_n$; *that is, we can choose the standard basis* $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ *of* $\mathbf{R}^n$ *as a basis of* L.

(2) *If* $k < n$, *then there exist a non-negative integer* t *and a partition* $(m_1, \dots, m_q)$ *of* k *into* q $(= n - k)$ *summands such that we have* $M \sim I_t \oplus$

$X_1 \oplus \cdots \oplus X_q$ with $X_i \in \tilde{R}(m_i)$ for each $i = 1, \dots, q$. These $X_1, \dots, X_q$ are uniquely determined from L, up to numbering.

PROOF. Here we let $\Phi = \Phi(\mathbf{B}_n)$; the case of $\mathbf{C}_n$-type can be proved similarly. If $k = n$, or if $k < n$ and M is irreducible, then our assertion follows from Proposition 3.13, or 3.14, respectively. Suppose that M is reducible; then decompose M into a direct sum of irreducible matrices by appropriate right-elementary transformation and replacing rows. Then, if there exist some square blocks, we gather ones as the first block. Thus we have $M \sim I_t \oplus M_1 \oplus \cdots \oplus M_q$, where each M_i is of size $n_i \times m_i$ $(1 \le m_i < n_i)$, full-rank, sincere, and irreducible. Here we rewrite it as

$$(3.12)\quad \begin{aligned} &PMQ = I_t \oplus M_1 \oplus \cdots \oplus M_q \text{ with permutation } P \text{ and invertible } Q \text{, and} \\ &{}^t(PM) = [{}^tY_0 \,|\, {}^tY_1 \,|\, \cdots \,|\, {}^tY_q] \text{ where each } Y_i \text{ is of size } n_i \times k \text{ and } n_0 = t. \end{aligned}$$

Then each column of PM is a root of type $\mathbf{B}_n$, and each Y_i is irreducible by Lemma 2.8. Moreover, each column of Y_i satisfies the condition that it is either the zero vector or a root of type $\mathbf{B}_{n_i}$. Therefore, applying Proposition 3.14 to each M_i, we can choose $X_i \in \tilde{R}(m_i)$ $(i = 1, \dots, q)$ such that

$$(3.13)\qquad M \sim I_t \oplus M_1 \oplus \cdots \oplus M_q \simeq I_t \oplus X_1 \oplus \cdots \oplus X_q.$$

Since each $X_i \in \tilde{R}(m_i)$ has $n_i = m_i + 1$ rows, the number of the right-hand side of (3.13) is equal to $n = t + m_1 + \cdots + m_q + q$; so that we have $q = n - k$, because M has $k = t + m_1 + \cdots + m_q$ columns. The uniqueness of the decomposition follows from Lemma 3.11. Thus we obtain our theorem. □

COROLLARY 3.16. *The number $b^0(n,k)$ of distinct k-dimensional sincere subspaces generated by some roots in $\Phi \subset \mathbf{R}^n$ of type $\mathbf{B}_n$ is given by*

$$(3.14)\qquad b^0(n,k) = \sum_{t=0}^{k} \binom{n}{t} \cdot 2^{k-t} \cdot S_2(n-t, n-k).$$

PROOF. We count the number of decompositions as the right-hand side of (3.13). First we choose and fix t rows where the identity matrix of degree t is placed. Next we put q blocks $X_1, \dots, X_q$, each of which has at least two rows. Since $\#R(m_i) = 2^{m_i}$, the number of such candidates is given by

$$\binom{n}{t} \cdot 2^{m_1 + \cdots + m_q} \cdot S_2(n-t, q).$$

Here we note that $m_1 + \cdots + m_q = k - t$ and $q = n - k$. Since the size t of square matrix runs from zero to k, we obtain our assertion. □

