TSUKUBA J. MATH.
Vol. 42 No. 1 (2018), 65–95

# CONSTANT NEGATIVE GAUSSIAN CURVATURE TORI AND THEIR SINGULARITIES

By

Tatsumasa Ura

**Abstract.** We construct constant negative Gaussian curvature tori with one family of planar curvature lines in Euclidean 3-space. We show that these tori are wave fronts. The singularities of these tori are studied.

## 1 Introduction

In this paper we construct constant negative Gaussian curvature tori with singularities in Euclidean 3-space $\mathbf{R}^3$.

In contrast, in 1986, Wente [14] discovered constant mean curvature tori. A constant mean curvature surface, away from its umbilics, possesses an isothermic coordinate system, that is, a conformal curvature line coordinate system $(x, y)$. Using such coordinates, the Gauss-Codazzi equation satisfies the sinh-Gordon equation

$$\phi_{xx} + \phi_{yy} + \sinh \phi = 0, \tag{1.1}$$

where $\phi$ is derived from the conformal factor of the first fundamental form $\mathrm{I} = (e^{\phi}/4)(\mathrm{d}x^2 + \mathrm{d}y^2)$. Wente showed the existence of constant mean curvature tori by analyzing the doubly periodic solutions of (1.1). After Wente's discovery, Abresch [1] found that if one assumes one family of curvature lines to be planar, then $\phi$ in (1.1) satisfies

$$\tanh \frac{\phi(x, y)}{4} = f(x) \cdot g(y), \tag{1.2}$$

which induces a separation of variables in (1.1). Then he obtained the doubly periodic solutions of (1.1) explicitly by using Jacobi's theta functions. In the same

2010 *Mathematics Subject Classification*: Primary 53A05; Secondary 57R45, 35L70.
*Key words and phrases*: constant negative Gaussian curvature torus, sine-Gordon equation, singularity.
Received February 23, 2018.
Revised July 12, 2018.

way, Walter [13] gave an explicit parametrization of constant mean curvature tori which have one family of planar curvature lines. Furthermore, Spruck [11] showed that the surfaces obtained by Abresch and Walter are indeed the same as the ones obtained by Wente.

For a constant mean curvature surface $p : M \to \mathbf{R}^3$, where $M$ is a 2-manifold, the parallel surface

$$p' = p - \frac{1}{2H}\nu : M \to \mathbf{R}^3 \tag{1.3}$$

gives a constant positive Gaussian curvature surface which possibly has singularities, where $H$ and $\nu$ are the mean curvature and the unit normal vector field of $p$, respectively. Kimura [7] studied the singularities of constant positive Gaussian curvature tori that are the parallel surfaces of constant mean curvature tori obtained by Abresch [1] and Walter [13].

On the other hand, for surfaces with constant negative Gaussian curvature, there exist curvature line coordinate systems $(x, y)$, and taking these coordinate systems, the Gauss-Codazzi equation satisfies the sine-Gordon equation

$$\phi_{xx} - \phi_{yy} - \sin\phi = 0, \tag{1.4}$$

where $\phi$ is the angle between the asymptotic directions. Surfaces with constant negative Gaussian curvature have been studied in Enneper [4], Inoguchi [6], Brander [2], Dorfmeister and Sterling [3], Goulart and Tenenblat [5], and others. In particular, Melko and Sterling [9] exhibits many such examples, including tori, by applying soliton theory.

In this paper, as an analogy to the works of Abresch and Walter, we construct constant negative Gaussian curvature tori with singularities which have one family of planar curvature lines. Even though (1.4) is a hyperbolic partial differential equation, while (1.1) is an elliptic partial differential equation, the argument and calculations parallel to [13] do work well for constant negative Gaussian curvature tori. We also apply the work of Saji, Umehara, and Yamada [10] to investigate the shape of the singularities for the tori we have constructed.

This paper is organized as follows. In Section 2, the local theory of constant negative Gaussian curvature surfaces is given. In Section 3, we study the properties of $\phi$ when $x$-curve is planar. When one family of curvature lines are plane curves, the other are spherical curves in general. In Section 4, we study the properties of the planes and spheres in which the curvature lines lie on. In Section 5, we give the parametrizations of the surfaces. For surfaces with singularities, a notion of (wave) fronts has been well-investigated in Saji, Umehara, and Yamada [10] and elsewhere. In Section 6, we show that the surfaces we construct here are indeed fronts. In Sections 7 and 8, we give the necessary and sufficient conditions for the surfaces to be tori. In Section 9, we give the explicit parametrizations of these tori in terms of trigonometric functions. In Section 10, we study the singularities of these tori.

## 2 Preliminaries

Here, we recall the local theory of constant negative Gaussian curvature surfaces. Let $D$ be a domain in $\mathbf{R}^2(x, y)$, and $p : D \to \mathbf{R}^3$ a $K$ surface, that is, an immersion with constant negative Gaussian curvature $-1$. It is known that a surface with negative Gaussian curvature always has a curvature line coordinate system. So throughout this paper, we assume $(x, y)$ is a curvature line coordinate. Let $\nu : D \to S^2$ be a unit normal vector field along $p$, then the first, second, and third fundamental forms $\mathrm{I} = \langle \mathrm{d}p, \mathrm{d}p \rangle$, $\mathrm{II} = -\langle \mathrm{d}p, \mathrm{d}\nu \rangle$, $\mathrm{III} = \langle \mathrm{d}\nu, \mathrm{d}\nu \rangle$ are given by

$$\mathrm{I} = \cos^2 \frac{\phi}{2}\, \mathrm{d}x^2 + \sin^2 \frac{\phi}{2}\, \mathrm{d}y^2, \tag{2.1}$$

$$\mathrm{II} = \frac{1}{2} \sin \phi (\mathrm{d}x^2 - \mathrm{d}y^2), \tag{2.2}$$

$$\mathrm{III} = \sin^2 \frac{\phi}{2}\, \mathrm{d}x^2 + \cos^2 \frac{\phi}{2}\, \mathrm{d}y^2, \tag{2.3}$$

where $\phi : D \to \mathbf{R}$ is the angle between asymptotic directions and $\langle\, ,\, \rangle$ is the standard metric of $\mathbf{R}^3$. The Gauss-Weingarten formulas are

$$p_{xx} = -\frac{\phi_x}{2} \tan \frac{\phi}{2} p_x + \frac{\phi_y}{2} \cot \frac{\phi}{2} p_y + \frac{1}{2} \sin \phi \nu, \tag{2.4}$$

$$p_{xy} = -\frac{\phi_y}{2} \tan \frac{\phi}{2} p_x + \frac{\phi_x}{2} \cot \frac{\phi}{2} p_y, \tag{2.5}$$

$$p_{yy} = -\frac{\phi_x}{2} \tan \frac{\phi}{2} p_x + \frac{\phi_y}{2} \cot \frac{\phi}{2} p_y - \frac{1}{2} \sin \phi \nu, \tag{2.6}$$

$$\nu_x = -\tan \frac{\phi}{2} p_x, \tag{2.7}$$

$$\nu_y = \cot \frac{\phi}{2} p_y. \tag{2.8}$$

Under this setting, the Codazzi equation is trivial and the Gauss equation is the sine-Gordon equation

$$\phi_{xx} - \phi_{yy} = \sin \phi. \tag{2.9}$$

For later use, we give the following equations which can be verified by direct computations from the Gauss-Weingarten formulas

$$\det(p_x, p_{xx}, p_{xxx}) = \frac{1 + \cos \phi}{16} \{-2\phi_{xy} \sin \phi + (1 + \cos \phi)\phi_x \phi_y\}, \tag{2.10}$$

$$\det(p_y, p_{yy}, p_{yyy}) = \frac{1 - \cos \phi}{16} \{2\phi_{xy} \sin \phi + (1 - \cos \phi)\phi_x \phi_y\}. \tag{2.11}$$

Therefore

$x$-curve is a plane curve if and only if $-2\phi_{xy} \sin \phi + (1 + \cos \phi)\phi_x \phi_y = 0$,

$y$-curve is a plane curve if and only if $2\phi_{xy} \sin \phi + (1 - \cos \phi)\phi_x \phi_y = 0$.

Throughout this paper, we assume that none of $\phi_x$, $\phi_y$ vanishes identically,

$$\phi_x \not\equiv 0, \quad \phi_y \not\equiv 0. \tag{2.12}$$

Remark 2.1. It is well-known that if $\phi : D \to \mathbf{R}$ in (2.9) depends only on $x$ or $y$, then $K$ surface $p : D \to \mathbf{R}^3$ is a surface of revolution.

For the global theory of constant negative Gaussian curvature surfaces, the following theorem is well-known.

Theorem 2.2 (Hilbert). *There is no isometric immersion from a* 2 *dimensional complete constant negative Gaussian curvature Riemannian manifold $M$ into* $\mathbf{R}^3$.

By this theorem, it is natural to consider the class of surfaces for which certain kinds of singularities occur, and to study global properties of constant negative Gaussian curvature surfaces within that class.

Definition 2.3. *Let $M$ be a smooth* 2*-manifold and $p : M \to \mathbf{R}^3$ a smooth map. $p$ is called a* (wave) front *if there exists $\nu : M \to S^2$ such that*

$$\langle v, \nu(q) \rangle = 0$$

*for any $q \in M$ and $v \in T_q M$, and*

$$\langle dp, dp \rangle + \langle d\nu, d\nu \rangle \tag{2.13}$$

*gives a positive definite metric on $M$. When $p : M \to \mathbf{R}^3$ is a front, we say $p$ is* complete *if* (2.13) *is complete.*

Definition 2.4. *Let $M$ be a smooth 2-manifold and $p : M \to \mathbf{R}^3$ a front. $p$ is called a* K front *if there exists an open dense subset $W$ of $M$ such that $p|_W : W \to \mathbf{R}^3$ is an immersion with constant negative Gaussian curvature $-1$.*

## 3 *K* Surfaces for Which *x* Curves are Planar

In this section, as an analogue of Section 2 in [13], we consider a $K$ surface with one family of planar curvature lines. Let $D$ be a domain in $\mathbf{R}^2(x, y)$ and $p : D \to \mathbf{R}^3$ a $K$ surface with curvature line coordinates $(x, y)$. From now on, we assume that the $x$ curves are planar. Then by (2.10), we have the following lemma.

