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A GENERAL CRITERION FOR LINEARLY UNRELATED SEQUENCES*

By

Jaroslav HANČL and Simona SOBKOVÁ

Abstract. The main result of this paper is a general criterion for linearly unrelated sequences which does not depend on divisibility. A criterion for irrational sequences is presented as a consequence. Applications and several examples are included.

1. Introduction

Irrationality is a special case of linear independence and many new results on the irrationality of infinite series appeared after the second world war. For instance the criteria of Erdős and Strauss [6], [7], of Oppenheim [13] and especially the result of Apery in [1] which proves the irrationality of $\xi(3)$. Most of them depend on divisibility. The second wave of these papers comes from the eighties and nineties of the last century. Among them let us mention Badea [2], [3], Duverney [4] and Huttner [11]. Other results can be found in Nishioka's book [12], which contains a nice survey of Mahler theory. In 1975 Erdős defined the irrationality of a sequence in [5] and this definition is generalized in the following way.

DEFINITION 1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$$

is an irrational number, then the sequence $\{a_n\}_{n=1}^{\infty}$ is irrational. If $\{a_n\}_{n=1}^{\infty}$ is not an irrational sequence, then it is a rational sequence.

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In the same paper Erdős proved the irrationality of the sequence $\{2^{2^n}\}_{n=1}^{\infty}$ and also the following result.

THEOREM 1. *Let $n_1 < n_2 < \cdots$ be an infinite sequence of integers satisfying*

$$\lim_{k\to\infty} \sup n_k^{1/2^k} = \infty$$

and

$$n_k > k^{1+\varepsilon}$$

for some fixed $\varepsilon > 0$ and $k > k_0(\varepsilon)$. Then

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{n_k}$$

is an irrational number.

In 1993 in [9] Hančl proved.

THEOREM 2. *Let $\{r_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of positive real numbers such that $\lim_{n\to\infty} r_n = \infty$, let B be a positive integer, and let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers such that*

$$b_{n+1} \leq r_n^B$$

and

$$a_n \geq r_n^{2^n}$$

holds for every large n. Then the series

$$A = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$$

and the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ are irrational.

There is also a criterion for rational sequences.

THEOREM 3. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that* $\lim_{n\to\infty} \sup(\log_2 \log_2 a_n/n) < 1$. *Then $\{a_n\}_{n=1}^{\infty}$ is a rational sequence.*

This theorem, together with the proof of a more general result, can be found in [8]. Let us only note that the above sequence consists of positive real numbers.

The definition of linearly unrelated sequences was introduced in [10].

DEFINITION 2. Let $\{a_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \ldots, K)$ be sequences of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the numbers $\sum_{n=1}^{\infty} 1/a_{1,n}c_n, \sum_{n=1}^{\infty} 1/a_{2,n}c_n, \ldots, \sum_{n=1}^{\infty} 1/a_{K,n}c_n$, and 1 are linearly independent over rational numbers, then the sequences $\{a_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \ldots, K)$ are linearly unrelated.

The same paper contains the following theorem.

THEOREM 4. *Let* $\{a_{i,n}\}_{n=1}^{\infty}$, $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \ldots, K-1)$ *be sequences of positive integers and* $\varepsilon > 0$ *such that*

$$\frac{a_{1,n+1}}{a_{1,n}} \geq 2^{K^{n-1}}, \quad a_{1,n} \text{ divides } a_{1,n+1}$$

$$b_{i,n} < 2^{K^{n-(\sqrt{2}+\varepsilon)\sqrt{n}}}, \quad i = 1, \ldots, K-1$$

$$\lim_{n\to\infty} \frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}} = 0 \quad \text{for all } i, j = 1, \ldots, K-1, \ i > j$$

$$a_{i,n}2^{-K^{n-(\sqrt{2}+\varepsilon)\sqrt{n}}} < a_{1,n} < a_{i,n}2^{K^{n-(\sqrt{2}+\varepsilon)\sqrt{n}}}, \quad i = 1, \ldots, K-1$$

hold for every sufficiently large natural number n. Then the sequences $\{a_{i,n}/b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \ldots, K-1)$ *are linearly unrelated.*

However this criterion depends on divisibility. Theorem 5 in Section 2 does not depend on arithmetical properties but only on the speed of convergence of rational numbers.