Theorem 3.17. *Let $L = \mathbf{L}(w_1, w_2, \ldots, w_k)$ be the subspace generated by k roots $w_1, w_2, \ldots, w_k \in \Phi$ of type $\mathbf{B}_n$ $(n \geq 2)$, and $M = \mathbf{M}(w_1, w_2, \ldots, w_k)$ the corresponding matrix. Assume that M is full-rank and has s zero rows. Then there exist a non-negative integer t and a partition $(m_1, \ldots, m_q)$ of k into q $(= n - k - s)$ summands such that we have*

$$M \sim \left[\begin{array}{c} I_t \oplus X_1 \oplus X_2 \oplus \cdots \oplus X_q \\ \hline O \end{array}\right] \tag{3.15}$$

with $X_i \in \tilde{R}(m_i)$ for each $i = 1, \ldots, q$. These $X_1, \ldots, X_q$ are uniquely determined from L, up to numbering. Here, if $t = 0$ or $q = 0$, then we should consider the corresponding things to be trivial.

Proof. By appropriate permutation of rows, we may assume that the last s rows of M are zeros. The matrix consisting of the remaining rows is sincere; so that our assertion follows from Theorem 3.15. □

Corollary 3.18. *The number $b(n,k)$ of distinct k-dimensional subspaces generated by some roots in $\Phi \subset \mathbf{R}^n$ of type $\mathbf{B}_n$ is given by*

$$b(n,k) = \sum_{s=0}^{n-k} \binom{n}{s} \cdot b^0(n-s,k) \tag{3.16}$$

$$= \sum_{t=0}^{k} \binom{n}{t} \cdot 2^{k-t} \cdot S_1(n-t, n-k). \tag{3.17}$$

Proof. The idea of the proof for the first identity is similar to that of Theorem 3.17. The second can be proved by using a similar way to the proof of Corollary 3.16. □

Remark 3.19. Let $c^0(n,k)$ (resp. $c(n,k)$) be the number of distinct k-dimensional subspaces, which is sincere (resp. may not be sincere), generated by some roots in $\Phi \subset \mathbf{R}^n$ of type $\mathbf{C}_n$. Then we have $c(n,s) = b(n,s)$ and $c^0(n,s) = b^0(n,s)$, since the standard forms of $\mathbf{C}_n$-type are identical with that of $\mathbf{B}_n$-type.

Here we will put $\tilde{b}(n,k) = b(n,n-k)$ and $\tilde{b}^0(n,k) = b^0(n,n-k)$, and rewrite (3.17) and (3.14), respectively. Thus we have the following:

$$\tilde{b}(n,k) = \sum_{j=k}^{n} \binom{n}{j} \cdot 2^{j-k} \cdot S_1(j,k) = f_1(n,k;0,2,1), \tag{3.18}$$

$$\tilde{b}^0(n,k) = \sum_{j=2k}^{n} \binom{n}{j} \cdot 2^{j-k} \cdot S_2(j,k) = f_2(n,k;0,2,1). \tag{3.19}$$

By using the explicit formula (1.15) for unified Stirling numbers, we can rewrite (3.18), (3.17) as follows; this is the explicit formula for the number of subspaces of type $\mathbf{B}_n$:

$$\tilde{b}(n,k) = \frac{1}{2^k \cdot k!} \cdot \sum_{l=0}^{k} (-1)^{k+l} \cdot \binom{k}{l} \cdot (2l+1)^n,$$

$$b(n,k) = \frac{1}{2^{n-k} \cdot (n-k)!} \cdot \sum_{l=0}^{n-k} (-1)^{n-k+l} \cdot \binom{n-k}{l} \cdot (2l+1)^n.$$

On the other hand, from (3.18), (3.19) we obtain the exponential GF

$$\sum_{n=0}^{\infty} \frac{\tilde{b}(n,k)}{n!} x^n = \frac{1}{k!} \cdot e^x \cdot \frac{1}{2^k} \cdot (e^{2x} - 1)^k,$$

$$\sum_{n=0}^{\infty} \frac{\tilde{b}^0(n,k)}{n!} x^n = \frac{1}{k!} \cdot e^x \cdot \frac{1}{2^k} \cdot (e^{2x} - 1 - 2x)^k,$$

respectively. In addition, from (3.16), we obtain the following formula

$$\tilde{b}(n,k) = \sum_{s=0}^{k} \binom{n}{s} \cdot \tilde{b}^0(n-s, k-s). \tag{3.20}$$