Lemma 3.1. *If the $x$ curves are planar, namely* $\det(p_x, p_{xx}, p_{xxx}) = 0$, *then*

$$\phi_{xy} = \frac{\phi_x \phi_y}{2} \cot \frac{\phi}{2}. \tag{3.1}$$

Lemma 3.2. *The integral of* (3.1) *is*

$$\tan \frac{\phi}{4} = f(x) \cdot g(y), \tag{3.2}$$

*where $f$ and $g$ are functions of $x$ and $y$ alone, respectively.*

Here "′" and "˙" denote the derivatives with respect to $x$ and $y$ on $D$, respectively. Taking derivatives of (3.2), we have

$$\phi_x = \frac{4f'g}{1 + f^2 g^2}, \qquad \phi_y = \frac{4f\dot{g}}{1 + f^2 g^2}, \tag{3.3}$$

$$\phi_{xx} = \frac{4f''g}{1 + f^2 g^2} - \frac{8ff'^2 g^3}{(1 + f^2 g^2)^2}, \quad \phi_{yy} = \frac{4f\ddot{g}}{1 + f^2 g^2} - \frac{8f^3 g\dot{g}^2}{(1 + f^2 g^2)^2}. \tag{3.4}$$

Furthermore, we get

$$\sin \phi = \frac{4 \tan \frac{\phi}{4} \left(1 - \tan^2 \frac{\phi}{4}\right)}{\left(1 + \tan^2 \frac{\phi}{4}\right)^2} = \frac{4fg(1 - f^2 g^2)}{(1 + f^2 g^2)^2}. \tag{3.5}$$

Thus, substituting (3.4) and (3.5) into (2.9) yields

$$\frac{1}{f^2}A + \frac{f''}{f^3}B = C + D, \tag{3.6}$$

where

$$A = \frac{\ddot{g}}{g^3} + \frac{1}{g^2}, \qquad B = -\frac{1}{g^2},$$

$$C = \frac{f''}{f} - 2\frac{f'^2}{f^2} + \frac{1}{2}, \quad D = -\frac{\ddot{g}}{g} + 2\frac{\dot{g}^2}{g^2} + \frac{1}{2}.$$

Two further differentiations of (3.6) with respect to $x$ yield

$$\left(\frac{1}{f^2}\right)' A + \left(\frac{f''}{f^3}\right)' B = C', \tag{3.7}$$

$$\left(\frac{1}{f^2}\right)'' A + \left(\frac{f''}{f^3}\right)'' B = C''. \tag{3.8}$$

Consider (3.7), (3.8) as a system of linear equations for $A$, $B$. If its determinant would not vanish, then $A$, $B$ would be functions of $x$, in particular $g = \text{const}$, contradicting (2.12). Hence,

$$\left(\frac{1}{f^2}\right)' \left(\frac{f''}{f^3}\right)'' - \left(\frac{f''}{f^3}\right)' \left(\frac{1}{f^2}\right)'' = 0. \tag{3.9}$$

Integration by quadrature leads to $f'^2 = c_4 f^4 + c_2 f^2 + c_0$ with constant coefficients. The same procedure applies to $g$ and leads to $\dot{g}^2 = d_4 g^4 + d_2 g^2 + d_0$. Entering these in (2.9) yields relations between the constants which are expressed in

$$c_4 = -d_0, \quad d_4 = -c_0, \quad c_2 - d_2 - 1 = 0.$$

Thus we have proved the following theorem.

Theorem 3.3. *A function $\phi$ of the form* (3.2) *satisfies the Gauss equation* (2.9) *if and only if $f$ and $g$ satisfy elliptic differential equations of the form*

$$2f'^2 = -\beta f^4 + 2\lambda f^2 + \alpha, \tag{3.10}$$

$$2\dot{g}^2 = -\alpha g^4 + 2\mu g^2 + \beta, \tag{3.11}$$

*with real constants $\alpha$, $\beta$, $\lambda$, and $\mu$, where*

$$\lambda - \mu - 1 = 0. \tag{3.12}$$

Observe that (3.10) and (3.11) imply

$$\frac{f''}{f} = -\beta f^2 + \lambda, \quad \frac{\ddot{g}}{g} = -\alpha g^2 + \mu. \tag{3.13}$$

From (3.2), we can change $f$ and $g$ to $cf$ and $g/c$, respectively, for any $c \in \mathbf{R}\backslash\{0\}$. Taking a suitable choice of $c$ ($c = \sqrt[4]{\alpha/\beta}$), we may assume that $\alpha = \beta$ in (3.10) and (3.11). That is, $f$ and $g$ satisfy

$$2f'^2 = -\alpha f^4 + 2\lambda f^2 + \alpha, \tag{3.14}$$

$$2\dot{g}^2 = -\alpha g^4 + 2(\lambda - 1)g^2 + \alpha, \tag{3.15}$$

with real constants $\alpha$ and $\lambda$.

## 4 The Generating Planes

In this section, as an analogue of Section 3 in [13], we study the properties of the planes of the $x$-curves. We call these planes the "generating planes". Let $\mathfrak{L} : \mathbf{R} \to \mathbf{R}^3$ be normal vector fields of generating planes.

Proposition 4.1. *The family of generating planes has a common 1-dimensional subspace.*

Proof. $p_x \times p_{xx}$ is a normal vector of the generating plane. It would be better to use the multiple of this by $1/\langle p_x, p_x\rangle$.

$$\mathfrak{L} = (p_x \times p_{xx})\frac{1}{\cos^2\frac{\phi}{2}} = -p_y + \frac{\phi_y}{2}\nu. \tag{4.1}$$

Let $x_0 \in \mathbf{R}$ be a point satisfying $f(x_0) \neq 0$. The direction of $\mathfrak{L}$ is independent of $x$. We differentiate (4.1), and use (2.6), (2.8), (2.5), to obtain

$$\mathfrak{L}_y = \frac{\phi_x}{2}\tan\frac{\phi}{2}p_x + \frac{\phi_{xx}}{2}\nu, \tag{4.2}$$

$$\mathfrak{L}_{yy} = \frac{1}{4}\left(2\phi_{xy}\tan\frac{\phi}{2} + \phi_x\phi_y\right)p_x + \left(\frac{\phi_x^2}{4} + \frac{\phi_{xx}}{2}\cot\frac{\phi}{2}\right)p_y + \frac{1}{2}\phi_{xxy}\nu. \tag{4.3}$$

Moreover, the cross product of $\mathfrak{L}$ and $\mathfrak{L}_y$ is

$$\mathfrak{L} \times \mathfrak{L}_y = -\frac{\phi_{xx}}{2}\tan\frac{\phi}{2}p_x + \frac{\phi_x\phi_y}{4}p_y + \frac{\phi_x}{2}\sin^2\frac{\phi}{2}\nu. \tag{4.4}$$

Also, when we multiply $\phi_x$ to (3.1) and differentiate, we get

$$\phi_{xxy}\phi_x - \phi_{xx}\phi_{xy} + \frac{\phi_x^3\phi_y}{4} = 0. \tag{4.5}$$

Thus, we get $\det(\mathfrak{L}, \mathfrak{L}_y, \mathfrak{L}_{yy}) = 0$.

Now, if $g(y_0) \neq 0$, then, from (3.2), $\tan(\phi(x_0, y)/4) \neq 0$ in a neighborhood of $y_0$, so, from (4.1), (4.2), $\mathfrak{L}(x_0, y)$ and $\mathfrak{L}_y(x_0, y)$ are linearly independent. By $\det(\mathfrak{L}, \mathfrak{L}_y, \mathfrak{L}_{yy}) = 0$, the span of $\mathfrak{L}(x_0, y)$, $\mathfrak{L}_y(x_0, y)$ is independent of $y$. Hence, for all $y$, there is a fixed vector $e \in \mathbf{R}^3 \setminus \{0\}$ with unique direction such that $\langle \mathfrak{L}(x_0, y), e \rangle = 0$. This direction spanned by $e$ is the 1-dimensional subspace. □

Remark 4.2. Proposition 4.1 is an analogue of Proposition 3.A of [13], but for a constant mean curvature surface, the generating planes are just parallel to a fixed 1-dimensional subspace, so they have no common 1-dimensional subspace as claimed in Proposition 4.1.

Proposition 4.3. *The y-curves are spherical curves where the centers are on the* 1*-dimensional subspace of Proposition* 4.1. *We call these spheres the "generating spheres", and the* 1*-dimensional subspace of Proposition* 4.1 *the "axis" of surface.*

Proof. We look for a point $\mathfrak{l}$ common to the normal planes of a fixed $y$-curve. We set $\mathfrak{l}(x) = p + B(x, y)p_x + C(x, y)\nu$. From $\mathfrak{l}_y = 0$,

$$\mathfrak{l}_y = \left(-\frac{B\phi_y}{2}\tan\frac{\phi}{2} + B_y\right)p_x + \left(1 + \frac{B\phi_x}{2}\cot\frac{\phi}{2} + C\cot\frac{\phi}{2}\right)p_y + C_y\nu = 0. \tag{4.6}$$

Hence, we get $B = X_1(x)/\cos(\phi/2)$, $C = X_2(x)$ with suitable functions $X_1$ and $X_2$ of one variable. Substituting them into the second terms of the right hand side of (4.6), we have

$$\sin\frac{\phi}{2} + \frac{\phi_x}{2}X_1 + X_2\cos\frac{\phi}{2} = 0. \tag{4.7}$$

From (3.3), we obtain

$$(4.7) \Leftrightarrow 1 + \frac{f'}{f}X_1 + X_2\cot\frac{\phi}{2} = 0.$$

On the other hand, if $\phi$ depends on $y$, then $X_2 = 0$, and $X_1 = -f/f'$. Hence,

$$\mathfrak{l} = p - \frac{f}{f' \cos\frac{\phi}{2}} p_x. \tag{4.8}$$

For the function $\tau := \|\mathfrak{l} - p\|^2$, we deduce $\tau_y = 0$. So every $y$-curve lies on a sphere with center $\mathfrak{l}(x)$ and radius $R(x)$, where

$$R^2 = \|\mathfrak{l} - p\|^2 = \frac{f^2}{f'^2 \cos^2\frac{\phi}{2}} \cdot \cos^2\frac{\phi}{2} = \frac{f^2}{f'^2}. \tag{4.9}$$

Moreover, from (4.1), (4.2),

$$\langle \mathfrak{l}_x, \mathfrak{L} \rangle = \langle \mathfrak{l}_x, \mathfrak{L}_y \rangle = 0, \tag{4.10}$$

so, $\mathfrak{l}$ is on the straight line that is parallel to $\mathfrak{L} \times \mathfrak{L}_y$. We can easily verify $\langle \mathfrak{l} - p, \mathfrak{L} \rangle = 0$, so, $\mathfrak{l} - p$ is included in the generating plane when we fix $y$. The generating plane of $x$-curve depends only on $y$, and $\mathfrak{l}$ depends only on $x$, so, $\mathfrak{l}$ is included in the 1-dimensional subspace of Proposition 4.1. □

From now on, we set this 1-dimensional subspace to be the $x_3$-axis in $\mathbf{R}^3$.

## 5 Parametrization of the Surface

In this section, as an analogue of Section 4 in [13], we give a parametrization of the position vector $p$.