2. Main Results

THEOREM 5. *Let K be a positive integer,* α, ε *be positive real numbers such that* $0 < \alpha < 1$ *and let* $\{a_{i,n}\}_{n=1}^{\infty}$, $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \ldots, K)$ *be sequences of positive integers with* $\{a_{1,n}\}_{n=1}^{\infty}$ *nondecreasing, such that*

$$\lim_{n\to\infty} \sup a_{1,n}^{1/(K+1)^n} = \infty \tag{1}$$

$$a_{1,n} \geq n^{1+\varepsilon} \tag{2}$$

$$b_{i,n} \leq 2^{(\log_2 a_{1,n})^\alpha}, \quad i = 1, \dots, K \tag{3}$$

$$\lim_{n\to\infty} \frac{a_{i,n} b_{j,n}}{b_{i,n} a_{j,n}} = 0, \quad i, j = 1, \dots, K, \ i > j \tag{4}$$

$$a_{i,n} 2^{-(\log_2 a_{1,n})^\alpha} \leq a_{1,n} \leq a_{i,n} 2^{(\log_2 a_{1,n})^\alpha} \tag{5}$$

hold for every sufficiently large n. Then the sequences $\{a_{i,n}/b_{i,n}\}_{n=1}^\infty$ $(i = 1, \dots, K)$ are linearly unrelated.

THEOREM 6. *Let α and ε be positive real numbers such that $0 < \alpha < 1$ and let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ be sequences of positive integers with $\{a_n\}_{n=1}^\infty$ nondecreasing, such that*

$$\lim_{n\to\infty} \sup a_n^{1/2^n} = \infty,$$

$$a_n \geq n^{1+\varepsilon}$$

and

$$b_n \leq 2^{(\log_2 a_n)^\alpha}$$

hold for every sufficiently large n. Then the sequence $\{a_n/b_n\}_{n=1}^\infty$ and the series $\sum_{n=1}^\infty b_n/a_n$ are irrational.

This theorem is an immediate consequence of Theorem 5. It is enough to put $K = 1$, $\{a_{1,n}\}_{n=1}^\infty = \{a_n\}_{n=1}^\infty$ and $\{b_{1,n}\}_{n=1}^\infty = \{b_n\}_{n=1}^\infty$.

REMARK 1. It is clear that our Theorem 6 is a stronger result then Theorem 2 above (see [9] also) because the restrictions on a_n and b_n $(n = 1, \dots)$ are weaker.

REMARK 2. In Theorem 6 put $b_n = 1$ for each natural number n. Then we obtain Erdős Theorem 1 above (see [5] also).

Open problem 1. Let M and K be positive integers. Are the sequences $\{M^{(K+1)^n} + i\}_{n=1}^\infty$ $(i = 1, \dots, K)$ linearly unrelated?

Open problem 2. Let M and S be positive integers such that $M > 1$. Is the sequence $\{M^{2^n} + S\}_{n=1}^\infty$ irrational?

EXAMPLE 1. Let $a_{i,n} = n^{9^n+i} + 3^n$, $b_{i,n} = n^{3^n} + i$ $(i = 1, 2, \ldots, 8)$. Then the sequences $\{a_{i,n}/b_{i,n}\}_{n=1}^{\infty}$ are linearly unrelated.

EXAMPLE 2. The sequences

$$\left\{\frac{n^{2^n} + 3^n}{[n^{2^{n/2}}] + 2^n}\right\}_{n=1}^{\infty}$$

and

$$\left\{\frac{2^{n2^n} + n!}{[3^{n2^{3n/4}} + 5^{n2^{3n/4}}]}\right\}_{n=1}^{\infty}$$

where $[x]$ means the greatest integer less then or equal to x, are irrational sequences.