We can obtain from Theorem 1.14, specializing parameters $r = 1$, $\alpha = 0$, $\beta = 2$ and $\gamma = 1$ (hence $\delta_1 = 1$ and $C_1(n,s) = \binom{n}{s}$):

$$f_1(n,k;0,2,1) = \sum_{s=0}^{k} \delta_1^s \cdot C_1(n,s) \cdot f_2(n-s, k-s; 0,2,1).$$

3.3. $\mathbf{D}_n$-type. In this subsection, we denote by $\Phi = \Phi(\mathbf{D}_n)$ the root system of type $\mathbf{D}_n$, which can be regarded as a finite subset of $\mathbf{R}^n$:

$$\Phi = \Phi(\mathbf{D}_n) = \{\pm(\boldsymbol{e}_i^{(n)} \pm \boldsymbol{e}_j^{(n)});\ 1 \leq i < j \leq n\}.$$

Therefore we can consider the root system $\Phi(\mathbf{D}_n)$ to be a subset of $\Phi(\mathbf{B}_n)$. We note that Lemma 3.12 and Propositions 3.13, 3.14 are still valid for $\mathbf{D}_n$-type. Hence the standard forms of $\mathbf{D}_n$-type are almost the same as those of $\mathbf{B}_n$-type:

THEOREM 3.20. *Let $L = \mathbf{L}(w_1, w_2, \ldots, w_k)$ be the subspace generated by k roots $w_1, w_2, \ldots, w_k \in \Phi$ of type $\mathbf{D}_n$ $(n \geq 4)$, and $M = \mathbf{M}(w_1, w_2, \ldots, w_k)$ the corresponding matrix. Assume that M is full-rank and sincere.*

(1) *If $k = n$, then we have $M \simeq I_n$; that is, we can choose the standard basis $e_1, e_2, \ldots, e_n$ of $\mathbf{R}^n$ as a basis of L.*
(2) *If $k < n$, then there exist a non-negative integer t $(\neq 1)$ and a partition $(m_1, \ldots, m_q)$ of k into q $(= n - k)$ summands such that we have $M \sim I_t \oplus X_1 \oplus \cdots \oplus X_q$ with $X_i \in \tilde{R}(m_i)$ for each $i = 1, \ldots, q$. These $X_1, \ldots, X_q$ are uniquely determined from L, up to numbering.*

PROOF. If $k = n$, or if $k < n$ and M is irreducible, then our assertion follows from Proposition 3.13, or 3.14, respectively. Suppose that M is reducible; then, as in the proof of Theorem 3.15, we have an irreducible decomposition $M \sim I_t \oplus M_1 \oplus \cdots \oplus M_q$, where each M_i is of size $n_i \times m_i$ $(1 \leq m_i < n_i)$, full-rank, sincere, and irreducible. Here we rewrite it as (3.12). Then each column of PM is a root of type $\mathbf{D}_n$, and each Y_i is irreducible by Lemma 2.8. Moreover, each column of Y_i satisfies the condition that it is either the zero vector or a root of type $\mathbf{D}_{n_i}$. Therefore, applying Proposition 3.14 to each M_i, we can choose $X_i \in \tilde{R}(m_i)$ $(i = 1, \ldots, q)$ satisfying (3.13). Thus we have $q = n - k$ as in the proof of Theorem 3.15. The uniqueness of the decomposition can be proved by using Lemma 3.11.

Suppose that $t = 1$ in (3.13). Then the ending of the first column of PM in (3.12) is grater than 1. In addition, the first column of a unique one of $Y_1, \ldots, Y_q$ is not zero, and its starting is identical with its ending. This contradicts to the condition that Y_i is irreducible, because of $n_i \geq 2$ (see Remark 2.7). Thus we conclude that $t \neq 1$ and we obtain our theorem. □