Proposition 5.1. *The unit normal vector field $\nu$ satisfies*

$$\nu_{xx} - \nu_{yy} = \cos\phi\nu. \tag{5.1}$$

Proof. By direct calculations from (2.7), (2.8), we have

$$\nu_{xx} = -\frac{\phi_x}{2} p_x - \frac{\phi_y}{2} p_y - \sin^2\frac{\phi}{2}\nu, \quad \nu_{yy} = -\frac{\phi_x}{2} p_x - \frac{\phi_y}{2} p_y - \cos^2\frac{\phi}{2}\nu.$$

This completes the proof. □

We set $p = (p_1, p_2, p_3)$, $\nu = (\nu_1, \nu_2, \nu_3)$, $\mathfrak{L} = (L_1, L_2, L_3)$. Then, by $L_3 = \langle \mathfrak{L}, e_3 \rangle = 0$ and (4.1), we have

$$0 = \langle \mathfrak{L}, e_3 \rangle = -p_{3y} + \frac{\phi_y}{2}\nu_3 = -\tan\frac{\phi}{2}\nu_{3y} + \frac{\phi_y}{2}\nu_3. \tag{5.2}$$

Observe that $v_3$ cannot vanish identically, otherwise, by (2.7) and (2.8), we would have $p_3 = \text{const}$. Combining (5.2) with (3.1) yields $v_{3y}/v_3 = \phi_{xy}/\phi_x$, and this can be integrated to

$$v_3 = B(x)\phi_x \tag{5.3}$$

with $B$ depending only on $x$. Also, we can easily calculate $v_{3xx} = B''\phi_x + 2B'\phi_{xx} + B\phi_{xxx}$, $v_{3yy} = B\phi_{xyy}$. Substituting these into (5.1), we have

$$B''\phi_x + 2B'\phi_{xx} + B(\phi_{xxx} - \phi_{xyy}) = B\phi_x \cos\phi. \tag{5.4}$$

By the sine-Gordon equation (2.9), we get $\phi_{xxx} - \phi_{xyy} = \phi_x \cos\phi$, so,

$$B''\phi_x + 2B'\phi_{xx} = 0, \tag{5.5}$$

which integrates to $B' = C(y)/\phi_x^2$. If $B'(x) \not\equiv 0$, i.e. $B$ is not constant, then $C(y) \not\equiv 0$ and $\phi_x^2 = C(y)/B'(x)$, hence

$$2\phi_x\phi_{xy} = \frac{\dot{C}(y)}{B'(x)},$$

thus $2\phi_{xy}/\phi_x = \dot{C}(y)/C(y)$. From this follows, by (3.1), $2\phi_{xy}/\phi_x = \phi_y \cot(\phi/2)$,

$$\left(\phi_y \cot\frac{\phi}{2}\right)_x = \left(\frac{2\phi_{xy}}{\phi_x}\right)_x = \left(\frac{\dot{C}(y)}{C(y)}\right)_x = 0,$$

hence,

$$\phi_{xy} \cot\frac{\phi}{2} - \frac{\phi_y}{\sin^2\frac{\phi}{2}}\frac{\phi_x}{2} = 0, \tag{5.6}$$

and using (3.1) again,

$$\phi_x\phi_y \equiv 0,$$

contradicting (2.12). Thus

$$B = B_0 \neq 0 : \text{const}, \quad v_3 = B_0\phi_x. \tag{5.7}$$

With this, (2.7) and (2.8) imply

$$p_{3x} = -B_0\phi_{xx} \cot\frac{\phi}{2}, \tag{5.8}$$

$$p_{3y} = B_0\phi_{xy} \tan\frac{\phi}{2}. \tag{5.9}$$

The right-hand sides can be explicitly expressed, using (3.2) and its second derivatives. In addition, it is useful to write the last quotient in (5.8) using

$(\phi_x \cot \phi/2)_x$. This results in:

$$p_{3x} = -2B_0 \left\{ \left( \frac{f'(1-f^2g^2)}{f(1+f^2g^2)} \right)_x + \frac{f'^2}{f^2} \right\} \tag{5.10}$$

$$= -2B_0 \left\{ \left( \frac{-2ff'g^2}{1+f^2g^2} \right)_x + \frac{f''}{f} \right\}, \tag{5.11}$$

$$p_{3y} = 8B_0 \frac{ff'g\dot{g}}{(1+f^2g^2)^2}. \tag{5.12}$$

Note that (5.10) and (5.11) are equivalent by $(f'/f)' = f''/f - f'^2/f^2$. Hence, we get

$$p_3 = -2B_0 \left\{ \frac{-2ff'g^2}{1+f^2g^2} + \int^x \frac{f''}{f}\,\mathrm{d}x \right\}. \tag{5.13}$$

In Section 7, we will see that $f$ is elliptic. So, there exists a zero $x_0$ of $f'$ where the sign of $f$ may be arranged such that $f_0 = f(x_0) > 0$.

From (4.9), the $y$-curve will be planar where $f' = 0$. First, we consider the plane $y$-curve corresponding $x = x_0$ and take its plane as the $(x_1, x_2)$-plane of the coordinate system in $\mathbf{R}^3$. The restrictions of $p, v, \ldots$ onto $x = x_0$ will be marked subsequently by a bar (i.e. $\bar{p}$, $\bar{v}$, and so on). We orient this plane by $\bar{v}$, $\bar{p}_y$ and find the norm $w$ of the tangent vector $\bar{p}_y$ and plane curvature $\kappa = \det(\bar{p}_y, \bar{p}_{yy})/w^3$ of the $y$-curve via

$$\det(\bar{p}_y, \bar{p}_{yy}) = \det\left( \bar{p}_y, \frac{\bar{\phi}_y}{2} \cot \frac{\bar{\phi}}{2} \bar{p}_y - \frac{1}{2} \sin \bar{\phi}\bar{v} \right) = -\frac{1}{2} \sin \bar{\phi} \det(\bar{p}_y, \bar{v}),$$

$\bar{p}_y/w$ and $\bar{v}$ are unit vectors and orthogonal each other, so, $\det(\bar{p}_y/w, \bar{v}) = -1$. Hence,

$$w = \|p_y\| = \sin \frac{\bar{\phi}}{2}, \quad \kappa = \frac{1}{w^2} \det\left( \frac{\bar{p}_y}{w}, \bar{v} \right) = \cot \frac{\bar{\phi}}{2}. \tag{5.14}$$

Let $s$ be an arc length parameter of the planar curve $\bar{p}$. There exists a function $\omega = \omega(s)$ so that $\mathrm{d}\bar{p}/\mathrm{d}s = (\cos \omega(s), \sin \omega(s))$, $\bar{v} = (-\sin \omega(s), \cos \omega(s))$. So, $\kappa = \det(\mathrm{d}\bar{p}/\mathrm{d}s, \mathrm{d}^2\bar{p}/\mathrm{d}s^2) = \mathrm{d}\omega/\mathrm{d}s$. Also, since $\mathrm{d}s/\mathrm{d}y = \|\bar{p}_y\| = w$, the angle change rate of $\bar{v}$ is given by

$$\frac{\mathrm{d}\omega}{\mathrm{d}y} = \frac{\mathrm{d}\omega}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}y} = \kappa w = \cos \frac{\bar{\phi}}{2}. \tag{5.15}$$

By (3.2),

$$\omega = \int_0^y \frac{1 - f_0^2 g^2}{1 + f_0^2 g^2}\, dy. \tag{5.16}$$

We set $\bar{\mathfrak{L}}^0 := \bar{\mathfrak{L}}/\|\bar{\mathfrak{L}}\|$, then there exists a function $\eta = \eta(y)$ such that it can be expressed as $\bar{\mathfrak{L}}^0 = (\cos\eta(y), \sin\eta(y), 0)$. Also, we denote by $\iota = \iota(y)$ the angle between $\bar{v}$ and $\bar{\mathfrak{L}}^0$. By definition,

$$\iota = \eta - \omega. \tag{5.17}$$

From (4.1) and $\cos\iota = \langle \bar{v}, \bar{\mathfrak{L}}/\|\bar{\mathfrak{L}}\| \rangle$,

$$\cos\iota = \frac{\dot{g}}{\sqrt{g^2 + \dot{g}^2}}, \quad \sin\iota = \varepsilon_1 \frac{g}{\sqrt{g^2 + \dot{g}^2}}, \tag{5.18}$$

where $\varepsilon_1 = \pm 1$. The three angles, $\omega$, $\iota$, and $\eta$ are completely determined by (5.16), (5.18), (5.17), and an initial condition for $\omega$. The coordinate representations of $\bar{v}$ and the unit normal vectors $\bar{\mathfrak{L}}^0$, $\bar{\mathfrak{L}}^{0\perp}$, positively proportional to $\bar{\mathfrak{L}}$, $\bar{\mathfrak{L}}^\perp$, are

$$\bar{v} = \begin{pmatrix} \cos\omega \\ \sin\omega \\ 0 \end{pmatrix}, \quad \bar{\mathfrak{L}}^0 = \begin{pmatrix} \cos\eta \\ \sin\eta \\ 0 \end{pmatrix}, \quad \bar{\mathfrak{L}}^{0\perp} = \begin{pmatrix} -\sin\eta \\ \cos\eta \\ 0 \end{pmatrix}, \tag{5.19}$$

where the operator $\perp$ meaning rotation in the positive sense by $\pi/2$. We also determine the constant $B_0$ in (5.7). From (5.11),

$$\overline{p_3}_x = -2B_0 \left\{ \frac{f_0''}{f_0} - \frac{2f_0 f_0'' g^2}{1 + f_0^2 g^2} \right\}. \tag{5.20}$$

Also, $\bar{p}_x = (0, 0, \bar{p}_{3_x})$, so by $\|\bar{p}_x\|^2 = \overline{p_3}_x^2$ and (5.20),

$$\frac{1 - f_0^2 g^2}{1 + f_0^2 g^2} = 4B_0^2 \left\{ \frac{f_0''}{f_0} - \frac{2f_0 f_0'' g^2}{1 + f_0^2 g^2} \right\}^2. \tag{5.21}$$

Since $\dot{g} \not\equiv 0$, this is an identity in $g$, and setting $g = 0$ gives the following equation, using (3.14),

$$B_0^2 = \frac{1}{4(\lambda^2 + \alpha^2)}. \tag{5.22}$$

We determine the first and second components $p_1$ and $p_2$. We denote the plane $\Pi(c)$ by

$$\Pi(c) = \{(x_1, x_2, c) \in \mathbf{R}^3 \mid x_1, x_2 \in \mathbf{R}\}. \tag{5.23}$$