3. Proofs

LEMMA 1. *Let K, α, ε and the sequences $\{a_{i,n}\}_{n=1}^{\infty}$, $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \ldots, K)$ satisfy all conditions stated in Theorem 5. Then there is a positive real number $B = B(K, \alpha, \varepsilon)$ which does not depend on n such that*

$$\sum_{i=1}^{K} \sum_{j=0}^{\infty} \frac{b_{i,n+j}}{a_{i,n+j}} < \frac{1}{a_{1,n}^{B}} \tag{6}$$

holds for every sufficiently large n.

PROOF (of Lemma 1). From (3) and (5) we obtain

$$\sum_{j=0}^{\infty} \frac{b_{i,n+j}}{a_{i,n+j}} \leq \sum_{j=0}^{\infty} \frac{2^{(\log_2 a_{1,n+j})^{\alpha}} 2^{(\log_2 a_{1,n+j})^{\alpha}}}{a_{1,n+j}} \leq \sum_{j=0}^{\infty} \frac{2^{(\log_2 a_{1,n+j})^{\beta}}}{a_{1,n+j}} \tag{7}$$

for every sufficiently large n, where β $(1 > \beta > \alpha)$ is a constant which does not depend on n.

Now we have

$$\sum_{j=0}^{\infty} \frac{2^{(\log_2 a_{1,n+j})^{\beta}}}{a_{1,n+j}} = \sum_{n+j<a_{1,n}^{1/(1+\varepsilon)}} \frac{2^{(\log_2 a_{1,n+j})^{\beta}}}{a_{1,n+j}} + \sum_{n+j\geq a_{1,n}^{1/(1+\varepsilon)}} \frac{2^{(\log_2 a_{1,n+j})^{\beta}}}{a_{1,n+j}}. \tag{8}$$

We will estimate the first summand in the right hand side of (8). The sequence $\{a_{1,n}\}_{n=1}^{\infty}$ is nondecreasing and the function $x^{-1}2^{(\log_2 x)^{\beta}}$ is decreasing for every sufficiently large x. It follows

$$\sum_{n+j<a_{1,n}^{1/(1+\varepsilon)}} \frac{2^{(\log_2 a_{1,n+j})^\beta}}{a_{1,n+j}} \le \frac{2^{(\log_2 a_{1,n})^\beta}}{a_{1,n}} a_{1,n}^{1/(1+\varepsilon)} \le \frac{1}{a_{1,n}^{B_1}}. \tag{9}$$

Here B_1 $(0 < B_1 < \varepsilon/(1+\varepsilon))$ is a positive real number which does not depend on n.

Now we will estimate the second summand in the right hand side of (8).

From (2) and the fact that the function $x^{-1}2^{(\log_2 x)^\beta}$ is decreasing for every sufficiently large x we obtain

$$\sum_{n+j\ge a_{1,n}^{1/(1+\varepsilon)}} \frac{2^{(\log_2 a_{1,n+j})^\beta}}{a_{1,n+j}} \le \sum_{n+j\ge a_{1,n}^{1/(1+\varepsilon)}} \frac{2^{(\log_2 (n+j)^{1+\varepsilon})^\beta}}{(n+j)^{1+\varepsilon}} \le \sum_{n+j\ge a_{1,n}^{1/(1+\varepsilon)}} \frac{1}{(n+j)^{1+\varepsilon/2}}$$

$$\le \int_{a_{1,n}^{1/(1+\varepsilon)}}^{\infty} \frac{dx}{x^{1+\varepsilon/3}} \le \frac{1}{(a_{1,n}^{1/(1+\varepsilon)})^{\varepsilon/4}} = \frac{1}{a_{1,n}^{B_2}}, \tag{10}$$

where $B_2 = \varepsilon/4(1+\varepsilon)$ is a positive real constant which does not depend on n. So (7), (8), (9) and (10) imply

$$\sum_{i=1}^{K}\sum_{j=0}^{\infty} \frac{b_{i,n+j}}{a_{i,n+j}} \le \sum_{i=1}^{K}\sum_{j=0}^{\infty} \frac{2^{(\log_2 a_{1,n+j})^\beta}}{a_{1,n+j}}$$

$$\le \left(\frac{1}{a_{1,n}^{B_1}} + \frac{1}{a_{1,n}^{B_2}}\right)K \le \frac{1}{a_{1,n}^{B}},$$

where $B = 1/2\min(B_1, B_2)$ is a positive real constant which does not depend on n and (6) follows. □