COROLLARY 3.21. *The number $d^0(n,k)$ of distinct sincere k-dimensional subspaces generated by some roots in $\Phi \subset \mathbf{R}^n$ of type $\mathbf{D}_n$ is given as follows*:

$$d^0(n,k) = b^0(n,k) - n \cdot 2^{k-1} \cdot S_2(n-1, n-k). \tag{3.21}$$

PROOF. The standard forms of $\mathbf{D}_n$-type are almost the same as those of $\mathbf{B}_n$-type, except for the square matrix of size $t = 1$; thus (3.21) follows immediately from (3.14). □

THEOREM 3.22. *Let $L = \mathbf{L}(w_1, w_2, \dots, w_k)$ be the subspace generated by k roots $w_1, w_2, \dots, w_k \in \Phi$ of type $\mathbf{D}_n$ $(n \geq 4)$, and $M = \mathbf{M}(w_1, w_2, \dots, w_k)$ the corresponding matrix. Assume that M is full-rank and has s zero rows. Then there exist a non-negative integer t $(\neq 1)$ and a partition $(m_1, \dots, m_q)$ of k into q $(= n - k - s)$ summands such that we have (3.15) with $X_i \in \tilde{R}(m_i)$ for each $i = 1, \dots, q$. These $X_1, \dots, X_q$ are uniquely determined from L, up to numbering. Here, if $t = 0$ or $q = 0$, then we should consider the corresponding things to be trivial.*

PROOF. By appropriate permutation of rows, we may assume that the last s rows of M are zeros. The matrix consisting of the remaining rows is sincere; so that our assertion follows from Theorem 3.20. □

COROLLARY 3.23. *The number $d(n,k)$ of distinct k-dimensional subspaces generated by some roots in $\Phi \subset \mathbf{R}^n$ of type $\mathbf{D}_n$ is given as follows*:

$$d(n,k) = \sum_{s=0}^{n-k} \binom{n}{s} \cdot d^0(n-s,k) \tag{3.22}$$

$$= b(n,k) - n \cdot 2^{k-1} \cdot S_1(n-1, n-k). \tag{3.23}$$

PROOF. The idea of the proof is similar to that of $\mathbf{B}_n$-type. □

Here, putting $\tilde{d}(n,k) = d(n, n-k)$ and $\tilde{d}^0(n,k) = d^0(n, n-k)$, as in $\mathbf{B}_n$ or $\mathbf{A}_n$-type, we rewrite (3.23) and (3.21) as follows:

$$\tilde{d}(n,k) = \tilde{b}(n,k) - n \cdot 2^{n-k-1} \cdot S_1(n-1,k), \tag{3.24}$$

$$\tilde{d}^0(n,k) = \tilde{b}^0(n,k) - n \cdot 2^{n-k-1} \cdot S_2(n-1,k). \tag{3.25}$$

By using the explicit formula for $\mathbf{A}_n$ and $\mathbf{B}_n$-type, we will rewrite (3.24), (3.23), respectively; thus we obtain the explicit formulas for $\mathbf{D}_n$-type:

$$\tilde{d}(n,k) = \frac{1}{2^k \cdot k!} \cdot \sum_{l=0}^{k} (-1)^{k+l} \cdot \binom{k}{l} \cdot \{(2l+1)^n - n \cdot (2l)^{n-1}\},$$

$$d(n,k) = \frac{1}{2^{n-k} \cdot (n-k)!} \cdot \sum_{l=0}^{n-k} (-1)^{n-k+l} \cdot \binom{n-k}{l} \cdot \{(2l+1)^n - n \cdot (2l)^{n-1}\}.$$

On the other hand, from (3.24), (3.25), we obtain the exponential GF

$$\sum_{n=0}^{\infty} \frac{\tilde{d}(n,k)}{n!} x^n = \frac{1}{2^k \cdot k!} \cdot (e^x - x) \cdot (e^{2x} - 1)^k, \tag{3.26}$$

$$\sum_{n=0}^{\infty} \frac{\tilde{d}^0(n,k)}{n!} x^n = \frac{1}{2^k \cdot k!} \cdot (e^x - x) \cdot (e^{2x} - 1 - 2x)^k, \tag{3.27}$$

respectively. In addition, from (3.22), we obtain the following formula

$$\tilde{d}(n,k) = \sum_{s=0}^{k} \binom{n}{s} \cdot \tilde{d}^0(n-s, k-s). \tag{3.28}$$

