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HYPOELLIPTICITY AND LOCAL SOLVABILITY OF PSEUDOLOCAL CONTINUOUS LINEAR OPERATORS IN GEVREY CLASSES

By

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Abstract. In this paper we extend a well-known result concerning hypoellipticity and local solvability of linear partial differential operators on Schwartz distributions (see [14] and [19]) to the framework of pseudolocal continuous linear maps T acting on Gevrey classes. Namely we prove that the Gevrey hypoellipticity of T implies the Gevrey local solvability of the transposed operator ${}^tT$. As an application, we identify some classes of non-Gevrey-hypoelliptic operators. A fundamental kernel is also constructed for any Gevrey hypoelliptic partial differential operator.

1. Introduction

We are concerned with hypoellipticity and local solvability of continuous linear maps in Gevrey classes. Let us start by recalling that for a real number $s \geq 1$ and an open set $\Omega \subset \boldsymbol{R}^n$, the space $G^s(\Omega)$ of the *Gevrey functions of order s* is defined to be the class of all functions $f \in C^\infty(\Omega)$ such that for any compact set $K \subset \Omega$ there is a constant $C = C(K) > 0$ for which the following estimates are fulfilled:

$$\max_{x \in K} |\partial^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s, \quad \alpha \in \boldsymbol{Z}_+^n. \tag{1.1}$$

Moreover, for $s > 1$, $G_0^s(\Omega) := G^s(\Omega) \cap C_0^\infty(\Omega)$ is the space of the Gevrey functions of order s with compact support in Ω. We provide the Gevrey classes $G_0^s(\Omega)$ and $G^s(\Omega)$ with the following usual locally convex topologies

$$G_0^s(\Omega) = \mathrm{indlim}_{K \subset\subset \Omega}(\mathrm{indlim}_{\eta \to 0}\ G_0^s(K;\eta)),$$

$$G^s(\Omega) = \mathrm{projlim}_{K \subset\subset \Omega}(\mathrm{indlim}_{\eta \to 0}\ G^s(K;\eta)).$$

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For any compact set $K \subset \Omega$ and every positive number η, $G^s(K;\eta)$ is the Banach space of all functions $f \in C^\infty(K)$, the space of the Whitney infinitely differentiable functions in K (see [17] and [18]), such that $\|f\|_{K,\eta} := \sup_{\alpha \in Z_+^n} \eta^{|\alpha|}(\alpha!)^{-s} \max_{x \in K}|D^\alpha f(x)|$ is finite; $G_0^s(K;\eta) := G^s(K;\eta) \cap C_0^\infty(K)$ and indlim$_*$, projlim$_*$ (here $*$ stands for either $K \subset\subset \Omega$ or $\eta \to 0$) means that we are taking the inductive or projective limit of the preceding Banach spaces as K varies over the family of all compact subsets of Ω and η over the set of all positive real numbers (see [4] and the references there for a definition of the projective and inductive limit topologies in an abstract functional setting and [5] for an exhaustive description of Gevrey spaces from the topological point of view). Lastly we define the spaces of *s-ultradistributions* $\mathscr{D}_s'(\Omega)$ and $\mathscr{E}_s'(\Omega)$, for $s > 1$, as the strong duals of the locally convex spaces $G_0^s(\Omega)$ and $G^s(\Omega)$ respectively, namely $\mathscr{D}_s'(\Omega) := (G_0^s(\Omega))'$ and $\mathscr{E}_s'(\Omega) := (G^s(\Omega))'$. Let $P = P(x,D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ be a linear partial differential operator, where $D_j = -i\partial_j$ $(j = 1,\dots,n)$, $D^\alpha := D^{\alpha_1} \cdots D^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ for any multi-index $\alpha = (\alpha_1,\dots,\alpha_n)$ and $m \in N$. Let us suppose the coefficients $a_\alpha(x)$ of P belong to the Gevrey class $G^s(\Omega)$; then P continuously maps every one of the spaces $G^s(\Omega)$, $G_0^s(\Omega)$, $\mathscr{D}_s'(\Omega)$ and $\mathscr{E}_s'(\Omega)$ into itself (cf. [5], [13]). Under the above assumptions, we say that P is *s-hypoelliptic* in Ω if $s - \text{singsupp}\, u = s - \text{singsupp}(Pu)$ for $u \in \mathscr{D}_s'(\Omega)$. Let us recall that for an ultradistribution $u \in \mathscr{D}_s'(\Omega)$ the *s-singular support* of u, denoted by $s - \text{singsupp}\, u$, is the smallest closed subset of Ω in the complement of which u is a G^s function. P is said to be *s-locally solvable* at a point $x_0 \in \Omega$ if there is an open neighbourhood $U \subset \Omega$ of x_0 such that for any $f \in G_0^s(U)$ there is a s-ultradistribution $u \in \mathscr{D}_s'(U)$ solving the equation $Pu = f$ in U. Moreover P is said to be *s-locally solvable* in Ω if it is s-locally solvable at any point $x_0 \in \Omega$. In [1] it is proved that if P is s-hypoelliptic in Ω then its transposed operator ${}^tP$ is s-locally solvable in Ω. This result extends to the framework of the Gevrey classes a well-known result for partial differential operators in the C^∞ case (cf. [14] and [19]). The same has been recently obtained by Wakabayashi in the spaces of the hyperfunctions (see [16]). In §2 of this paper we state an analogous property concerning s-hypoellipticity and s-local solvability of a *s-pseudolocal* linear continuous operator from $G_0^s(\Omega)$ into $G^s(\Omega)$ which extends to a linear continuous operator from $\mathscr{E}_s'(\Omega)$ into $\mathscr{D}_s'(\Omega)$; the definitions of s-pseudolocality, s-hypoellipticity and s-local solvability in this context will be precised in the next section. In §3 we restrict ourselves to s-hypoelliptic partial differential operators for which, following [14], we will prove the existence of a fundamental kernel. The arguments used are closely related to those of [1] and [14], nevertheless we will give a self contained exposition of the matter. In §4 we apply the result of § 2 as a necessary condition of s-hypoellipticity. In this way we seek some classes of non s-hypoelliptic differential and pseudodifferential operators.

2. Hypoellipticity and Local Solvability of Linear Continuous Pseudolocal Operators

Throughout this work we denote by $\mathscr{L}(E,F)$ the space of all continuous linear operators from a locally convex space E into another locally convex space F; moreover we write E' for the strong dual of the locally convex space E. Let us recall also that given an operator $T \in \mathscr{L}(E,F)$ the *transposed operator* ${}^tT$ is defined as follows:

$$\langle {}^tTu, v\rangle := \langle u, Tv\rangle, \quad u \in F', v \in E,$$

where $\langle .,. \rangle$ is used to denote the duality between any locally convex space E and its strong dual E'. It turns out that ${}^tT \in \mathscr{L}(F', E')$.