LEMMA 2. *Let K, α, ε and the sequences $\{a_{i,n}\}_{n=1}^{\infty}$, $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \dots, K)$ satisfy all conditions stated in Theorem 5 and instead of* (2) *we have*

$$a_{1,n} > 2^n \tag{11}$$

for every sufficiently large n. Then there is a real number γ $(1 > \gamma > \alpha)$ such that

$$\sum_{i=1}^{K}\sum_{j=0}^{\infty} \frac{b_{i,n+j}}{a_{i,n+j}} \le \frac{2^{(\log_2 a_{1,n})^\gamma}}{a_{1,n}} \tag{12}$$

holds for every sufficiently large n.

PROOF (of Lemma 2). As in the proof of Lemma 1 there is a positive real constant β $(1 > \beta > \alpha)$ which does not depend on n such that (7) holds for every sufficiently large n. Then we have

$$\sum_{j=0}^{\infty} \frac{2^{(\log_2 a_{1,n+j})^\beta}}{a_{1,n+j}} = \sum_{n+j<\log_2 a_{1,n}} \frac{2^{(\log_2 a_{1,n+j})^\beta}}{a_{1,n+j}} + \sum_{n+j\geq \log_2 a_{1,n}} \frac{2^{(\log_2 a_{1,n+j})^\beta}}{a_{1,n+j}}. \tag{13}$$

We will estimate both sums in the right hand side of equation (13). For the first summand, the facts that the sequence $\{a_{1,n}\}_{n=1}^{\infty}$ is nondecreasing and the function $x^{-1}2^{(\log_2 x)^\beta}$ is decreasing for every sufficiently large x imply

$$\sum_{n+j<\log_2 a_{1,n}} \frac{2^{(\log_2 a_{1,n+j})^\beta}}{a_{1,n+j}} \leq \frac{2^{(\log_2 a_{1,n})^\beta} \log_2 a_{1,n}}{a_{1,n}} \leq \frac{2^{(\log_2 a_{1,n})^{\gamma_1}}}{a_{1,n}}, \tag{14}$$

where γ_1 $(1 > \gamma_1 > \beta)$ is a positive real constant which does not depend on n. Now we will estimate the second summand of equation (13). From (11) and the fact that the function $x^{-1}2^{(\log_2 x)^\beta}$ is decreasing for sufficiently large x we obtain

$$\begin{aligned}\sum_{n+j\geq \log_2 a_{1,n}} \frac{2^{(\log_2 a_{1,n+j})^\beta}}{a_{1,n+j}} &\leq \sum_{n+j\geq \log_2 a_{1,n}} \frac{2^{(\log_2 2^{n+j})^\beta}}{2^{n+j}} = \sum_{j\geq \log_2 a_{1,n}} \frac{1}{2^{j-j^\beta}} \\ &\leq \frac{1}{2^{\log_2 a_{1,n}-(\log_2 a_{1,n})^\beta}} C \leq \frac{2^{(\log_2 a_{1,n})^{\gamma_2}}}{a_{1,n}},\end{aligned} \tag{15}$$

where γ_2 $(1 > \gamma_2 > \beta)$ and C are positive real constants which does not depend on n. Therefore (7), (13), (14) and (15) imply