Thus we can enumerate the number of distinct subspaces for $\mathbf{D}_n$-type, by using that of distinct sincere subspaces, as in (3.11) or (3.20). In fact, this can be considered to be an inversion formula in the following way. So, putting $p(x) = e^x - x$ and

$$q_r(x) = \frac{1}{2}\left(e^{2x} - \sum_{j=0}^{r-1} \frac{1}{j!}(2x)^j\right) = \sum_{j=r}^{\infty} \frac{2^{j-1}}{j!} x^j,$$

we define the function

$$G_{r,k}(x) = \frac{1}{k!} \cdot p(x) \cdot q_r(x)^k = \sum_{n=0}^{\infty} \frac{g_r(n,k)}{n!} x^n$$

for $r = 1, 2$. Then it follows from (3.26) and (3.27) that $\tilde{d}(n,k) = g_1(n,k)$ and $\tilde{d}^0(n,k) = g_2(n,k)$. For the functions $q_1(x)$ and $q_2(x)$, we note that the condition for difference (1.11) is satisfied; so we conclude that (3.28) is an inversion formula for $g_1(n,k)$ and $g_2(n,k)$.

4. Counting Subspaces for Exceptional Type

4.1. A Strategy for Enumerating the Numbers for Exceptional Type. Let Φ be a root system of an arbitrary exceptional-type (except for $\mathbf{G}_2$-type) and Φ^+ the subset consisting of all positive roots with respect to a fixed lexicographical order. Our strategy for enumerating subspaces for exceptional-type is similar as in the case of co-dimension one; here we recall it from [5].

Let L be a subspace generated by some roots in Φ, and k its dimension, and put $\Psi^+ = L \cap \Phi^+$. We will choose k positive roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ as follows:

$$\alpha_1 = \min \Psi^+, \quad \text{and} \quad \alpha_p = \min(\Psi^+ \backslash \langle \alpha_1, \ldots, \alpha_{p-1} \rangle_{\mathbf{R}}) \quad \text{for } p = 2, 3, \ldots, k.$$

We call $\alpha_1, \alpha_2, \dots, \alpha_k$ the *refined basis* of L, which is nothing but the simple roots of the subsystem $\Psi = L \cap \Phi$ with respect to the lexicographical order.

We number the positive roots as $\Phi^+ = \{a_1, a_2, \dots, a_l\}$ with $a_1 < a_2 < \cdots < a_l$, where l denotes the number of positive roots of Φ. For $p = 1, 2, \dots, l-1$, we put $B(p) = \{j;\ a_j - a_p \notin \Phi^+ \text{ and } p < j \le l\}$. If k roots $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ consist with a refined basis, then we must necessarily have

$$i_p \in B(i_1) \cap \cdots \cap B(i_{p-1}) \quad \text{for } p = 2, 3, \dots, k. \tag{4.1}$$

Since each subspace has a unique refined basis, to classify subspaces generated by some roots in Φ, it is sufficient to classify the refined bases.

To do this, first we find out all k-tuples $I = (i_1, i_2, \dots, i_k)$ with $i_1 < i_2 < \cdots < i_k$ satisfying the condition (4.1). The number of such k-tuples is not so large as compared with the binomial coefficient $\binom{l}{k}$. Next we want to check whether the corresponding roots $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ consist with the refined basis of $\mathbf{L}(a_{i_1}, a_{i_2}, \dots, a_{i_k})$, or not. In fact, putting $R(I;p) = \{a \in L \cap \Phi^+;\ a < a_{i_p}\}$ for $p = 2, 3, \dots, k$, we have an easy criterion: For a k-tuple $I = (i_1, i_2, \dots, i_k)$, the corresponding roots $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ consist with a refined basis if and only if

$$\dim \langle R(I;p) \rangle_{\mathbf{R}} = p - 1 \quad \text{for } p = 2, 3, \dots, k. \tag{4.2}$$

We can easily check the conditions (4.1) and (4.2) by using computer. Thus we obtain the numbers for exceptional-type that are presented in Theorems 4.1 and 4.2. Note that our assertion for $\mathbf{G}_2$-type is trivial.