Hereafter we fix the attention on a continuous linear operator $T : G_0^s(\Omega) \to G^s(\Omega)$, extending to a continuous linear operator $T : \mathscr{E}_s'(\Omega) \to \mathscr{D}_s'(\Omega)$, $s > 1$, i.e. $T \in \mathscr{L}(G_0^s(\Omega), G^s(\Omega)) \cap \mathscr{L}(\mathscr{E}_s'(\Omega), \mathscr{D}_s'(\Omega))$; then we have also ${}^tT \in \mathscr{L}(G_0^s(\Omega), G^s(\Omega)) \cap \mathscr{L}(\mathscr{E}_s'(\Omega), \mathscr{D}_s'(\Omega))$.

Definition 2.1. *An operator* $T \in \mathscr{L}(G_0^s(\Omega), G^s(\Omega)) \cap \mathscr{L}(\mathscr{E}_s'(\Omega), \mathscr{D}_s'(\Omega))$, $s > 1$, *is said to be s-pseudolocal in* Ω *if for every ultradistribution* $u \in \mathscr{E}_s'(\Omega)$ *it holds that*

$$s - \text{singsupp}(Tu) \subset s - \text{singsupp}\, u.$$

An example of an operator as in Definition 2.1 is given by any linear partial differential operator with Gevrey coefficients (see [13]). The classes of the Gevrey finite-order pseudodifferential operators, s-ultradifferential operators and pseudodifferential operators of infinite order, studied in [13], [5] and [20] (see also the references there), also satisfy the assumptions of the preceding definition. We are going now to look at the properties of the s-hypoellipticity and s-local solvability of an operator $T \in \mathscr{L}(G_0^s(\Omega), G^s(\Omega)) \cap \mathscr{L}(\mathscr{E}_s'(\Omega), \mathscr{D}_s'(\Omega))$. Let us rigorously state the above notions in the present context.

Definition 2.2. *An operator* $T \in \mathscr{L}(G_0^s(\Omega), G^s(\Omega)) \cap \mathscr{L}(\mathscr{E}_s'(\Omega), \mathscr{D}_s'(\Omega))$, $s > 1$, *is said to be s-hypoelliptic in* Ω *if*

$$s - \text{singsupp}\, u = s - \text{singsupp}(Tu)$$

for all s-ultradistributions $u \in \mathscr{E}_s'(\Omega)$. *Note that s-hypoelliptic operators are s-pseudolocal in* Ω. *We say that* $T \in \mathscr{L}(G_0^s(\Omega), G^s(\Omega)) \cap \mathscr{L}(\mathscr{E}_s'(\Omega), \mathscr{D}_s'(\Omega))$, $s > 1$, *is s-locally solvable at a point* $x_0 \in \Omega$ *if there is an open neighbourhood* $U \subset \Omega$ *of* x_0 *such that for every function* $f \in G_0^s(U)$ *there exists an ultradistribution* $u \in \mathscr{E}_s'(\Omega)$ *such that* $Tu = f$, *as an equality in* $\mathscr{D}_s'(U)$. *We say that T is s-locally solvable in* Ω *if it is s-locally solvable at every point of* Ω.

The basic result of this section is given by the following

PROPOSITION 2.1. *Assume that* $T \in \mathscr{L}(G_0^s(\Omega), G^s(\Omega)) \cap \mathscr{L}(\mathscr{E}_s'(\Omega), \mathscr{D}_s'(\Omega))$, $s > 1$, *is s-hypoelliptic in* Ω. *Then* ${}^tT$ *is s-locally solvable in* Ω.

PROOF. Let x_0 be an arbitrary point in Ω and take two compact subsets K and K' of Ω, such that $x_0 \in \operatorname{int}(K)$ and $K \subset \operatorname{int}(K')$ (here and later $\operatorname{int}(K)$ stands for the interior set of K). Moreover let $\chi \in G_0^s(K')$ be a cut-off function, $\chi \equiv 1$ in an open neighbourhood U of K. Let $C_0(K';K)$ be the Banach space of all the continuous functions in K' compactly supported in K, with norm $|u|_{\infty,K} := \sup_{x \in K'}|u(x)| = \sup_{x \in K}|u(x)|$. Setting $D_\chi := \{u \in C_0(K';K) : \chi Tu \in G_0^s(K')\}$, we define $T_\chi : D_\chi \to G_0^s(K')$ by

$$T_\chi u := \chi Tu, \quad u \in D_\chi. \tag{2.1}$$

First, we prove that T_χ is closed, that is the graph $\mathscr{G}_\chi := \{(u, T_\chi u), u \in D_\chi\} \subset C_0(K';K) \times G_0^s(K')$ of T_χ is a closed subspace of $C_0(K';K) \times G_0^s(K')$, with respect to the product topology. For this purpose, let $\{(u_\alpha, T_\chi u_\alpha)\}_{\alpha \in A}$ be a net of points $(u_\alpha, T_\chi u_\alpha) \in \mathscr{G}_\chi$ converging to a point $(u, v) \in C_0(K';K) \times G_0^s(K')$; this means that $u_\alpha \to u$ in $C_0(K';K)$ and $T_\chi u_\alpha \to v$ in $G_0^s(K')$. Since $C_0(K';K)$ is continuously imbedded in $\mathscr{E}_s'(\Omega)$ and T is continuous on $\mathscr{E}_s'(\Omega)$, $u_\alpha \to u$ in $C_0(K';K)$ yields $T_\chi u_\alpha \to T_\chi u$ in $\mathscr{D}_s'(\Omega)$; on the other hand $T_\chi u_\alpha \to v$ in $G_0^s(K')$ yields $T_\chi u_\alpha \to v$ in $\mathscr{D}_s'(\Omega)$. So we have $T_\chi u = v$ which shows the closedness of $\mathscr{G}_\chi$. Using the hypothesis of s-hypoellipticity of T, we prove now that

$$D_\chi = G_0^s(K). \tag{2.2}$$

Of course we have $G_0^s(K) \subset D_\chi$. Conversely, suppose $u \in D_\chi$. From $T_\chi u \in G_0^s(K')$ it follows that $(T_\chi u)_{|U} = (\chi Tu)_{|U} = (Tu)_{|U} \in G^s(U)$; because of the s-hypoellipticity of T we derive $u_{|U} \in G^s(U)$ and then $u \in G_0^s(K)$, since $\operatorname{supp} u \subset K$. Thanks to the equality (2.2), we may define $J : \mathscr{G}_\chi \to G_0^s(K)$ by