$$\begin{aligned}\sum_{i=1}^{K}\sum_{j=0}^{\infty} \frac{b_{i,n+j}}{a_{i,n+j}} &\leq \sum_{i=1}^{K}\sum_{j=0}^{\infty} \frac{2^{(\log_2 a_{1,n+j})^\beta}}{a_{1,n+j}} \\ &\leq K\left(\frac{2^{(\log_2 a_{1,n})^{\gamma_1}}}{a_{1,n}} + \frac{2^{(\log_2 a_{1,n})^{\gamma_2}}}{a_{1,n}}\right) \leq \frac{2^{(\log_2 a_{1,n})^{\gamma}}}{a_{1,n}}\end{aligned}$$

and here γ $(1 > \gamma > \max(\gamma_1, \gamma_2) > \beta)$ is a positive real constant which does not depend on n. So (12) follows. □

Proof (of Theorem 5). Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then the sequences $\{a_{i,n}c_n\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \dots, K)$ also satisfy conditions (1)–(5) and if in addition we reorder the sequence $\{a_{1,n}c_n\}_{n=1}^{\infty}$ to be nondecreasing then the new sequence together with the relevant reordered sequences $\{a_{i,n}c_n\}_{n=1}^{\infty}$ $(i = 1, \dots, K)$ and $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \dots, K)$ will satisfy (1)–(5) also. It follows that they will satisfy all conditions stated in Theorem 5. Thus it suffices to prove if

K, α, ε and the sequences $\{a_{i,n}\}_{n=1}^{\infty}$, $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, \dots, K)$ satisfy all conditions stated in Theorem 5 then the numbers $\sum_{n=1}^{\infty} b_{1,n}/a_{1,n}, \dots, \sum_{n=1}^{\infty} b_{K,n}/a_{K,n}$ and the number 1 are linearly independent over the rational numbers. To establish this we will prove that for every K-tuple of integers $\alpha_1, \alpha_2, \dots, \alpha_K$ (not all zero) the sum

$$I = \sum_{i=1}^{K} \alpha_i \sum_{n=1}^{\infty} \frac{b_{i,n}}{a_{i,n}}$$

is an irrational number. Suppose that I is a rational number. Now we will continue in a natural and usual way. Let R be a maximal index such that $\alpha_R \neq 0$. Then we have

$$I = \sum_{i=1}^{K} \alpha_i \sum_{n=1}^{\infty} \frac{b_{i,n}}{a_{i,n}} = \sum_{n=1}^{\infty} \sum_{i=1}^{R} \alpha_i \frac{b_{i,n}}{a_{i,n}} = \sum_{n=1}^{\infty} \frac{b_{R,n}}{a_{R,n}} \left(\sum_{i=1}^{R-1} \alpha_i \frac{b_{i,n} a_{R,n}}{a_{i,n} b_{R,n}} + \alpha_R \right).$$

Now (4) implies that the number

$$\sum_{i=1}^{R-1} \alpha_i \frac{b_{i,n} a_{R,n}}{a_{i,n} b_{R,n}} + \alpha_R$$

and the number α_R have the same sign for every sufficiently large n. Without loss of generality assume

$$\sum_{i=1}^{K} \alpha_i \frac{b_{i,n}}{a_{i,n}} > 0 \tag{16}$$

for every sufficiently large n. Since I is a rational number there must be integers p, q, $(q > 0)$ such that

$$I = \frac{p}{q} = \sum_{i=1}^{K} \alpha_i \sum_{n=1}^{\infty} \frac{b_{i,n}}{a_{i,n}}.$$

From this and (16) we obtain that

$$\begin{aligned} C_N &= \left(p - q \sum_{i=1}^{K} \alpha_i \sum_{n=1}^{N-1} \frac{b_{i,n}}{a_{i,n}} \right) \prod_{n=1}^{N-1} \prod_{i=1}^{K} a_{i,n} \\ &= q \left(\prod_{n=1}^{N-1} \prod_{i=1}^{K} a_{i,n} \right) \sum_{i=1}^{K} \alpha_i \sum_{n=N}^{\infty} \frac{b_{i,n}}{a_{i,n}} \end{aligned} \tag{17}$$