4.2. $\mathbf{E}_n$-type. The root system $\Phi(\mathbf{E}_6)$ of type $\mathbf{E}_6$ can be regarded as a finite subset of $V^{(6)}$:

$$\begin{aligned}\Phi(\mathbf{E}_6) = {} & \{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j \in V^{(6)};\ 1 \le i < j \le 5\} \\ & \cup \left\{ \pm \frac{1}{2} \left(\boldsymbol{e}_8 - \boldsymbol{e}_7 - \boldsymbol{e}_6 + \sum_{i=1}^{5} (-1)^{v_i} \boldsymbol{e}_i \right) \in V^{(6)};\ \sum_{i=1}^{5} v_i \text{ is even} \right\},\end{aligned}$$

where $V^{(6)}$ is the ortho-complement of the subspace spanned by ${}^t(0, \dots, 0, 1, 0, 1)$ and ${}^t(0, \dots, 0, 0, 1, 1) \in \mathbf{R}^8$; that is, $V^{(6)} = \{\boldsymbol{x} = (x_i)_{i=1}^8 \in \mathbf{R}^8;\ x_6 = x_7 = -x_8\}$.

The root system $\Phi(\mathbf{E}_7)$ of type $\mathbf{E}_7$ can be regarded as a finite subset of $V^{(7)}$:

$$\begin{aligned}\Phi(\mathbf{E}_7) = {} & \{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j \in V^{(7)};\ 1 \le i < j \le 6\} \cup \{\pm(\boldsymbol{e}_7 - \boldsymbol{e}_8)\} \\ & \cup \left\{ \pm \frac{1}{2} \left(\boldsymbol{e}_7 - \boldsymbol{e}_8 + \sum_{i=1}^{6} (-1)^{v_i} \boldsymbol{e}_i \right) \in V^{(7)};\ \sum_{i=1}^{6} v_i \text{ is odd} \right\},\end{aligned}$$

$e(n,k)$	$n=6$	$n=7$	$n=8$
$k=1$	36	63	120
2	390	1281	4900
3	1530	10395	85680
4	2001	33411	661542
5	639	36435	2091600
6	1	8821	2221780
7		1	440880
8			1

Table 4.1. The values of $e(n,k)$

$e^0(n,k)$	$n=6$	$n=7$	$n=8$
$k=1$	16	32	64
2	280	976	3808
3	1340	9200	76384
4	1920	31560	627536
5	638	35696	2053072
6	1	8814	2214856
7		1	440872
8			1

Table 4.2. The values of $e^0(n,k)$

where $V^{(7)}$ is the hyperplane of vectors orthogonal to ${}^t(0,\dots,0,1,1) \in \mathbf{R}^8$; that is, $V^{(7)} = \{\boldsymbol{x} = (x_i)_{i=1}^8 \in \mathbf{R}^8;\ x_7 = -x_8\}$.

The root system $\Phi(\mathbf{E}_8)$ of type $\mathbf{E}_8$ can be regarded as a finite subset of $\mathbf{R}^8$:

$$\Phi(\mathbf{E}_8) = \{\pm\boldsymbol{e}_i \pm \boldsymbol{e}_j \in \mathbf{R}^8;\ 1 \le i < j \le 8\}$$
$$\cup \left\{\frac{1}{2}\sum_{i=1}^{8}(-1)^{v_i}\boldsymbol{e}_i \in \mathbf{R}^8;\ \sum_{i=1}^{8} v_i \text{ is even}\right\}.$$

Theorem 4.1. *We have Table* 4.1 *for the number* $e(n,k)$ *of distinct k-dimensional subspaces generated by some roots in* $\Phi(\mathbf{E}_n)$ $(n = 6,7,8)$, *and also we have Table* 4.2 *for the number* $e^0(n,k)$ *of distinct k-dimensional sincere subspaces.*