$$J(u, T_\chi u) := u, \quad u \in D_\chi. \tag{2.3}$$

The operator J is closed; in fact let $\{(u_\alpha, T_\chi u_\alpha, u_\alpha)\}_{\alpha\in A}$ be a net of points $(u_\alpha, T\chi u_\alpha, u_\alpha) \in \mathscr{G}_J$ converging to a point $(u, v, w) \in C_0(K'; K) \times G_0^s(K') \times G_0^s(K)$ with respect to the product topology; $\mathscr{G}_J$ is the graph of J, that is the set $\mathscr{G}_J := \{(u, T_\chi u, u) : u \in D_\chi = \mathscr{G}_0^s(K)\} \subset C_0(K'; K) \times G_0^s(K') \times G_0^s(K)$. Since $G_0^s(K)$ is continuously imbedded in $C_0(K'; K)$ from $u_\alpha \to u$ in $C_0(K'; K)$ and $u_\alpha \to w$ in $G_0^s(K)$ it follows that $u = w \in G_0^s(K)$; furthermore, by the same argument as before, $T_\chi u = v \in G_0^s(K')$. So $(u, v, w) \in \mathscr{G}_J$ and the closedness of J is then proved. The next step will be to prove that the topology τ induced on $\mathscr{G}_\chi$ by the product topology in $C_0(K'; K) \times G_0^s(K')$ is identical to the topology τ_1 defined by

$$(\mathscr{G}_\chi, \tau_1) := \operatorname{indlim}_{\eta\to 0} \mathscr{G}_\chi \cap [C_0(K'; K) \times G_0^s(K'; \eta)]. \tag{2.4}$$

Clearly $\mathscr{G}_\chi \cap [C_0(K'; K) \times G_0^s(K'; \eta)]$ is a closed subspace of the Banach space $C_0(K'; K) \times G_0^s(K'; \eta)$ and then, in its turn, a Banach space for each $\eta > 0$. The inclusion $\tau \subset \tau_1$ follows by observing that the maps

$$\mathscr{G}_\chi \cap [C_0(K'; K) \times G_0^s(K'; \eta)] \hookrightarrow (\mathscr{G}_\chi, \tau) \tag{2.5}$$

are also continuous for all $\eta > 0$ and using the definition of inductive topology. To show the converse inclusion $\tau_1 \subset \tau$, we use the same argument as in [1]. Notice that $\mathscr{G}_\chi$ is closed in $C_0(K'; K) \times G_0^s(K')$ and $G_0^s(K) \times G_0^s(K') \hookrightarrow C_0(K'; K) \times G_0^s(K')$ with continuous imbedding; then $\mathscr{G}_\chi$ is also closed in $G_0^s(K) \times G_0^s(K')$; moreover the maps

$$G_0^s(K; \eta) \times G_0^s(K'; \eta) \hookrightarrow G_0^s(K; \eta') \times G_0^s(K'; \eta')$$

are compact for $\eta > \eta' > 0$ (see [5]). Since

$$G_0^s(K) \times G_0^s(K') \cong \operatorname{indlim}_{\eta\to 0} G_0^s(K; \eta) \times G_0^s(K'; \eta),$$

by [4] Theorem 7' we derive the following isomorphism

$$(\mathscr{G}_\chi, \tau_2) \cong \operatorname{indlim}_{\eta\to 0} \mathscr{G}_\chi \cap [G_0^s(K; \eta) \times G_0^s(K'; \eta)],$$

denoting by τ_2 the topology induced on $\mathscr{G}_\chi$ by $G_0^s(K) \times G_0^s(K')$. As a consequence, $(\mathscr{G}_\chi, \tau_2)$ is a dual Fréchet-Schwartz space and then a semi-Montel space. Since the inclusion map

$$(\mathscr{G}_\chi, \tau_2) \hookrightarrow (\mathscr{G}_\chi, \tau_1)$$

is obviously continuous, from the Open mapping Theorem for LF-spaces ([11] Theorem 8.4.11) it is also open; so $\tau_2 = \tau_1$ and $(\mathscr{G}_\chi, \tau_1)$ is a semi-Montel space. Using now Proposition 8.6.8 (v) in [11], we obtain that $\tau = \tau_1$. Since J is a closed map between two inductive limits of Banach spaces, by the closed graph Theorem of Köthe-Grothendieck for LF-spaces (see again [11], Corollary 1.2.20), we have that J is also continuous. Let $\{p_\alpha\}_{\alpha\in A}$ be a fundamental system of continuous seminorms defining the topology of $G_0^s(K')$ (we refer to [5] and [10] for explicit systems of seminorms defining the topology of the Gevrey classes). Since the inclusion $G_0^s(K)\subset G_0^s(K')$ is an isomorphism (cf. [2] Lemma 4.6), the system $\{p_\alpha\}_{\alpha\in A}$ also defines the topology of $G_0^s(K)$. So the continuity of J can be restated as follows: for any $\alpha\in A$ there exist $\alpha'\in A$ and a constant $C_\alpha>0$ such that

$$p_\alpha(u)\le C_\alpha(|u|_{\infty,K}+p_{\alpha'}(T_\chi u)), \tag{2.6}$$

for all $u\in G_0^s(K)$. Let us denote by $H^{1,\infty}(K)$ the Sobolev space of all the functions $u\in L^\infty(K)$ with partial derivatives $D_ju\in L^\infty(K)$, $j=1,\dots,n$. $H^{1,\infty}(K)$ is a Banach space with norm $|u|_{\infty,K}+\sum_{j=1}^n|D_ju|_{\infty,K}$ and $G_0^s(K)$ is continuously imbedded in $H^{1,\infty}(K)$. Thus we can find an index $\alpha_0\in A$ and a positive constant c_0, such that the following estimate holds

$$|u|_{\infty,K}+\sum_{j=1}^n|D_ju|_{\infty,K}\le c_0p_{\alpha_0}(u) \tag{2.7}$$

for every $u\in G_0^s(K)$. On the other hand we have for any arbitrary compact set $H\subset K$

$$|u|_{\infty,H}\le d(H)\sum_{j=1}^n|D_ju|_{\infty,H},\quad u\in G_0^s(H), \tag{2.8}$$

where $d(H)$ is the diameter of H. Taking now α equal to α_0 in formula (2.6) and using estimates (2.7) and (2.8) we get

$$\begin{aligned}p_{\alpha_0}(u)&\le C_{\alpha_0}\left(d(H)\sum_{j=1}^n|D_ju|_{\infty,H}+p_{\alpha_0'}(T_\chi u)\right)\\&\le C_{\alpha_0}d(H)c_0p_{\alpha_0}(u)+C_{\alpha_0}p_{\alpha_0'}(T_\chi u),\end{aligned} \tag{2.9}$$

for any $u\in G_0^s(H)$. Notice that the constants C_{α_0} and c_0 involved in the preceding inequality are independent of H. Choosing then a compact H_1 such that $x_0\in\mathrm{int}(H_1)$ and $C_{\alpha_0}d(H_1)c_0<1/2$, we derive from (2.9)

$$p_{\alpha_0}(u)\le 2C_{\alpha_0}p_{\alpha_0'}(T_\chi u),\quad u\in G_0^s(H_1). \tag{2.10}$$