We fix the generating plane and the generating sphere, and set the intersection between $\Pi(p_3)$ and generating plane as

$$z = Z\bar{\mathfrak{L}}^{0\perp} + p_3 e_3, \quad Z \in \mathbf{R}.$$

The point $z$ is on the generating sphere, so $z$ satisfies $\|z - \mathfrak{l}\|^2 = R^2$, and $\mathfrak{l} = (0, 0, \ell_3)$,

$$Z^2 - R^2 + (p_3 - \ell_3)^2 = 0. \tag{5.24}$$

Hence

$$Z = \pm\sqrt{R^2 - (p_3 - \ell_3)^2}. \tag{5.25}$$

The calculation for the part inside the root is, from (4.8), (4.9), (3.2), and (5.10),

$$\begin{aligned} R^2 - (p_3 - \ell_3)^2 &= \frac{f^2}{f'^2} - \frac{f^2}{f'^2 \cos^2 \frac{\phi}{2}} {p_3}_x^2 \\ &= \frac{4}{\lambda^2 + \alpha^2} \frac{f^2(\alpha + 2\lambda g^2 - \alpha g^4)}{(1 + f^2 g^2)^2}. \end{aligned}$$

So, (5.25) takes the form

$$Z = \varepsilon_2 W, \tag{5.26}$$

where $\varepsilon_2 = \pm 1$ and

$$W := \frac{2}{\sqrt{\lambda^2 + \alpha^2}} \frac{f\sqrt{\alpha + 2\lambda g^2 - \alpha g^4}}{1 + f^2 g^2}. \tag{5.27}$$

We thus have the following theorem:

Theorem 5.2. *If there is a zero $x_0$ of $f'$ with $f(x_0) > 0$, then the position vector function $p : \mathbf{R}^2 \to \mathbf{R}^3$ with planar $x$-curves can be brought to the explicit form*

$$p = Z\mathfrak{L}^{0\perp} + p_3 e_3, \tag{5.28}$$

*where $Z$, $\mathfrak{L}^{0\perp}$, $p_3$ are given by* (5.19), (5.26), (5.27), *and* (5.13).

By (5.26) and (5.28), we may set $\varepsilon_2 = -1$.

Remark 5.3. The turning angle $\eta$ of the generating planes is determined by (5.16), (5.18), (5.17). We have also proven a direct formula for $\eta$ which is generally available from (4.1), (4.2). In the setting for $f'(x_0) = 0$, $f_0 := f(x_0)$, $f_0'' := f''(x_0)$, described above, the result is

$$\frac{d\eta}{dy} = -\frac{f_0''}{f_0} \sin^2 \iota. \tag{5.29}$$

In fact, by (4.1), we have

$$\mathfrak{L}^{\perp} = -\frac{\phi_y}{2 \sin \frac{\phi}{2}} p_y - \sin \frac{\phi}{2} \nu. \tag{5.30}$$

Also, we have

$$\frac{d\eta}{dy} = \langle (\bar{\mathfrak{L}}^0)_y, \bar{\mathfrak{L}}^{0\perp} \rangle. \tag{5.31}$$

Hence, we get (5.29) from (5.19).

## 6 Fronts

Now we extend the domain $D$ to the entire $(x, y)$-plane $\mathbf{R}^2$. The purpose of this section is to show that this extension $p : \mathbf{R}^2 \to \mathbf{R}^3$ in (5.28) gives a front. By (5.26), (5.27),

$$Z_x = Z \frac{f'}{f} \cos \frac{\phi}{2}, \tag{6.1}$$

$$Z_y = Z \frac{\dot{g} U}{f(\alpha + 2\lambda g^2 - \alpha g^4)} \sin \frac{\phi}{2}, \tag{6.2}$$

where

$$U = U(x, y) = \lambda - \lambda f^2 g^2 - \alpha f^2 - \alpha g^2.$$

Also, by (5.17), (5.18),

$$Z\dot{\eta} = \frac{Z(1 + f^2 g^2)}{2f} \frac{g\overline{U}}{(1 + f_0^2)(g^2 + \dot{g}^2)} \sin \frac{\phi}{2}. \tag{6.3}$$

Hence, by (5.10), (5.12), (5.28),

$$\langle p_x, p_x \rangle = Z_x^2 + p_{3x}^2 = \left( Z^2 \frac{f'^2}{f^2} + B_0^2 \phi_{xx}^2 \frac{1}{\sin^2 \frac{\phi}{2}} \right) \cos^2 \frac{\phi}{2}, \tag{6.4}$$

$$\langle p_y, p_y \rangle = Z_y^2 + Z^2\dot{\eta}^2 + p_{3y}^2$$
$$= \left( \frac{Z^2(1+f^2g^2)^2}{4f^2} \frac{g^2\overline{U}^2}{(1+f_0^2)^2(g^2+\dot{g}^2)^2} \right.$$
$$\left. + Z^2 \frac{\dot{g}^2 U^2}{f^2(\alpha + 2\lambda g^2 - \alpha g^4)^2} + B_0^2 \phi_{xy}^2 \frac{1}{\cos^2 \frac{\phi}{2}} \right) \sin^2 \frac{\phi}{2}. \tag{6.5}$$

From (3.4), we see $|\phi_{xx}/\sin(\phi/2)| < \infty$, and also $|\phi_{xy}/\cos(\phi/2)| < \infty$ as well. Hence, we set

$$e_1 := \frac{p_x}{\cos \frac{\phi}{2}}, \quad e_2 := \frac{p_y}{\sin \frac{\phi}{2}}, \tag{6.6}$$

and then $e_1$, $e_2$ are smooth vector fields which are defined on $\mathbf{R}^2$, and $\langle e_i, e_j \rangle = \delta_{ij}$. Thus, we establish

$$\nu = e_1 \times e_2, \tag{6.7}$$

and then $\nu$ is defined on $\mathbf{R}^2$, and $\langle w, \nu \rangle = 0$ for all tangent vectors $w$. Since

$$\nu_x = -\sin \frac{\phi}{2} e_1, \quad \nu_y = \cos \frac{\phi}{2} e_2, \tag{6.8}$$

we have

$$p_x = 0 \Rightarrow \nu_x \neq 0, \quad p_y = 0 \Rightarrow \nu_y \neq 0, \tag{6.9}$$

therefore $p$ is a front.

By the above arguments, hereafter we consider the domain of $p$ to be the entire $(x, y)$-plane $\mathbf{R}^2$, and $p : \mathbf{R}^2 \to \mathbf{R}^3$ to be a front. Note that $\sin \phi \nu$ with $\nu$ in (6.7) coincides with $\sin \phi \nu$ defined in the beginning of Section 2 whenever $p$ is an immersion. Hence, using this $\nu$ in (6.7), the Gauss-Weingarten formulas (2.4)–(2.8) can be extended to the entire $\mathbf{R}^2$.

## 7 Necessary Conditions for Double Periodicity of the Position Vector Function

As analogues of Section 5 in [13], we give here necessary conditions for the position vector $p$ to become doubly periodic. The symbols $\mathrm{cn}_k$ and $\mathrm{sn}_k$ denote the cosine and sine amplitude of Jacobi with modulus $k \in (0, 1)$.

Proposition 7.1. *If* $p : \mathbf{R}^2 \to \mathbf{R}^3$ *in* (5.28) *is doubly periodic with respect to some fundamental parallelogram* $\mathscr{D} \subset \mathbf{R}^2$, *then*:

(1) $p : \mathbf{R}^2 \to \mathbf{R}^3$ *is in fact doubly periodic with respect to* $x$ *and* $y$ (*so* $f$ *and* $g$ *are both periodic*).
(2) $f$ *has at least one zero.*
(3) $\alpha$ *in* (3.14) *is positive.*

Proof. (1) Let $\tilde{\pi}$ denote the canonical projection from $\mathbf{R}^2$ to the torus $\mathscr{T} = \mathbf{R}^2/\mathscr{D}$ and $\tilde{p} : \mathscr{T} \to \mathbf{R}^3$ the induced immersion from $\tilde{\pi}$, so $p = \tilde{p} \circ \tilde{\pi}$. First, we show that $p$ is periodic in $x$. In fact, any mapping $x \mapsto \tilde{\pi}(x, y_0)$ has either a dense image in $\mathscr{T}$ or is periodic with a common period for all $y_0 \in \mathbf{R}$. If it is dense, by $\overline{\tilde{\pi}(\mathbf{R} \times y_0)} = \mathscr{T}$,

$$\tilde{p}(\mathscr{T}) = \tilde{p}(\overline{\tilde{\pi}(\mathbf{R} \times y_0)}) \subset \overline{\tilde{p}(\tilde{\pi}(\mathbf{R} \times y_0))} = \overline{p(\mathbf{R} \times y_0)}.$$

Since $p(\mathbf{R} \times y_0)$ is assumed to be contained in a generating plane $E_0$, we would have $\tilde{p}(\mathscr{T}) \subset E_0$. But this is not possible because the Gaussian curvature of $E_0$ is 0. Thus, $p$ is periodic in $x$. The same argument applies with respect to $y$. From the periodicity of $p$ in $x$ and $y$, we obtain the periodicity of $f$ and $g$ using (3.2).

(2) Assume $f$ has no zero. Then the function

$$-\frac{1}{2B_0} p_3 - \frac{f'(1 - f^2 g^2)}{f(1 + f^2 g^2)}$$

is $C^\omega$ and periodic in $x$ and, by (5.10), has nonnegative derivative $f'^2/f^2$ for $x$. This is only possible if $f' \equiv 0$, which contradicts (2.12).

(3) Evaluation of (3.14) at a zero $x_1$ of $f$ gives $\alpha \geq 0$, and if we have $\alpha = 0$ then $f(x_1) = f'(x_1) = 0$, hence $f \equiv 0$. Thus $\alpha > 0$. □

From periodicity of $f$ and $g$, we obtain $\alpha > 0$. This follows from the next proposition.