4.3. $\mathbf{F}_4$ or $\mathbf{G}_2$-type. The root system $\Phi(\mathbf{F}_4)$ of type $\mathbf{F}_4$ can be regarded as a finite subset of $\mathbf{R}^4$:

	$f(k)$		$g(k)$
$k=1$	24	$k=1$	6
2	122	2	1
3	120		
4	1		

Table 4.3. The values of $f(k)$ and $g(k)$

	$f^0(k)$		$g^0(k)$
$k=1$	8	$k=1$	3
2	76	2	1
3	116		
4	1		

Table 4.4. The values of $f^0(k)$ and $g^0(k)$

$$\Phi(\mathbf{F}_4) = \{\pm \boldsymbol{e}_i \in \mathbf{R}^4;\ 1 \le i \le 4\} \cup \{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j \in \mathbf{R}^4;\ 1 \le i < j \le 4\}$$
$$\cup \left\{\frac{1}{2}(\pm \boldsymbol{e}_1 \pm \boldsymbol{e}_2 \pm \boldsymbol{e}_3 \pm \boldsymbol{e}_4)\right\}.$$

The root system $\Phi(\mathbf{G}_2)$ of type $\mathbf{G}_2$ can be regarded as a finite subset of $V^{(3)}$:

$$\Phi(\mathbf{G}_2) = \{\pm(\boldsymbol{e}_i - \boldsymbol{e}_j) \in V^{(3)};\ 1 \le i < j \le 3\}$$
$$\cup \{\pm(2\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3), \pm(2\boldsymbol{e}_2 - \boldsymbol{e}_1 - \boldsymbol{e}_3), \pm(2\boldsymbol{e}_3 - \boldsymbol{e}_1 - \boldsymbol{e}_2)\}.$$

where $V^{(3)}$ is the hyperplane of vectors orthogonal to ${}^t(1,1,1) \in \mathbf{R}^3$; that is, $V^{(3)} = \{\boldsymbol{x} = (x_i)_{i=1}^3 \in \mathbf{R}^3;\ x_1 + x_2 + x_3 = 0\}$.

THEOREM 4.2. *We have Table* 4.3 *for the number $f(k)$ (resp. $g(k)$) of distinct k-dimensional subspaces generated by some roots in $\Phi(\mathbf{F}_4)$ (resp. $\Phi(\mathbf{G}_2)$). We also have Table* 4.4 *for the number $f^0(k)$ and $g^0(k)$ of distinct k-dimensional sincere subspaces.*

References

[1] Comtet L., Advanced combinatorics, D. Reidel Publishing Co., Dordrecht, 1974.
[2] He, T.-X., Expression and computation of generalized Stirling numbers, J. Combin. Math. Combin. Comput. **86** (2013), 239–268.
[3] Howard, F. T., Associated Stirling numbers, Fibonacci Quart. **18** (1980), no. 4, 303–315.
[4] Hsu, L. and Shiue, P. J.-S., A unified approach to generalized Stirling numbers, Adv. in Appl. Math. **20** (1998), no. 3, 366–384.

[5] Kamiyoshi, T., Nagura, M. and Otani, S., Counting one-codimensional subspaces generated by subsets of a root system, Int. J. Algebra **5** (2011), no. 12, 591–604.
[6] Riordan, J., An Introduction to Combinatorial Analysis, Wiley, New York, 1958.
[7] Riordan, J., Combinatorial identities, Krieger, New York, 1979.
[8] Wall, H. S., On the *n*th derivative of $f(x)$, Bull. Amer. Math. Soc. **44** (1938), no. 6, 395–398.

Tomohiro Kamiyoshi
Department of Science
Matsue College
National Institute of Technology
Matsue, Shimane 690-8518, Japan
E-mail: kamiyoshi@matsue-ct.jp

Makoto Nagura
Department of Liberal Studies
Nara College
National Institute of Technology
Yamato-Koriyama, Nara 639-1080, Japan
E-mail: nagura@libe.nara-k.ac.jp

Shin-ichi Otani
School of Science and Engineering
Kanto-Gakuin University
Yokohama, Kanagawa 236-8501, Japan
E-mail: hocke@kanto-gakuin.ac.jp