Let us take an arbitrary function $f\in G_0^s(U_1)$, where $U_1\subset H_1$ is an open neighbourhood of x_0 and define a linear form L on the linear space $\chi T(G_0^s(H_1)):=\{\chi Tu:u\in G_0^s(H_1)\}\subset G_0^s(K')$ by

$$\langle L, \chi T u\rangle := \int_{H_1} u(x) f(x)\, dx, \quad u \in G_0^s(H_1). \tag{2.11}$$

The above definition is well posed; in fact if $\chi T u_1 = \chi T u_2$ in $G_0^s(K')$, with $u_1, u_2 \in G_0^s(H_1)$, then $p_{\alpha_0}(u_1 - u_2) = 0$ because of (2.10) for $u = u_1 - u_2$; but this last equality yields $u_1 = u_2$ taking into account formula (2.7) and the fact that $|u|_{\infty, H_1} + \sum_{j=1}^n |D_j u|_{\infty, H_1}$ is a norm on $H^{1,\infty}(H_1)$. We obtain

$$|\langle L, \chi T u\rangle| \leq \int_{H_1} |u(x)|\, |f(x)|\, dx \leq |u|_{\infty, H_1} \|f\|_{L^1(H_1)}, \tag{2.12}$$

for all $u \in G_0^s(H_1)$. On the other hand, from (2.7) we deduce that $|u|_{\infty, H_1} \leq c_0 p_{\alpha_0}(u)$ for $u \in G_0^s(H_1)$. Therefore from (2.10) we conclude that

$$|\langle L, \chi T u\rangle| \leq 2 c_0 C_{\alpha_0} \|f\|_{L^1(H_1)} p_{\alpha_0'}(\chi T u), \tag{2.13}$$

for every $u \in G_0^s(H_1)$. This shows the continuity of L with respect to the topology induced on $\chi T(G_0^s(H_1))$ by $G_0^s(K')$. From the Hanh-Banach theorem, L extends to a continuous linear form $L_0 \in (G_0^s(K'))'$. We can now define a continuous linear form on $G^s(\Omega)$ by setting

$$\langle \bar{L}_0, u\rangle := \langle L_0, \chi u\rangle, \quad u \in G^s(\Omega). \tag{2.14}$$

Actually, $\bar{L}_0$ is a linear form on $G^s(\Omega)$. Let us consider now a net $\{u_\alpha\}_{\alpha \in A}$ of points $u_\alpha \in G^s(\Omega)$ converging to 0 in $G^s(\Omega)$. By the continuity of L_0 in $G_0^s(K')$ it follows that $\langle L_0, \chi u_\alpha\rangle \to 0$ and then $\bar{L}_0 \in \mathscr{E}_s'(\Omega)$. Taking an arbitrary function $\varphi \in G_0^s(U_1)$ we have

$$\langle {}^t T \bar{L}_0, \varphi\rangle = \langle \bar{L}_0, T\varphi\rangle = \langle L_0, \chi T \varphi\rangle = \int f(x)\varphi(x)\, dx = \langle f, \varphi\rangle. \tag{2.15}$$

This shows the equality ${}^t T \bar{L}_0 = f$ in U_1 and completes the proof. Q.E.D.

3. Gevrey Fundamental Kernels of Partial Differential Operators

The theory of the kernels in the frame of s-ultradistributions can be developed in a manner analogous to the Schwartz distributions case (see [6], [7] and [13]); essentially it relies on the basic isomorphism

$$\mathscr{D}_s'(\Omega \times \Omega') \cong \mathscr{L}(G_0^s(\Omega'), \mathscr{D}_s'(\Omega)), \tag{3.1}$$

where $\Omega, \Omega' \subset \boldsymbol{R}^n$ are open sets, which associates to any ultradistribution $K = K(x, y) \in \mathscr{D}_s'(\Omega \times \Omega')$ the operator $K \in \mathscr{L}(G_0^s(\Omega'), \mathscr{D}_s'(\Omega))$ defined by

$$\langle K(\psi), \varphi\rangle = \langle K, \varphi \otimes \psi\rangle, \quad \varphi \in G_0^s(\Omega), \psi \in G_0^s(\Omega'). \tag{3.2}$$

Recall that $(\varphi \otimes \psi)(x, y) := \varphi(x)\psi(y)$ is the tensor product of φ and ψ. We will make use of the formal integral

$$K(\psi)(x) = \int K(x, y)\psi(y)\, dy \tag{3.3}$$

to denote the ultradistribution $K(\psi)$ obtained by applying the operator K to the function ψ. As in the C^∞ case, the following properties of a Gevrey distribution kernel can be stated (see for example [13]).

Definition 3.1. *We say that a Gevrey distribution kernel $K(x, y)$ is s-semiregular in x if* (3.2) *gives a continuous linear operator from $G_0^s(\Omega')$ to $G^s(\Omega)$. We say that $K(x, y)$ is s-semiregular in y if K can be extended as a continuous linear map from $\mathscr{E}_s'(\Omega')$ to $\mathscr{D}_s'(\Omega)$ (or equivalently the transposed map ${}^tK$ is s-semiregular in x). We call $K(x, y)$ s-regular if it is s-semiregular both in x and y. K is called s-regularizing kernel if it is an element of $G^s(\Omega \times \Omega')$ (or equivalently the corresponding map can be extended to a continuous linear map of $\mathscr{E}_s'(\Omega')$ into $G^s(\Omega)$). Finally $K(x, y) \in \mathscr{D}_s'(\Omega \times \Omega)$ is said to be s-very regular if it is s-regular and, moreover, is a G^s function in the complement of the diagonal $\Delta := \{(x, y) \in \Omega \times \Omega : x = y\}$ in $\Omega \times \Omega$.*

Remark 1. Concerning the above definition, let us observe in addition that the following isomorphism $\mathscr{L}(G_0^s(\Omega'), G^s(\Omega)) \cong G^s(\Omega; \mathscr{D}_s'(\Omega'))$ holds (see [7], Theorem 5.2); so the kernels s-semiregular in x can be identified with the G^s functions of $x \in \Omega$ valued in the space $\mathscr{D}_s'(\Omega')$, while the kernels s-semiregular in y are identified with the G^s functions of $y \in \Omega'$ valued in $\mathscr{D}_s'(\Omega)$.