is a positive integer for every sufficiently large N. So (17) implies

$$1 \le Q_1 \left(\prod_{n=1}^{N-1} \prod_{i=1}^{K} a_{i,n} \right) \sum_{i=1}^{K} \sum_{n=N}^{\infty} \frac{b_{i,n}}{a_{i,n}} \tag{18}$$

for every sufficiently large N, where $Q_1 = q \max_{i=1,\dots,K} |\alpha_i|$ is a positive integer constant which does not depend on N. From (5) we obtain

$$\prod_{n=1}^{N-1} \prod_{i=1}^{K} a_{i,n} \le Q_2 \left(\prod_{n=1}^{N-1} a_{1,n} \right)^K 2^{(K-1)\sum_{n=1}^{N-1} (\log_2 a_{1,n})^\alpha} \tag{19}$$

for every sufficiently large N, where Q_2 is a positive real constant which does not depend on N. Then (18) and (19) imply

$$1 \le Q \left(\prod_{n=1}^{N-1} a_{1,n} \right)^K 2^{(K-1)\sum_{n=1}^{N-1} (\log_2 a_{1,n})^\alpha} \sum_{i=1}^{K} \sum_{n=N}^{\infty} \frac{b_{i,n}}{a_{i,n}} \tag{20}$$

for every sufficiently large N, where Q is a positive real constant which does not depend on N. Now the proof falls into several cases.

1. Let us assume that (11) holds for every sufficiently large n and there is a $\delta > 0$ such that

$$\lim_{n\to\infty} \sup a_{1,n}^{1/(K+1+\delta)^n} = \infty. \tag{21}$$

This implies that there exist infinitely many N such that

$$a_{1,N}^{1/(K+1+\delta)^N} > \max_{k=1,\dots,N-1} a_{1,k}^{1/(K+1+\delta)^k}.$$

It follows that

$$\begin{aligned} a_{1,N} &> \left(\max_{k=1,\dots,N-1} a_{1,k}^{1/(K+1+\delta)^k} \right)^{(K+1+\delta)^N} \\ &> \left(\max_{k=1,\dots,N-1} a_{1,k}^{1/(K+1+\delta)^k} \right)^{(K+\delta)((K+1+\delta)^{N-1}+(K+1+\delta)^{N-2}+\dots+1)} \\ &> \left(\prod_{n=1}^{N-1} a_{1,n} \right)^{K+\delta}. \end{aligned}$$

From this we obtain

$$a_{1,N}^{(K+\delta/2)/(K+\delta)} > \left(\prod_{n=1}^{N-1} a_{1,n} \right)^{K+\delta/2}. \tag{22}$$

Lemma 2, (20) and (22) imply

$$
\begin{aligned}
1 &\leq Q\left(\prod_{n=1}^{N-1} a_{1,n}\right)^K 2^{(K-1)\sum_{n=1}^{N-1}(\log_2 a_{1,n})^\alpha} \sum_{i=1}^{K}\sum_{n=N}^{\infty}\frac{b_{i,n}}{a_{i,n}} \\
&\leq Q\left(\prod_{n=1}^{N-1} a_{1,n}\right)^K \prod_{n=1}^{N-1} 2^{(K-1)(\log_2 a_{1,n})^\alpha}\frac{2^{(\log_2 a_{1,N})^\gamma}}{a_{1,N}} \\
&= Q\frac{(\prod_{n=1}^{N-1} a_{1,n})^{K+\delta/2}}{a_{1,N}^{(K+\delta/2)/(K+\delta)}}\prod_{n=1}^{N-1}\left(\frac{2^{(K-1)(\log_2 a_{1,n})^\alpha}}{a_{1,n}^{\delta/2}}\right)\frac{2^{(\log_2 a_{1,N})^\gamma}}{a_{1,N}^{(\delta/2)/(K+\delta)}} \\
&< \frac{2^{(\log_2 a_{1,N})^\gamma}}{a_{1,N}^{(\delta/2)/(K+\delta)}}
\end{aligned}
$$

for infinitely many sufficiently large N. This is a contradiction for a sufficiently large $a_{1,n}$.