Proposition 7.2. *The set of solutions of* (3.14) *and* (3.15) *with* $\alpha > 0$ *can be described by*

$$f(x) = \gamma \operatorname{cn}_k(ax), \quad g(y) = \bar{\gamma} \operatorname{cn}_{\bar{k}}(\bar{a}y), \tag{7.1}$$

*where* $0 < k < 1$, $0 < \bar{k} < 1$, *and constants* $\gamma$, $\bar{\gamma}$, $a$, $\bar{a}$, $k$, $\bar{k}$ *satisfy*

$$\gamma^2 = \frac{\lambda}{\alpha} + \sqrt{\frac{\lambda^2}{\alpha^2} + 1}, \quad \bar{\gamma}^2 = \frac{\lambda - 1}{\alpha} + \sqrt{\frac{(\lambda - 1)^2}{\alpha^2} + 1}, \tag{7.2}$$

$$a^2 = \frac{\alpha}{2}\left(\gamma^2 + \frac{1}{\gamma^2}\right), \quad \bar{a}^2 = \frac{\alpha}{2}\left(\bar{\gamma}^2 + \frac{1}{\bar{\gamma}^2}\right), \tag{7.3}$$

$$k^2 = \frac{\gamma^4}{\gamma^4 + 1}, \qquad \bar{k}^2 = \frac{\bar{\gamma}^4}{\bar{\gamma}^4 + 1}. \tag{7.4}$$

*Moreover, if we set* $k = \sin\theta$, $\bar{k} = \sin\bar{\theta}$, *then*

$$\theta - \bar{\theta} < \frac{\pi}{2}, \tag{7.5}$$

$$\gamma^2 = \tan\theta, \quad \bar{\gamma}^2 = \tan\bar{\theta}, \tag{7.6}$$

$$a^2 = \frac{\sin 2\bar{\theta}}{\sin 2(\theta - \bar{\theta})}, \quad \bar{a}^2 = \frac{\sin 2\theta}{\sin 2(\theta - \bar{\theta})}. \tag{7.7}$$

Proof. Since $f$ and $g$ satisfy (3.14) and (3.15) with $\alpha > 0$ (see Proposition 7.1), it is known that $f$ and $g$ can be described by (7.1) for some constants $\gamma$, $\bar{\gamma}$, $a$, $\bar{a}$, $k$, $\bar{k}$. In fact, $f$ and $g$ in (7.1) satisfy

$$f'^2 = -\frac{a^2k^2}{\gamma^2}f^4 + a^2(2k^2 - 1)f^2 + a^2\gamma^2,$$

$$\dot{g}^2 = -\frac{\bar{a}^2\bar{k}^2}{\bar{\gamma}^2}g^4 + \bar{a}^2(2\bar{k}^2 - 1)g^2 + \bar{a}^2\bar{\gamma}^2.$$

Then by (3.14), we have the evaluations in the left-hand sides of (7.2)–(7.4). Inversely, given $(a, \gamma, k)$ with (7.2), (7.3), (7.4), there exists $(\alpha, \lambda)$ with $\alpha > 0$, namely

$$\lambda = a^2\frac{\gamma^4 - 1}{\gamma^4 + 1}, \quad \alpha = \frac{2a^2\gamma^2}{\gamma^4 + 1}. \tag{7.8}$$

Similar equations are available for $\bar{a}$, $\bar{\gamma}$, $\bar{k}$. Since $\alpha$ is common to both equations, and by observing (3.12), we have two further couplings.

$$a^2kk' = \bar{a}^2\bar{k}\overline{k'}, \tag{7.9}$$

$$a^2(k^2 - k'^2) - \bar{a}^2(\bar{k}^2 - \overline{k'}^2) - 1 = 0, \tag{7.10}$$

where, as usual,

$$k' := \sqrt{1 - k^2}, \quad \overline{k'} := \sqrt{1 - \bar{k}^2}. \tag{7.11}$$

The system (7.9), (7.10) can be uniquely solved for $a^2$, $\bar{a}^2$, which produces (7.6), (7.7). The positiveness of $a^2$, $\bar{a}^2$ is just expressed by (7.5). □

Together with the corresponding equations for $\bar{a}$, $\bar{\gamma}$, $\bar{k}$ we obtain in addition

$$\lambda = -\frac{\cos 2\theta \sin 2\bar{\theta}}{\sin 2(\theta - \bar{\theta})}, \quad \mu = -\frac{\sin 2\theta \cos 2\bar{\theta}}{\sin 2(\theta - \bar{\theta})}, \tag{7.12}$$

$$\alpha = \frac{\sin 2\theta \sin 2\bar{\theta}}{\sin 2(\theta - \bar{\theta})}. \tag{7.13}$$

PROPOSITION 7.3. *If* $\alpha > 0$, *then* $\varepsilon_1 = -1$.

PROOF. By (4.1), $\bar{\mathfrak{L}}^0$ is a negative multiple of $\bar{p}_y$, so $\angle(\bar{v}, \bar{\mathfrak{L}}) \equiv -\pi/2 \pmod{2\pi}$, hence from the continuity of $\iota$, $\varepsilon_1 = -1$. □

To verify the necessary condition that $p_3$ is periodic, from (5.13) we see that only $f''/f$ has to be handled. We use following equation:

$$\int \mathrm{sn}_k^2\, v \,\mathrm{d}v = \frac{1}{k^2}\left[\left(1 - \frac{\mathbf{E}}{\mathbf{K}}\right)v - \frac{1}{2\mathbf{K}}\frac{\vartheta_4'\left(\frac{v}{2\mathbf{K}}\right)}{\vartheta_4\left(\frac{v}{2\mathbf{K}}\right)}\right], \tag{7.14}$$

where $\mathbf{K}$, $\mathbf{E}$ are the first and second complete elliptic integrals with modulus $k$,

$$\mathbf{K}(k) = \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad \mathbf{E}(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi}\, \mathrm{d}\varphi, \tag{7.15}$$

and $\vartheta_4$ is one of the elliptic $\vartheta$ functions,

$$\vartheta_4(v) = q_0 \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2\pi v + 2^{4n-2}),$$

$$q_0 = \prod_{n=1}^{\infty} (1 - q^{2n}), \quad q = e^{-\pi \mathbf{K}'/\mathbf{K}}. \tag{7.16}$$

See [12, p. 257]. Inserting this into (5.13) gives

$$p_3 = 2B_0 \left\{ \frac{2ff'g^2}{1+f^2g^2} - a^2\left(1 - 2\frac{\mathbf{E}}{\mathbf{K}}\right)x + \frac{a}{\mathbf{K}} \frac{\vartheta_4'\left(\frac{ax}{2\mathbf{K}}\right)}{\vartheta_4\left(\frac{ax}{2\mathbf{K}}\right)} \right\} + \text{const.} \tag{7.17}$$

## 8 Necessary and Sufficient Conditions for Double Periodicity of the Position Vector Function

In this section, as an analogue of Section 6 in [13], we consider the closedness conditions for the position vector $p$. In Proposition 7.1, we saw that the condition $\alpha > 0$ is necessary for the double periodicity of the position vector function $p$. In Proposition 7.2, we saw that the condition $\alpha > 0$ is sufficient for the periodicity of the first fundamental form. In order to establish the necessary and sufficient conditions for the double periodicity of $p$, it only remains to discuss the periodicity behavior of the functions $p_3$ and $\omega$. This will be done here, assuming of course $\alpha > 0$.

In the case of $p_3$, the periodicity is fulfilled if and only if

$$\frac{\mathbf{K}}{\mathbf{E}} = 2. \tag{8.1}$$

Since $\mathbf{K}(k)/\mathbf{E}(k) \in (1, \infty)$ is strictly increasing, there is exactly one modulus $k$ satisfying (8.1). Numeric computation gives

$$\theta \approx 65.354\,955\,354^\circ \approx 1.140\,659\,153, \tag{8.2}$$

$$k = \sin\theta \approx 0.908\,908\,557\,55. \tag{8.3}$$

As for $\omega$, $p$ will become periodic in the parameter $y$ when a repeated period increment of $\omega$ is a multiple of $2\pi$. The reason is that only $\cos\omega$, $\sin\omega$, $\cos(\omega+\iota)$, $\sin(\omega+\iota)$ enter into the representation of $p$ in (5.28), and $\cos\iota$, $\sin\iota$ are expressed by $g$ in (5.18). The period increment $\omega_0$ of $\omega$ can be calculated from (5.16).

$$\omega_0 = 4\int_0^{\bar{\mathbf{K}}/\bar{a}} \frac{1 - \gamma^2\bar{\gamma}^2\,\mathrm{cn}^2_{\bar{k}}(\bar{a}y)}{1 + \gamma^2\bar{\gamma}^2\,\mathrm{cn}^2_{\bar{k}}(\bar{a}y)}\,\mathrm{d}y$$

$$= \frac{4}{\bar{a}}\int_0^{\pi/2} \frac{1 - \gamma^2\bar{\gamma}^2\cos^2\varphi}{1 + \gamma^2\bar{\gamma}^2\cos^2\varphi}\,\frac{\mathrm{d}\varphi}{\sqrt{1 - \bar{k}^2\sin^2\varphi}},$$

where $\overline{\mathbf{K}} = \mathbf{K}(\bar{k})$. For $\omega$ to be increased after $n$ periods of $y$ to $2\pi \cdot \ell$ ($\ell$, $n$ are integers), we must have $-n \cdot \omega_0 = \ell \cdot 2\pi$, which is equivalent to

$$\int_0^{\pi/2} \frac{1 - \tan\theta \tan\bar{\theta} \cos^2\varphi}{1 + \tan\theta \tan\bar{\theta} \cos^2\varphi} \frac{d\varphi}{\sqrt{1 - \sin^2\bar{\theta} \sin^2\varphi}} = \frac{\ell}{n} \frac{\pi}{2} \sqrt{\frac{\sin 2\theta}{\sin 2(\theta - \bar{\theta})}}. \tag{8.4}$$

In equation (8.4), $\theta$ is known by (8.2) and we have, for given $\ell/n$, to solve it for $\bar{\theta}$. We set

$$J(\bar{\theta}) := \int_0^{\pi/2} \frac{1 - \tan\theta \tan\bar{\theta} \cos^2\varphi}{1 + \tan\theta \tan\bar{\theta} \cos^2\varphi} \frac{d\varphi}{\sqrt{1 - \sin^2\bar{\theta} \sin^2\varphi}}, \quad S(\bar{\theta}) := \sqrt{\frac{\sin 2\theta}{\sin 2(\theta - \bar{\theta})}},$$

and we consider the quotient

$$\xi(\bar{\theta}) := J(\bar{\theta})/S(\bar{\theta}), \tag{8.5}$$

which is $C^\omega$ on $0 < \bar{\theta} < \theta$. Clearly, $\xi(0) = \pi/2$, and an elementary calculation shows

$$\xi'(0) = -\frac{\pi}{2}(\tan\theta + \cot 2\theta) < 0. \tag{8.6}$$

Lemma 8.1. *For any $\ell/n \in (0,1)$, there is exactly one solution $\bar{\theta} \in (0, \theta)$ of* (8.4), *and for any $\ell/n \notin (0,1)$, there is no solution $\bar{\theta} \in (0, \theta)$ of* (8.4).

Proof. From (5.17) and (5.18),

$$\frac{1}{4}\omega_0 = \int_0^{\overline{\mathbf{K}}/\bar{a}} \dot{\omega}\, dy = \int_0^{\overline{\mathbf{K}}/\bar{a}} \dot{\eta}\, dy - \int_0^{\overline{\mathbf{K}}/\bar{a}} \dot{\imath}\, dy = \int_0^{\overline{\mathbf{K}}/\bar{a}} \dot{\eta}\, dy - \frac{\pi}{2}. \tag{8.7}$$

Note that $\dot{\imath} = d\imath/dy$, not the ninth Latin alphabet. By $-n \cdot \omega_0 = \ell \cdot 2\pi$,

$$\int_0^{\overline{\mathbf{K}}/\bar{a}} \dot{\eta}\, dy = \left(-\frac{\ell}{n} + 1\right)\frac{\pi}{2}. \tag{8.8}$$

Also, from (5.29) and (5.18), we get

$$\dot{\eta} = a^2 \sin^2 \imath = a^2 \frac{g^2}{\dot{g}^2 + g^2} = \frac{a^2}{T^2 + 1}, \quad T = \frac{\dot{g}}{g}. \tag{8.9}$$

Obviously, $T$ is elliptic, so it must satisfy a corresponding differential equation

$$\dot{T}^2 = T^4 - 2\mu T^2 + \bar{a}^4. \tag{8.10}$$

Moreover, $T$ is strictly decreasing on $[0, \bar{\mathbf{K}}/\bar{a}]$ from 0 to $-\infty$, and thus can replace $y$ as parameter on this interval.