In this section we restrict ourselves to a linear partial differential operator

$$P(x, D) = P = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \tag{3.4}$$

with coefficients $a_\alpha \in G^s(\Omega)$, for a given open subset Ω of $\boldsymbol{R}^n$. Given an ultradistributions $K(x, y) \in \mathscr{D}_s'(\Omega_x \times \Omega_y')$ we may consider the operator P in (3.4) acting on $K(x, y)$ with respect to the x-variables:

$$P_x(K(x, y)) = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha K(x, y);$$

here P_x and D_x^α just mean that we are considering P and D^α acting on the x variables; with the same convention, we will write P_y and D_y^α. According to [14], we may give the following

DEFINITION 3.2. *Let P be a linear partial differential operator with G^s coefficients. A kernel $K(x,y) \in \mathscr{D}_s'(\Omega \times \Omega)$ is called a fundamental kernel of P if*

$$P_x K(x,y) - \delta(x-y) = 0,$$

where we write $\delta(x-y)$ for the Dirac measure on the diagonal of $\Omega \times \Omega$. We say that $K(x,y)$ is a two-sided fundamental kernel of P if it satisfies

$$P_x K(x,y) - \delta(x-y) = 0 \quad {}^t P_y K(x,y) - \delta(x-y) = 0$$

or equivalently

$$KP\psi = PK\psi = \psi$$

for all $\psi \in G_0^s(\Omega)$, being K the map in $\mathscr{L}(G_0^s(\Omega), \mathscr{D}_s'(\Omega))$ corresponding to $K(x,y)$ by (3.2).

The following proposition concerning the above properties of a distribution kernel can be proved easily, by using arguments analogous to those for the "singular support lemma" (see for example [15]).

PROPOSITION 3.1. *If the distribution kernel $K(x,y) \in \mathscr{D}_s'(\Omega \times \Omega)$ is s-very regular, then the continuous linear operator K corresponding to $K(x,y)$ by* (3.2) *is s-pseudolocal in Ω.*

The main result of this section is given by the following.

PROPOSITION 3.2. *Let P be a linear partial differential operator with coefficients in $G^s(\Omega)$, where Ω is an open subset of $\boldsymbol{R}^n$, and suppose P is s-hypoelliptic in Ω. Then every point of Ω has an open neighbourhood in which the transposed operator ${}^tP$ has a fundamental kernel. If ${}^tP$ is also s-hypoelliptic, every point of Ω has a neigbourhood where P has a two-sided s-very regular fundamental kernel.*

We need some preparation before proving the proposition. First of all we introduce the following

DEFINITION 3.3. *A continuous increasing function $\sigma : [0,\infty) \to \boldsymbol{R}$ is said to be a subordinate function if it satisfies*:

(i) $\sigma(0) = 0$;

(ii) $\frac{\sigma(\rho)}{\rho} \to 0$ *as* $\rho \to \infty$.

Hereafter we write Σ for the class of all the subordinate functions and for any $\sigma \in \Sigma$ we set, with abuse of notation, $\sigma(\xi) := \sigma(|\xi|)$. The importance of the above definition in our context is that the expressions

$$\|f\|_{s,\sigma} := \|\exp(\sigma(\xi)^{1/s})\hat{f}(\xi)\|_2 \tag{3.5}$$

give a fundamental system of continuous seminorms on $G_0^s(K)$, for any compact set $K \subset \boldsymbol{R}^n$, as σ varies in Σ (see [5], Theorem 9.4), where $\|.\|_2$ is the norm in the space $L^2(\boldsymbol{R}^n)$ and $\hat{f}(\xi) := \int \exp(-i\xi \cdot x) f(x)\, dx$ is the Fourier transform of $f \in G_0^s(K)$. For fixed $s > 1$ and $\sigma \in \Sigma$, we define the space Φ_σ^s by

$$\Phi_\sigma^s := \{u \in \mathscr{S}'(\boldsymbol{R}^n) : \|u\|_{s,\sigma} < \infty\},$$

where $\mathscr{S}'(\boldsymbol{R}^n)$ is the space of the tempered distributions in $\boldsymbol{R}^n$ and the Fourier transformation is defined on $\mathscr{S}'(\boldsymbol{R}^n)$ by duality. It is a standard argument to show that Φ_σ^s is a Hilbert space with inner product given by

$$(u, v)_{s,\sigma} := (\exp(\sigma(\xi)^{1/s})\hat{u}(\xi), \exp(\sigma(\xi)^{1/s})\hat{v}(\xi)), \quad u, v \in \Phi_\sigma^s, \tag{3.6}$$

denoting by $(.,.)$ the inner product in $L^2(\boldsymbol{R}^n)$. In view of Lemma 3.3 in [5] and Definition 3.3, it immediately follows that $G_0^s(\Omega) \subset \Phi_\sigma^s$ with continuous imbedding, for any open $\Omega \subset \boldsymbol{R}^n$, $s > 1$ and $\sigma \in \Sigma$. This allows us to define the space $G_\sigma^s(\Omega)$ as the closure of $G_0^s(\Omega)$ in Φ_σ^s. As a closed subspace of Φ_σ^s, $G_\sigma^s(\Omega)$ is in its turn a Hilbert space with the inner product (3.6).

REMARK 2. Let us observe that for any $s > 1$ and $\sigma \in \Sigma$, in view of the Parseval's identity we have for every $u \in \Phi_\sigma^s$:

$$\|u\|_2 = c_n\|\hat{u}\|_2 \leq c_n\|\exp(-\sigma(.)^{1/s})\|_\infty \|u\|_{s,\sigma} \leq C_n\|u\|_{s,\sigma},$$

where $c_n := (2\pi)^{-n/2}$ and $C_n := c_n\|\exp(-\sigma(.)^{1/s})\|_\infty < \infty$, since from Definition 3.3 it follows that $\exp(-\sigma(\xi)^{1/s}) \in L^\infty(\boldsymbol{R}^n)$. So the space Φ_σ^s is included in $L^2(\boldsymbol{R}^n)$ with continuous imbedding. As a subspace of Φ_σ^s, $G_\sigma^s(\Omega)$ is also included in $L^2(\boldsymbol{R}^n)$ with continuous imbedding for any $\Omega \subset \boldsymbol{R}^n$ open.