2. Let us assume that (11) holds for every sufficiently large n and there is no $\delta > 0$ such that (21) holds. From this we see that for every $\delta > 0$

$$a_{1,n} < 2^{(K+1+\delta)^n} \tag{23}$$

holds for every sufficiently large n. Let δ be sufficiently small. Equation (1) implies

$$a_{1,N}^{1/(K+1)^N} > \left(1+\frac{1}{N^2}\right)\max_{k=1,\dots,N-1} a_{1,k}^{1/(K+1)^k} \tag{24}$$

for infinitely many N otherwise there exists n_0 such that for every $n \geq n_0$

$$
\begin{aligned}
a_{1,n}^{1/(K+1)^n} &\leq \left(1+\frac{1}{n^2}\right)\max_{k=1,\dots,n-1} a_{1,k}^{1/(K+1)^k} \\
&\leq \left(1+\frac{1}{n^2}\right)\left(1+\frac{1}{(n-1)^2}\right)\max_{k=1,\dots,n-2} a_{1,k}^{1/(K+1)^k} \\
&\leq \cdots \leq \prod_{j=n_0+1}^{n}\left(1+\frac{1}{j^2}\right)a_{1,n_0}^{1/(K+1)^{n_0}} \\
&\leq \prod_{j=n_0+1}^{\infty}\left(1+\frac{1}{j^2}\right)a_{1,n_0}^{1/(K+1)^{n_0}} < \text{const.},
\end{aligned}
$$

which contradicts (1). Hence

$$a_{1,N} > \left(1+\frac{1}{N^2}\right)^{(K+1)^N} \left(\max_{k=1,\dots,N-1} a_{1,k}^{1/(K+1)^k}\right)^{(K+1)^N}$$

$$> \left(1+\frac{1}{N^2}\right)^{(K+1)^N} \left(\max_{k=1,\dots,N-1} a_{1,k}^{1/(K+1)^k}\right)^{K((K+1)^{N-1}+\dots+1)}$$

$$> \left(1+\frac{1}{N^2}\right)^{(K+1)^N} \left(\prod_{n=1}^{N-1} a_{1,n}\right)^K .$$

Using (20), Lemma 2 and (23) we obtain

$$1 \le Q\left(\prod_{n=1}^{N-1} a_{1,n}\right)^K 2^{(K-1)\sum_{n=1}^{N-1}(\log_2 a_{1,n})^\alpha} \sum_{i=1}^{K}\sum_{n=N}^{\infty} \frac{b_{i,n}}{a_{i,n}}$$

$$\le Q\left(\prod_{n=1}^{N-1} a_{1,n}\right)^K 2^{(K-1)\sum_{n=1}^{N-1}(\log_2 a_{1,n})^\alpha} \frac{2^{(\log_2 a_{1,N})^\gamma}}{a_{1,N}}$$

$$< Q\frac{2^{(K-1)\sum_{n=1}^{N-1}(\log_2 a_{1,n})^\alpha}2^{(\log_2 a_{1,N})^\gamma}}{(1+1/N^2)^{(K+1)^N}}$$

$$< P\frac{2^{(K-1)\sum_{n=1}^{N-1}(\log_2 2^{(K+1+\delta)^n})^\alpha}2^{(\log_2 2^{(K+1+\delta)^N})^\gamma}}{2^{(\log_2(1+1/N^2))(K+1)^N}}$$

$$= P\frac{2^{(K-1)\sum_{n=1}^{N-1}((K+1+\delta)^\alpha)^n}2^{((K+1+\delta)^\gamma)^N}}{2^{(\log_2(1+1/N^2))(K+1)^N}}$$

$$\le P2^{(K-1)/((K+1+\delta)^\alpha-1)((K+1+\delta)^\alpha)^N+((K+1+\delta)^\gamma)^N-(\log_2(1+1/N^2))(K+1)^N}$$

for infinitely many N, where P is a positive real constant which does not depend on N. This is a contradiction for a sufficiently large N and sufficiently small positive real number δ.