From this and (8.9),

$$\int_0^{\bar{\mathbf{K}}/\bar{a}} \dot{\eta}\, \mathrm{d}y = -\int_0^{-\infty} \frac{1}{\sqrt{(1/a^4)T^4 - 2(\mu/a^4)T^2 + (\bar{a}/a)^4}} \frac{\mathrm{d}T}{T^2+1}. \tag{8.11}$$

By (7.7) and (7.12), the integrand of the right hand side depends only on $\bar{\theta}$. It is easy to verify that

$$\sqrt{(1/a^4)T^4 - 2(\mu/a^4)T^2 + (\bar{a}/a)^4}$$

has limits $\infty$ when $\bar{\theta} \to 0$, and 1 when $\bar{\theta} \to \theta$ respectively, and is monotonically decreasing in $\bar{\theta} \in (0, \theta)$. Thus

$$0 < \int_0^{\bar{\mathbf{K}}/\bar{a}} \dot{\eta}\, \mathrm{d}y < -\int_0^{-\infty} \frac{\mathrm{d}T}{T^2+1} = \frac{\pi}{2}. \tag{8.12}$$

Hence, exactly for $\ell/n \in (0,1)$, (8.8) has a solution $\bar{\theta} \in (0, \theta)$, which is then unique. □

## 9 Expression of the Surface by Trigonometric Functions

To draw the graphics of the surfaces, as an analogue of (6.B′) in [13], here we change variables so that the surfaces are described by trigonometric functions.

We change variables as follows.

$$\mathrm{cn}_k(ax) = \cos u, \quad \mathrm{sn}_k(ax) = \sin u, \tag{9.1}$$

$$\mathrm{cn}_{\bar{k}}(\bar{a}y) = \cos v, \quad \mathrm{sn}_{\bar{k}}(\bar{a}y) = \sin v. \tag{9.2}$$

We set $(\ell, n)$ to be integers which satisfy $\ell/n \in (0,1)$. Then, we get the expression $p : \mathbf{R}^2 \to \mathbf{R}^3$.

$$p(u,v) = Z(u,v) \cdot \begin{pmatrix} \cos(\omega(v) - j(v)) \\ \sin(\omega(v) - j(v)) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ p_3(u,v) \end{pmatrix}, \tag{9.3}$$

where $\Gamma := \gamma\bar{\gamma}$, and

$$Z(u,v) = \frac{-2}{\sqrt{\lambda^2+\alpha^2}} \frac{\gamma \cos u \sqrt{\alpha + 2\lambda\bar{\gamma}^2 \cos^2 v - \alpha\bar{\gamma}^4 \cos^4 v}}{1 + \Gamma^2 \cos^2 u \cos^2 v}, \tag{9.4}$$

$$\omega(v) = \frac{1}{\bar{a}} \int_0^v \frac{1 - \Gamma^2 \cos^2 t}{1 + \Gamma^2 \cos^2 t} \frac{\mathrm{d}t}{\sqrt{1 - \bar{k}^2 \sin^2 t}}, \tag{9.5}$$

$$\tan j(v) = \bar{a} \tan v \cdot \sqrt{1 - \bar{k}^2 \sin^2 v}, \tag{9.6}$$

$$p_3(u,v) = \frac{1}{a}\left\{ 2\Gamma^2 \frac{\cos u \sin u \cos^2 v \sqrt{1-k^2\sin^2 u}}{1+\Gamma^2\cos^2 u \cos^2 v} - \int_0^u \frac{1-2k^2\sin^2 t}{\sqrt{1-k^2\sin^2 t}}\,\mathrm{d}t \right\}, \tag{9.7}$$

where $k$ and $\theta$ are given by (8.2), (8.3), respectively, and $\bar{k}$ is the solution of (8.4), and other coefficients are given by (7.5)–(7.7), (7.12), and (7.13). Also, $j(v) = -\imath(v) - \pi/2$.

Figures 1–5 show the graphics of the surfaces for several values of $\ell/n$. In each figure, the left hand side is the whole surface ($0 \le u \le 2\pi$, $0 \le v \le n\pi$), and the right hand side is its intersection with the $(x_1, x_2)$-plane.

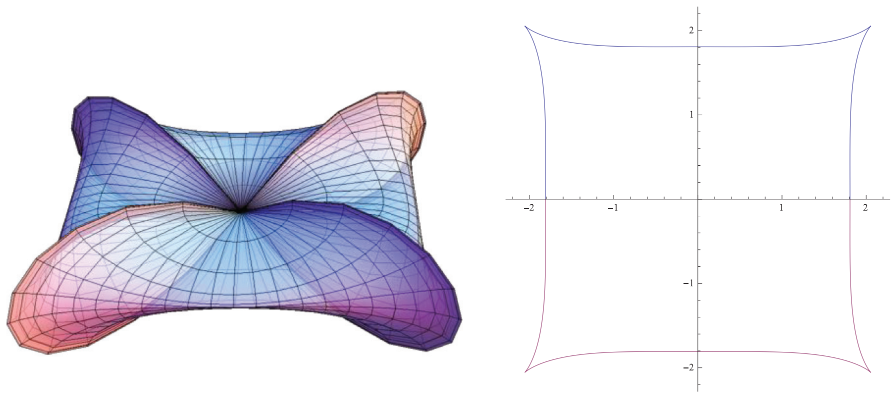



Figure 1: $\ell/n = 1/2$.

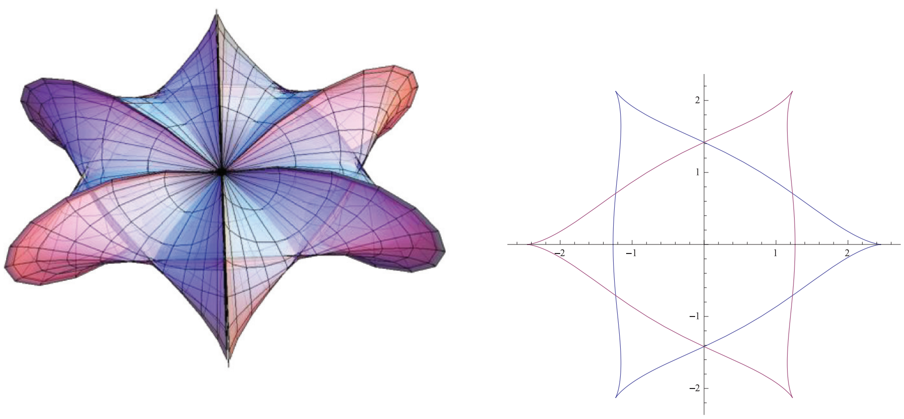



Figure 2: $\ell/n = 1/3$.

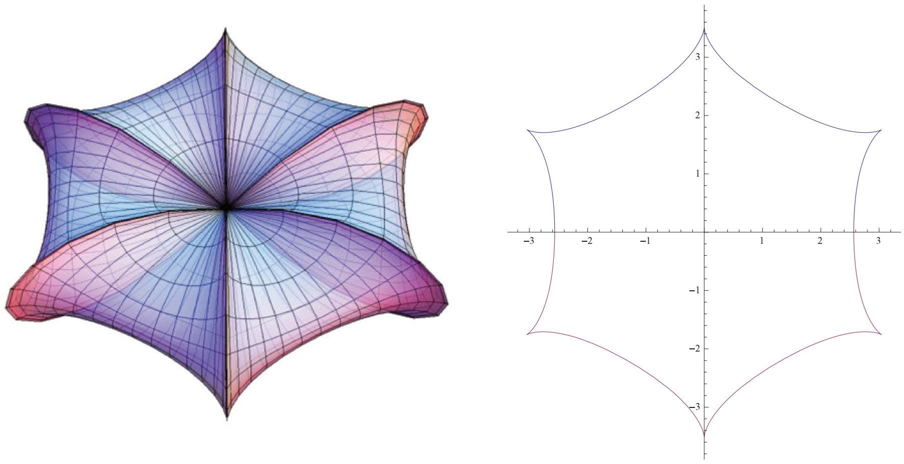


Figure 3: $\ell/n = 2/3$.

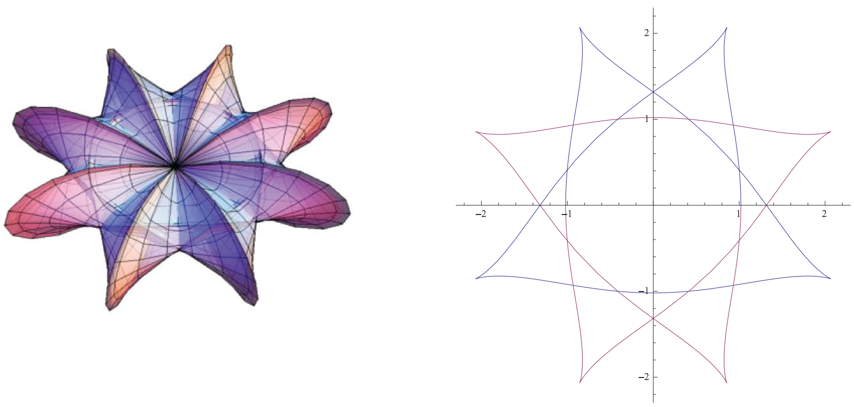


Figure 4: $\ell/n = 1/4$.

## 10 Singularities

In this section we study the singularities of the $K$ fronts given in (9.3). We call a point where $p$ is not an immersion a singularity of $p$. Moreover we consider

$$\Lambda(u, v) := \det(p_u, p_v, \nu) : \mathbf{R}^2 \to \mathbf{R}, \tag{10.1}$$

where $\nu : \mathbf{R}^2 \to S^2$ is defined in (6.7). Then, $q \in \mathbf{R}^2$ is a singular point if and only if $\Lambda = 0$ at $q$.