Hereafter, for a given subordinate function $\sigma \in \Sigma$ and a real number $s > 1$, we denote by j the operator which associates to each function $f \in L^2(\boldsymbol{R}^n)$ the linear form $j(f)$ on $G_\sigma^s(\Omega)$ defined by

$$\langle j(f), \psi \rangle = \int f(x)\psi(x)\, dx, \quad \psi \in G_\sigma^s(\Omega). \tag{3.7}$$

We also write J for the operator which associates to every linear form u, taken in the strong dual $(G_\sigma^s(\Omega))'$ of $G_\sigma^s(\Omega)$, the restriction of u to the space $G_0^s(\Omega)$. Concerning the preceding operators j and J, the following lemma is valid.

LEMMA 3.1. *For any open set* $\Omega \subset \boldsymbol{R}^n$, $s > 1$, $\sigma_1, \sigma_2 \in \Sigma$ *the following operators*

(a) $j : L^2(\boldsymbol{R}^n) \to (G^s_{\sigma_2}(\Omega))'$;

(b) $J : (G^s_{\sigma_1}(\Omega))' \to \mathscr{D}'_s(\Omega)$;

(c) $j : G^s_{\sigma_1}(\Omega) \to (G^s_{\sigma_2}(\Omega))'$

are continuous and injective.

PROOF. In view of Remark 2, the operator j in (a) is well-defined by means of (3.7). Moreover if $\psi_\nu \to 0$ in $G^s_{\sigma_2}(\Omega)$ as $\nu \to \infty$ then

$$|\langle j(u), \psi_\nu \rangle| \leq \|u\|_2 \|\psi_\nu\|_2 \tag{3.8}$$

so that $|\langle j(u), \psi_\nu \rangle| \to 0$, since $G^s_{\sigma_2}(\Omega)$ is continuously imbedded in $L^2(\boldsymbol{R}^n)$. This shows $j(u) \in (G^s_{\sigma_2}(\Omega))'$. It is easily seen that j is injective; finally from (3.8) we see that $u_\nu \to 0$ in $L^2(\boldsymbol{R}^n)$, as $\nu \to \infty$, yields $j(u_\nu) \to 0$ in $(G^s_{\sigma_2}(\Omega))'$. The continuity of the operator J in (b) plainly follows by observing that J is the transposed of the inclusion map $G^s_0(\Omega) \hookrightarrow G^s_{\sigma_1}(\Omega)$ which is obviously continuous. Moreover J is injective, since $G^s_0(\Omega)$ is dense in $G^s_{\sigma_1}(\Omega)$. The operator j in (c) is obviously injective and continuous since it is the restriction to the space $G^s_{\sigma_1}(\Omega)$ of the operator in (a) and $G^s_{\sigma_1}(\Omega)$ is continuously imbedded in $L^2(\boldsymbol{R}^n)$, in view of Remark 2. Q.E.D.

PROOF (of Proposition 3.2). Let us take an arbitrary point x_0 in Ω. The family $\{\|.\|_{s,\sigma}\}_{\sigma \in \Sigma}$, where $\|.\|_{s,\sigma}$ is defined by (3.5), is a fundamental system of continuous seminorms in $G^s_0(K)$ for any compact set K. Therefore by repeating the argument used to get the inequality (2.10) for P instead of T and without inserting the cut-off function χ (see also Lemma 2.3 in [1]), we may find two subordinate functions σ_0 and σ_1, an open neighbourhood $U \subset \Omega$ of x_0 and a positive constant C so that:

$$\|f\|_{s,\sigma_0} \leq C \|Pf\|_{s,\sigma_1}, \tag{3.9}$$

for every $f \in G^s_0(U)$. From the above inequality it comes that the map $Pf \mapsto f$ is well-defined on $P(G^s_0(U))$; indeed if $Pf_1 = Pf_2$, $f_1, f_2 \in G^s_0(U)$, writing (3.9) for $f_1 - f_2$ instead of f we derive $\|f_1 - f_2\|_{s,\sigma_0} = 0$ and then $f_1 = f_2$. Moreover the same inequality (3.9) tells us that the preceding map is a continuous linear operator from $P(G^s_0(U))$, as a subspace of $G^s_{\sigma_1}(U)$, into $G^s_{\sigma_0}(U)$. So, denoting by M the closure of $P(G^s_0(U))$ in $G^s_{\sigma_1}(U)$, we can extend $Pf \mapsto f$ to a continuous linear operator T' from M to $G^s_{\sigma_0}(U)$. Taking now advantage from the fact that $G^s_{\sigma_1}(U)$ is a Hilbert space, we can further extend T' to a continuous linear operator, say T, from the whole $G^s_{\sigma_1}(U)$ to $G^s_{\sigma_0}(U)$ putting it equal to 0 on the orthogonal $M^\perp$ of the space M. The operator $T : G^s_{\sigma_1}(U) \to G^s_{\sigma_0}(U)$ satisfies by definition:

$$T(Pf) = f, \tag{3.10}$$

for $f \in G^s_0(U)$. Let us consider now the transposed operator ${}^tT$; it is a continuous linear map ${}^tT : (G^s_{\sigma_0}(U))' \to (G^s_{\sigma_1}(U))'$. By virtue of the imbeddings given by the operators (a) and (b) in Lemma 3.1, we may regard ${}^tT$ as a continuous linear operator from $G^s_0(U)$ to $\mathscr{D}'_s(U)$. By means of the kernel theorem for ultra-distributions, the operator ${}^tT$ corresponds to a distribution kernel in $\mathscr{D}'_s(U \times U)$ giving the required fundamental kernel of ${}^tP$ in the neighbourhood U of the arbitrary point x_0. Indeed from equality (3.10), we readily deduce

$${}^tP({}^tTf) = f,$$

for all $f \in G^s_0(U)$. This proves the first assertion of the proposition. Let us assume now that even ${}^tP$ is s-hypoelliptic. Arguing as before we can find an open neighbourhood U' of x_0 (U' could be assumed to be equal to U in (3.9) without loss of generality) on which we get an estimate such as:

$$\|g\|_{s,\varepsilon_0} \le C\|{}^tPg\|_{s,\varepsilon_1}, \quad g \in G^s_0(U), \tag{3.11}$$

where $\varepsilon_0, \varepsilon_1 \in \Sigma$ are suitable subordinate functions. Since for any couple of subordinate functions σ_0 and σ_1 the functions η_0 and η_1 defined by:

$$\eta_0(\rho) := \min(\sigma_0(\rho), \sigma_1(\rho)), \quad \eta_1(\rho) := \max(\sigma_0(\rho), \sigma_1(\rho)), \quad \rho > 0$$

are subordinate functions too, taking into account formula (3.5), we can always assume $\sigma_i = \varepsilon_i$, $i = 0, 1$, in (3.9) and (3.11). So arguing as before (where P and ${}^tP$ are interchanged) we obtain a continuous linear operator $S : G^s_{\sigma_1}(U) \to G^s_{\sigma_0}(U)$ such that:

$$S({}^tPf) = f, \quad f \in G^s_0(U). \tag{3.12}$$

Following now [14], we define the operator

$$E := Tp_M + {}^tS(I - p_M),$$

where p_M is the projection of the Hilbert space $G^s_{\sigma_1}(U)$ onto the closed subspace M and I is the identity map. By use of the imbeddings given by the operators (a)–(c) in Lemma 3.1, we see that $E \in \mathscr{L}(G^s_0(U), \mathscr{D}'_s(U))$ (so it is a kernel distribution in $\mathscr{D}'_s(U \times U)$ again) and

$$E(Pf) = Tp_M Pf + {}^tS(I - p_M)Pf$$
$$= TPf + {}^tS(Pf - Pf) = f, \quad f \in G_0^s(U), \tag{3.13}$$

taking into account (3.10). Moreover we have:

$$P(Ef) = PTp_M f + P({}^tS(I - p_M)f)$$
$$= PTp_M f + (I - p_M)f, \quad f \in G_0^s(U), \tag{3.14}$$

since $P{}^tSf = f$ as a consequence of (3.12). Now we are going to show that $PTp_M f = p_M f$ for $f \in G_0^s(U)$. Indeed there is a sequence $\{f_\nu\}_{\nu=1}^{\infty}$ of functions $f_\nu \in G_0^s(U)$ such that $Pf_\nu \to p_M f$ in $G_{\sigma_1}^s(U)$, as $p_M f \in M$. Since T is a continuous operator on $G_{\sigma_1}^s(U)$ and satisfies (3.10), we derive then

$$f_\nu = TPf_\nu \to Tp_M f, \quad \nu \to \infty$$

in $G_{\sigma_0}^s(U)$ and then in $\mathscr{D}_s'(U)$, because of Lemma 3.1 (b) and (c). So we conclude that $Pf_\nu \to PTp_M f$ in $\mathscr{D}_s'(U)$. This just yields $PTp_M f = p_M f$ which gives

$$P(Ef) = f, \tag{3.15}$$

for all $f \in G_0^s(U)$. Equations (3.13) and (3.15) just mean that the distribution kernel in $\mathscr{D}_s'(U \times U)$ corresponding to E is a two-sided fundamental kernel of P in U. It remains to see that such a kernel E is s-very regular. This last assertion will be a consequence of the next lemma. Q.E.D.

LEMMA 3.2. *Let P be a linear partial differential operator on an open set $\Omega \subset \boldsymbol{R}^n$, with coefficients in $G^s(\Omega)$. If P and ${}^tP$ are s-hypoelliptic in Ω then every two sided fundamental kernel $T(x, y) \in \mathscr{D}_s'(\Omega \times \Omega)$ of P is s-very regular.*

PROOF. Let $T \in \mathscr{L}(G_0^s(\Omega), \mathscr{D}_s'(\Omega))$ be the operator corresponding to $T(x, y)$ by (3.2). Since $PTf = f$, for every $f \in G_0^s(\Omega)$, from the s-hypoellipticity of P it follows that $T(G_0^s(\Omega)) \subset G^s(\Omega)$. From that we deduce the graph of T, $\mathscr{G}(T) := \{(f, Tf), f \in G_0^s(\Omega)\}$ is a closed subspace of $G_0^s(\Omega) \times G^s(\Omega)$ and then we claim $T \in \mathscr{L}(G_0^s(\Omega), G^s(\Omega))$. To prove this last assertion, let K be any compact subset of Ω, $\chi \in G_0^s(\Omega)$ a cut-off function, such that $\chi \equiv 1$ on K, and for every $f \in G_0^s(\Omega)$ set $T_\chi f := \chi Tf$, as in the proof of Proposition 2.1. We have that $T_\chi \in \mathscr{L}(G_0^s(\Omega), \mathscr{E}_s'(\Omega)) \subset \mathscr{L}(G_0^s(\Omega), \mathscr{D}_s'(\Omega))$ and $T_\chi(G^s(\Omega)) \subset G_0^s(K) \subset G_0^s(\Omega)$. Arguing then on the graph $\mathscr{G}(T_\chi)$ of T_χ as before, it turns that $\mathscr{G}(T_\chi)$ is a closed subspace of $G_0^s(\Omega) \times G_0^s(\Omega)$. By Köthe-Grothendieck's closed graph theorem for

LF-spaces ([11], Corollary 1.2.20), it follows that $T_\chi \in \mathscr{L}(G_0^s(\Omega), G_0^s(\Omega))$. In view of definition of χ it comes that the map

$$\psi \mapsto (T\psi)_{|K}$$

associating to any $\psi \in G_0^s(\Omega)$ the restriction of $T\psi$ to K, is a continuous linear operator from $G_0^s(\Omega)$ to $G^s(K)$. Because of the arbitrary of K, this shows the continuity of $T : G_0^s(\Omega) \to G^s(\Omega)$ and then T is s-semiregular in x. Since the fundamental kernel T of P is two sided and ${}^tP$ is also s-hypoelliptic, arguing as before with ${}^tP$ and ${}^tT$ instead of P and T respectively, we prove that T is s-semiregular in y. Let us notice that, in view of Remark 1, $T(x,y) \in G^s(\Omega_y; \mathscr{D}_s'(\Omega_x))$. It remains now to show that $T(x,y)$ is a G^s function in the complement of the diagonal $\Delta \subset \Omega \times \Omega$. In order to do it, let U and V be two open subsets of Ω such that $U \cap V = \varnothing$ and $\alpha \in \mathbf{Z}_+^n$ an arbitrary multi-index. From now on $T(x,y)$ stands for its restriction to $U_x \times V_y$. Since $T(x,y)$ is a fundamental kernel we have