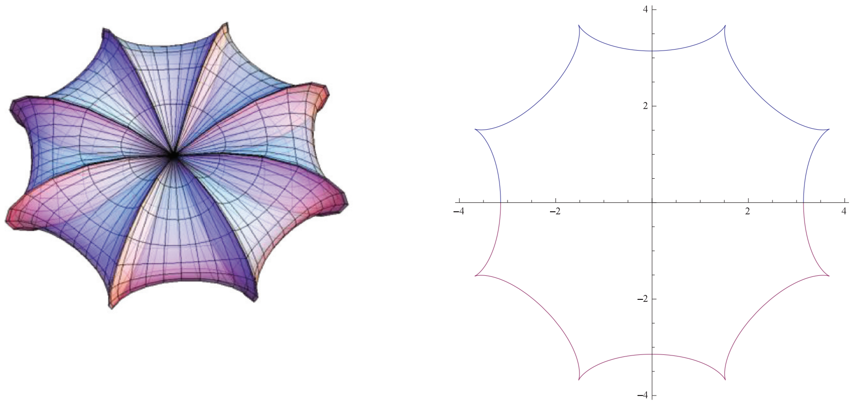



Figure 5: $\ell/n = 3/4$.

The singularity $q \in \mathbf{R}^2$ of a wave front $p : \mathbf{R}^2 \to \mathbf{R}^3$ is called non-degenerate if the differential $d\Lambda = \Lambda_u\, du + \Lambda_v\, dv$ of $\Lambda(u,v)$ does not vanish at $q$. We see that a connected component of the non-degenerate singular set is a regular curve on $\mathbf{R}^2$, by the implicit function theorem. We call this curve a singular curve. Furthermore, the direction of its tangent vector is called a singular direction. In addition, the direction of $\xi \in T_q\mathbf{R}^2$ $(\xi \neq 0)$ with $dp(\xi) = 0$ is called a null direction.

Definition 10.1 (cuspidal edges and swallowtails).

(1) *The map* $p_c : \mathbf{R}^2 \ni (u,v) \mapsto (u^2, u^3, v)$ *has singularities on the* $v$*-axis. We call the singularity* $q \in D$ *of a wave front* $p : D \to \mathbf{R}^3$ *a cuspidal edge if there exist* $U_1 \subset \mathbf{R}^2$ *with* $0 \in U_1$, $U_2 \subset D$ *with* $q \in U_2$, $\varphi : U_1 \to U_2$, $\Omega_1 \subset \mathbf{R}^3$ *with* $p_c(0) \in \Omega_1$, $\Omega_2 \subset \mathbf{R}^3$ *with* $p(q) \in \Omega_2$ *and* $\Phi : \Omega_1 \to \Omega_2$ *such that* $\varphi$ *and* $\Phi$ *are diffeomorphisms in* $\mathbf{R}^2$ *and* $\mathbf{R}^3$, *respectively, and* $\varphi(0) = q$ *and*

$$\Phi \circ p_c = p \circ \varphi. \tag{10.2}$$

(2) *The map* $p_s : \mathbf{R}^2 \ni (u,v) \mapsto (3u^4 + u^2v, 2u^3 + uv, v)$ *has singularities on the set* $\{(u,v) \in \mathbf{R}^2 \mid v = -6u^2\}$. *We call the singularity* $q \in D$ *of a wave front* $p : D \to \mathbf{R}^3$ *a swallowtail if there exist* $U_1 \subset \mathbf{R}^2$ *with* $0 \in U_1$, $U_2 \subset D$ *with* $q \in U_2$, $\varphi : U_1 \to U_2$, $\Omega_1 \subset \mathbf{R}^3$ *with* $p_s(0) \in \Omega_1$, $\Omega_2 \subset \mathbf{R}^3$ *with* $p(q) \in \Omega_2$ *and* $\Phi : \Omega_1 \to \Omega_2$ *such that* $\varphi$ *and* $\Phi$ *are diffeomorphisms in* $\mathbf{R}^2$ *and* $\mathbf{R}^3$, *respectively, and* $\varphi(0) = q$ *and*

$$\Phi \circ p_s = p \circ \varphi. \tag{10.3}$$

Proposition 10.2. *Let $I \subset \mathbf{R}$ be an interval. Let $D \subset \mathbf{R}^2$ be a domain. Suppose that $\gamma : I \to D$ is a (non-degenerate) singular curve of a wave front $p : D \to \mathbf{R}^3$.*

(1) *$\gamma(t_0)$ is a cuspidal edge if and only if its singular direction and null direction are linearly independent at $\gamma(t_0)$.*
(2) *$\gamma(t_0)$ is a swallowtail if and only if its singular direction $\dot\gamma(t_0)$ and null direction $\xi(t_0)$ are linearly dependent at $\gamma(t_0)$, and*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \det(\dot\gamma(t), \xi(t)) \neq 0.$$

Remark 10.3. The singularity set of $p_s$ is $\{(u,v) \in \mathbf{R}^2 \mid v = -6u^2\}$. $(u,v) = (0,0)$ is the swallowtail, and others are cuspidal edges.

We denote the set of singularities of $p$ by $\Sigma$.

Lemma 10.4. *The singular set $\Sigma$ of the K fronts $p$ defined by* (9.3)–(9.7) *is written as follows*:

$$\Sigma = \left\{ (u,v) \in \mathbf{R}^2 \;\middle|\; \cos u \cos v = 0,\ \cos u \cos v = \pm\frac{1}{\Gamma} \right\}. \tag{10.4}$$

Proof. By (6.6) and (10.1), we have

$$\begin{aligned} \Lambda(u,v) &= \det\left( p_u, p_v, \frac{1}{\cos\frac{\phi}{2}\sin\frac{\phi}{2}} p_u \times p_v \right) \\ &= \frac{2\Gamma \cos u \cos v (1 - \Gamma^2 \cos^2 u \cos^2 v)}{(1 + \Gamma^2 \cos^2 u \cos^2 v)^2}. \end{aligned}$$

Hence, $\Lambda = 0$ when $\Gamma \cos u \cos v = 0, \pm 1$. □

We set $\Sigma^c = \{(u,v) \in \mathbf{R}^2 \mid \cos u \cos v = c\}$, then

$$\Sigma = \Sigma^0 \cup \Sigma^{1/\Gamma} \cup \Sigma^{-1/\Gamma}, \tag{10.5}$$

and we have $p_v = 0$ on $\Sigma^0$, $p_u = 0$ on $\Sigma^{\pm 1/\Gamma}$. Also, we have $\Sigma^{\pm 1/\Gamma} = \varnothing$ if and only if $\Gamma < 1$. By definition of $\Sigma^{\pm 1/\Gamma}$ and (7.6), we have the following lemma.

Lemma 10.5. *$\Sigma^{\pm 1/\Gamma} = \varnothing$ if and only if $\bar\theta$ satisfies*

$$\tan\bar\theta < \cot\theta. \tag{10.6}$$

Remark 10.6. By Lemma 10.5, (8.1), and (8.2), we can find the condition of $\bar{\theta}$ for $\Sigma^{\pm 1/\Gamma} = \varnothing$. Then we can find the condition of $\ell$ and $n$ in (8.4) for $\Sigma^{\pm 1/\Gamma} = \varnothing$, since $\xi(\bar{\theta}) = (\pi/2)(\ell/n)$ in (8.5) is monotonically decreasing in $\bar{\theta} \in (0, \theta)$. Numerically, $\Sigma^{\pm 1/\Gamma} = \varnothing$ if and only if

$$\frac{\ell}{n} > 0.509896\cdots. \tag{10.7}$$

Lemma 10.7. *If $\Gamma > 1$, then the singularities on $\Sigma^{1/\Gamma}$ are non-degenerate, and when $u = n\pi$, singularities are swallowtails, and the others are cuspidal edges (see Figure 7).*

Proof. We set $\zeta(u, v) = \cos u \cos v - 1/\Gamma$, and then $\Sigma^{1/\Gamma} = \{(u, v) \in \mathbf{R}^2 \mid \zeta(u, v) = 0\}$. At each point on $\Sigma^{1/\Gamma}$, we have

$$\zeta_u \, du + \zeta_u \, dv \neq 0, \tag{10.8}$$

so, we have

$$\Lambda_u \, du + \Lambda_u \, dv \neq 0 \tag{10.9}$$

on $\Sigma^{1/\Gamma}$. Hence, at each point on $\Sigma^{1/\Gamma}$ is a non-degenerate singularity. The null direction is $\xi = (1, 0) = \partial/\partial u$, and the singular direction on $(u, v) \in \Sigma^{1/\Gamma}$ is $\zeta_v \cdot \partial/\partial u - \zeta_u \cdot \partial/\partial v$. We set $F := \det(\dot{\gamma}, \xi)$, and then

$$F = \det\begin{pmatrix} \zeta_v & 1 \\ -\zeta_u & 0 \end{pmatrix} = \zeta_u.$$

If $\zeta = 0$ and $F \neq 0$, then $(u, v)$ is a cuspidal edge, and if $\zeta = 0$, $F = 0$, and $dF \neq 0$, then $(u, v)$ is a swallowtail. So, since

$(u, v)$ is a cuspidal edge if and only if $\cos u \cos v = \dfrac{1}{\Gamma}$ and $\sin u \neq 0$,

$(u, v)$ is a swallowtail if and only if $\sin u = 0$,

we get the conclusion. The same argument works for $\Sigma^{-1/\Gamma}$ as well. □

Remark 10.8. By (2.1), we have

$$p_u \neq 0, \quad p_v = 0, \tag{10.10}$$

on $\Sigma^0$, and

$$p_u = 0, \quad p_v \neq 0 \tag{10.11}$$

on $\Sigma^{\pm 1/\Gamma}$.

We define

$$\gamma_m(t) := \left(\left(\frac{1}{2}+m\right)\pi, t\right), \quad \tilde{\gamma}_{\tilde{m}}(t) := \left(t, \left(\frac{1}{2}+\tilde{m}\right)\pi\right), \quad m, \tilde{m} \in \mathbf{Z} \tag{10.12}$$

Also, we set

$$\Sigma_1^0 = \{\gamma_m(t) \mid t \in \mathbf{R}\}_{m\in\mathbf{Z}}, \quad \Sigma_2^0 = \{\tilde{\gamma}_{\tilde{m}}(t) \mid t \in \mathbf{R}\}_{\tilde{m}\in\mathbf{Z}}. \tag{10.13}$$

Then,

$$\Sigma^0 = \Sigma_1^0 \cup \Sigma_2^0.$$

See Figure 6.

Lemma 10.9. *The singularities degenerate on*

$$\Sigma_1^0 \cap \Sigma_2^0 = \left\{ \left(\left(\frac{1}{2}+m\right)\pi, \left(\frac{1}{2}+\tilde{m}\right)\pi\right) \in \mathbf{R}^2 \,\middle|\, m\, \tilde{m} \in \mathbf{Z}\right\},$$

*where* $\Sigma_1^0$ *and* $\Sigma_2^0$ *are defined in* (10.13).

Proof. For any $t \in \mathbf{R}$,

$$\frac{\partial^i \zeta}{\partial v^i} = 0, \quad i \in \mathbf{N}, \tag{10.14}$$

on $\Sigma_1^0$, since $\zeta(\gamma_m(t)) = 0$. On the other hand, for any $t \in \mathbf{R}$,

$$\frac{\partial^i \zeta}{\partial u^i} = 0, \quad i \in \mathbf{N}, \tag{10.15}$$

on $\Sigma_2^0$, since $\zeta(\tilde{\gamma}_m(t)) = 0$. Hence,

$$\zeta_u = \zeta_v = 0 \tag{10.16}$$

on $\Sigma_1^0 \cap \Sigma_2^0$. Namely, the singularity degenerates on $\Sigma_1^0 \cap \Sigma_2^0$. □

Lemma 10.10. *The singularities are non-degenerate on* $\Sigma^0 \backslash (\Sigma_1^0 \cap \Sigma_2^0)$.

Proof. We have $\zeta_u = 0$ on $\Sigma_2^0$. Moreover, $\sin^2(\phi/2) = 0$ on the singularities. Hence, $\zeta_v = 0$ is equivalent to $\phi_y = 0$. This is equivalent to $f(x) \cdot \dot{g}(y) = 0$ by (3.2). Now, using $f = \gamma \cos u$, $g = \overline{\gamma} \cos v$,

$$\dot{g}(y) = \frac{\mathrm{d}g}{\mathrm{d}v} \cdot \frac{\mathrm{d}v}{\mathrm{d}y} = -\frac{\mathrm{d}v}{\mathrm{d}y} \overline{\gamma} \sin v.$$

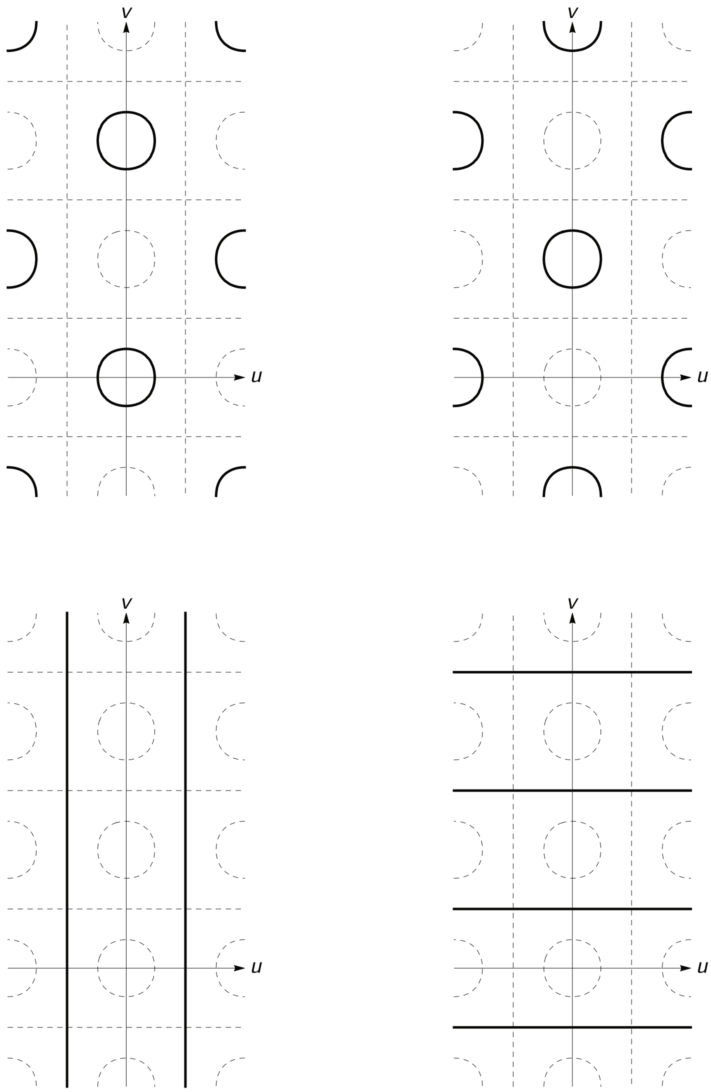



Figure 6: The singular set $\Sigma$ of $p$ for $\ell/n = 1/4$. The thick curves or the thick lines indicate $\Sigma^{1/\Gamma}$ (upper-left), $\Sigma^{-1/\Gamma}$ (upper-right), $\Sigma_1^0$ (lower-left), $\Sigma_2^0$ (lower-right).

Thus, if $\zeta_v = 0$ we see $\sin u \cos v = 0$, since $dv/dy \neq 0$. Since $\sin v = \pm 1$ on $\Sigma_1^0$, if $\zeta_v = 0$ we have $\cos u = 0$. Thus, the singularities are non-degenerate on $\Sigma_2^0 \setminus \Sigma_1^0$. Similarly, we see that the singularities are also non-degenerate on $\Sigma_1^0 \setminus \Sigma_2^0$. $\square$

Proposition 10.11. *The singularities on* $\Sigma_2^0 \setminus \Sigma_1^0$ *form cuspidal edges.*

Proof. The singularities on $\Sigma_2^0 \backslash \Sigma_1^0$ are non-degenerate by Lemma 10.10. Moreover, the singular direction is $\tilde{\gamma}' = (1,0) = \partial/\partial u$ by (10.12), and null direction is $\xi(t) = (0,1) = \partial/\partial v$ by (10.10). Hence, the singular direction and null direction are linearly independent. Therefore, the singularities on $\Sigma_2^0 \backslash \Sigma_1^0$ are cuspidal edges, by Proposition 10.2. □

Proposition 10.12. *The image of $p$ on $\Sigma_1^0$ is a single point* (*see Figure* 8).

Proof. By Proposition 4.3, a $y$-curve of $p$ lies on the sphere with center I and radius $R = |f/f'|$. Since $f = \gamma \cos u$,

$$f' = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = -\gamma \frac{du}{dx} \sin u.$$

Hence,

$$R^2 = \left( \frac{\cos u}{-\frac{du}{dx} \sin u} \right)^2. \tag{10.17}$$

Thus, if $u = (1/2 + m)\pi$, we see that $R^2 = 0$. Therefore, the image of $p$ in $\{((1/2 + m)\pi, t) \mid t \in \mathbf{R}\}$ is a single point for each $m \in \mathbf{Z}$.

Next, we will show that this single point does not depend on $m$. By (9.4), (9.7),

$$Z(u + 2\pi, v) = Z(u, v) = Z(-u, v), \quad p_3(u + 2\pi, v) = p_3(u, v) = p_3(-u, v). \tag{10.18}$$

Thus, we see that

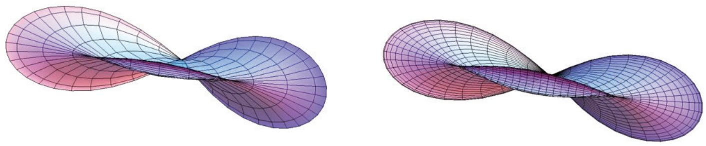
$$\cdots = p\left(-\frac{3}{2}\pi, v\right) = p\left(-\frac{1}{2}\pi, v\right) = p\left(\frac{1}{2}\pi, v\right) = p\left(\frac{3}{2}\pi, v\right) = \cdots.$$

Hence we get the conclusion. □

We have seen that the singularities are degenerate on $\Sigma_1^0 \cap \Sigma_2^0$ in Lemma 10.9. Therefore, we cannot apply any known criteria of singularities on $\Sigma_1^0 \cap \Sigma_2^0$. In addition, the shape of the singularities on $\Sigma_1^0 \cap \Sigma_2^0$ look like cone-like singularities, but nearby horizontal slices of the surfaces give curves that have a finite number of cusp points (see Figure 8).

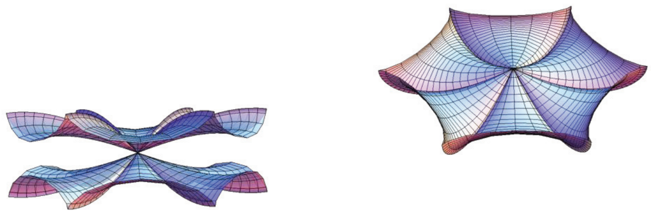
**Acknowledgement**

The author would like to thank to Professors Shoichi Fujimori, Jun-Ichi Inoguchi, Udo Hertrich-Jeromin, and the referee for their valuable comments.



$\ell/n = 1/3$ $\ell/n = 1/4$

Figure 7: $-\pi/2 \leq u < \pi/2,\ \pi/2 < v < 3\pi/2$



$\pi/4 \leq u \leq 3\pi/4$ $\pi/2 \leq u \leq \pi$

Figure 8: $\ell/n = 2/3,\ 0 \leq v < 6\pi$

## References


[1] Abresch, U., Constant Mean Curvature Tori in Terms of Elliptic Functions, J. Reine Angew. Math. (1985), 169–192.

[2] Brander, D., Pseudospherical surfaces with singularities, Ann. Mat. Pura Appl. (4) **196** (2017), no. 3, 905–928.

[3] Dorfmeister, J. F. and Sterling, I., Pseudospherical surfaces of low differentiability, Adv. Geom. **16** (2016), no. 1, 1–20.

[4] Enneper, A., Untersuchungen über die Flächen mit planen und sphärischen Krümmungslinien, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen 23 1878 and 26 1880.

[5] Goulart, C. and Tenenblat, K., On Bäcklund and Ribaucour transformations for surfaces with constant negative curvature, Geom. Dedicata **181** (2016), 83–102.

[6] Inoguchi, J., Constant negative Gaussian curvature surfaces and Sine-Gordon equation (in Japanese), Saitama Lecture Notes 1, 2012, Saitama University.

[7] Kimura, F., Constant positive Gaussian curvature tori, preprint.

[8] Kokubu, M., Rossman, W., Saji, K., Umehara, M., Yamada, K., Singularities of flat fronts in hyperbolic space, Pacific J. Math. **221** (2005), 303–351.

[9] Melko, M., Sterling, I., Application of Soliton Theory to the Construction of Pseudospherical Surfaces in $\mathbf{R}^3$, Ann. Global Anal. Geom. **11** (1993), 65–107.

[10] Saji, K., Umehara, M., Yamada, K., The geometry of fronts, Ann. of Math. (2). **169** (2009), 491–529.

[11] Spruck, J., The elliptic sinh-Gordon equation and the construction of toroidal soap bubbles, Lect. Notes in Math. **1340**, Springer Berlin (1988), 275–301.

[12] Tricomi, F., Funzioni Ellittiche, Bologna, Società Tipografica già Compositori, 1937.

[13] Walter, R., Explicit examples to the H-problem of Heinz Hopf, Geometriae Dedicata. **23** (1987), 187–213.

[14] Wente, H. C., Counterexample to a Conjecture of H. Hopf, Pacific J. Math. **121** (1986), 193–243.

Fuka Secondary School
P.O. Box 188
Sanya Juu
Siha
Kilimanjaro
Tanzania
Home Address:
Tsurajimacho Tsurushinden 1848-9
Kurashiki 712-8006
Japan
E-mail: p59n59we@s.okayama-u.ac.jp