$$P_x(\partial_y^\alpha T(x,y)) = 0, \quad \text{in } U_x \times V_y$$

so that the set of ultradistributions $\mathscr{R} := \{\partial_y^\alpha T(.\,,y) : \alpha \in \mathbf{Z}_+^n, y \in V_y\} \subset \mathscr{D}_s'(U_x)$ is contained in the null space of P, Ker P, where now we think about P as a continuous linear operator $P : \mathscr{D}_s'(U) \to \mathscr{D}_s'(U)$. On the other hand Ker $P \subset G^s(U_x)$, because of the s-hypoellipticity of P. Arguing as in the proof of Theorem 52.1 of [14] we see that the topologies of $G^s(U_x)$ and $\mathscr{D}_s'(U_x)$ coincide on Ker P (and then on $\mathscr{R}$). Thus it follows that $T(x,y) \in G^s(V_y; G^s(U_x)) \cong G^s(U_x \times V_y)$. Indeed, let $\{q_l\}_{l \in L}$ be a fundamental system of continuous seminorms on $\mathscr{D}_s'(U_x)$: we know that for any compact $K \subset V$ and $l \in L$ there is a constant $C_l = C_l(K) > 0$ such that

$$\max_{y \in K} q_l(\partial_y^\alpha T(.\,,y)) \le C_l^{|\alpha|+1}(\alpha!)^s, \quad \alpha \in \mathbf{Z}_+^n \tag{3.16}$$

(cf. [7] Definition 3.9). Moreover, let $\{p_j\}_{j \in J}$ be a fundamental system of continuous seminorms on $G^s(U_x)$. Then for all $j \in J$ there is a $l_j \in L$ and $C_j > 0$ such that

$$p_j(f) \le C_l q_{l_j}(f) \quad f \in \text{Ker } P. \tag{3.17}$$

From (3.16) and (3.17) we deduce that for any compact set $K \subset V_y$ and $j \in J$

$$\max_{y \in K} p_j(\partial_y^\alpha T(.\,,y)) \le C_j^{|\alpha|+1}(\alpha!)^s, \quad \alpha \in \mathbf{Z}_+^n,$$

for some positive constant $C_j = C_j(K)$, which proves $T(x,y) \in G^s(U_x \times V_y)$ and concludes the proof. Q.E.D.

4. Applications

As a necessary condition of s-hypoellipticity, Proposition 2.1 may be used to deduce some results of non s-hypoellipticity from known results of non s-local solvability. More precisely, analogously to the differential case studied in [1], we obtain that if an operator $T \in \mathscr{L}(G_0^s(\Omega), G^s(\Omega)) \cap \mathscr{L}(\mathscr{E}_s'(\Omega), \mathscr{D}_s'(\Omega))$, $s > 1$, is not s-locally solvable in Ω, then the transposed ${}^tT$ is non s-hypoelliptic in Ω. Assume in particular that T belongs to $\mathscr{L}(G_0^s(\Omega), G^s(\Omega)) \cap \mathscr{L}(\mathscr{E}_s'(\Omega), \mathscr{D}_s'(\Omega))$ for all $s > 1$ (this is the case when, for instance, T is an analytic pseudodifferential operator); we deduce that if T is not s-locally solvable then ${}^tT$ is neither s'-hypoelliptic for all $1 < s \leq s'$ nor hypoelliptic. In fact s'-hypoellipticity for ${}^tT$ would imply s'-local solvability of T, and hence s-local solvability for $s \leq s'$. A number of applications in this direction can be found in [1]; all of them concerns differential operators. Here we add few very elementary applications concerning Gevrey pseudolocal model operators which might be also not differential. We begin with the class of partial differential operators P in $\boldsymbol{R}^2_{x_1,x_2}$ of the following form

$$ {}^tP = (\partial_{x_1} + i\varphi(x_1)\partial_{x_2})^m + D_{x_2}^{m-1}, \tag{4.1} $$

where $\varphi(x_1) \in G^\theta(\boldsymbol{R})$, $\theta > 1$ and changes sign at $x_1 = 0$. From the Gevrey non solvability result in [8] Theorem 5.1, see also [9], we readily derive the following

PROPOSITION 4.1. *The operator* (4.1) *is neither s-hypoelliptic for all* $s \geq \theta$ *nor hypoelliptic.*

Another example is given by the pseudodifferential operators in $\boldsymbol{R}^2_{x_1,x_2}$ studied in [12]

$$ Q_\rho := D_{x_1} + ix_1^h|D_{x_2}|^\rho, \tag{4.2} $$

where $0 < \rho \leq 1$, h is an odd integer and $|D_{x_2}|^\rho$ is the pseudodifferential operator with symbol $|\xi_2|^\rho$. In [12] it is proved that the operator Q_ρ is not s-locally solvable for $s > 1/\rho$, whereas it is s-locally solvable for $1 < s < 1/\rho$. In particular $Q := D_{x_1} + ix_1^h|D_{x_2}|$ is not s-locally solvable for any $s > 1$. From Proposition 2.1 we deduce now the following Gevrey non hypoellipticity result.

PROPOSITION 4.2. *The transposed* ${}^tQ_\rho = -D_{x_1} + ix_1^h|D_{x_2}|^\rho$ *is neither s-hypoelliptic for all* $s > 1/\rho$, *nor hypoelliptic.*

Finally, let us consider the model operator in $\boldsymbol{R}^2_{x_1,x_2}$ with multiple characteristics

$$P_m = (D_{x_1} + ix_1^h|D_{x_2}|)^m + \text{lower order terms}, \quad m \geq 2, \tag{4.3}$$

where h is an odd positive integer. It is proved in [3], [9] that P_m is not s-locally solvable for each $s > 1$, while it is s-hypoelliptic in $\boldsymbol{R}^2_{x_1,x_2}$ for all $1 < s < m/(m-1)$. By use of Proposition 2.1 we derive then the following

Proposition 4.3. *The operator*

$${}^tP_m = (-D_{x_1} + ix_1^h|D_{x_2}|)^m + \textit{lower order terms}$$

is neither s-hypoelliptic, for all $s > 1$, *nor hypoelliptic in* $\boldsymbol{R}^2_{x_1,x_2}$. ${}^tP_m$ *is s-locally solvable in* $\boldsymbol{R}^2_{x_1,x_2}$ *for every* $1 < s < m/(m-1)$.

More general classes of operators containing the preceding models can be found in [3], [9] and the references there; results of Gevrey non-hypoellipticity for them can be deduced in a similar way.

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