3. Now let us assume that for infinitely many n

$$a_{1,n} \le 2^n \tag{25}$$

and there is a $\delta > 0$ such that (21) holds. Let A be a sufficiently large positive integer. From (21) we see that there exists n such that

$$a_{1,n}^{1/(K+1+\delta)^n} > A. \tag{26}$$

Let k be the least positive integer satisfying (26) and s be the greatest positive integer less than k such that (25) holds. So

$$a_{1,k} > A^{(K+1+\delta)^k} = 2^{(\log_2 A)(K+1+\delta)^k}. \tag{27}$$

Then there is a positive integer n such that

$$a_{1,n}^{1/(K+1+\delta)^n} > 2. \tag{28}$$

Let t be the least positive integer greater than s such that (28) holds. It follows that for every $r = s, s+1, \dots, t-1$

$$a_{1,r} < 2^{(K+1+\delta)^r} \tag{29}$$

and

$$a_{1,t} > 2^{(K+1+\delta)^t}. \tag{30}$$

From (29) and (30) we obtain

$$\begin{aligned} a_{1,t} &> 2^{(K+1+\delta)^t} > 2^{(K+\delta)((K+1+\delta)^{t-1}+(K+1+\delta)^{t-2}+\cdots+1)} \\ &> \left(\prod_{n=1}^{t-1} 2^{(K+1+\delta)^n}\right)^{(K+\delta)} > \left(\prod_{n=1}^{t-1} a_{1,n}\right)^{(K+\delta)} \left(\prod_{n=1}^{s} a_{1,n}\right)^{-(K+\delta)}. \end{aligned} \tag{31}$$

The sequence $\{a_{1,n}\}_{n=1}^{\infty}$ is nondecreasing and $a_{1,s} \le 2^s$. It follows that

$$\prod_{n=1}^{s} a_{1,n} < 2^{s^2}. \tag{32}$$

Together with (31) this implies

$$a_{1,t} > 2^{(K+1+\delta)^t} > \left(\prod_{n=1}^{t-1} a_{1,n}\right)^{(K+\delta)} 2^{-(K+\delta)s^2}. \tag{33}$$

Suppose that $a_{1,t}$ is sufficiently large. From this and the fact that

$$\lim_{x\to\infty} x^{\delta/3} 2^{-(K-1)(\log_2 x)^\alpha} = \infty$$

we obtain

$$\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{\delta/3} \ge 2^{(K-1)\sum_{n=1}^{t-1}(\log_2 a_{1,n})^\alpha}.$$

This and (33) imply

$$a_{1,t}^{(K+\delta/3)/(K+\delta)} > 2^{((K+\delta/3)/(K+\delta))(K+1+\delta)^t} > \left(\prod_{n=1}^{t-1} a_{1,n}\right)^{(K+\delta/3)} 2^{-(K+\delta/3)s^2}$$

$$> \left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K} 2^{(K-1)\sum_{n=1}^{t-1}(\log_2 a_{1,n})^\alpha} 2^{-(K+\delta)s^2}. \tag{34}$$

From Lemma 1 and Lemma 2 we obtain

$$\sum_{i=1}^{K}\sum_{n=t}^{\infty}\frac{b_{i,n}}{a_{i,n}} = \sum_{i=1}^{K}\sum_{n=t}^{k-1}\frac{b_{i,n}}{a_{i,n}} + \sum_{i=1}^{K}\sum_{n=k}^{\infty}\frac{b_{i,n}}{a_{i,n}} \le \frac{2^{(\log_2 a_{1,t})^\gamma}}{a_{1,t}} + \frac{1}{a_{1,k}^B}. \tag{35}$$

Now (20), (34) and (35) imply

$$\begin{aligned}
1 &\le Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^{K} 2^{(K-1)\sum_{n=1}^{t-1}(\log_2 a_{1,n})^\alpha} \sum_{i=1}^{K}\sum_{n=t}^{\infty}\frac{b_{i,n}}{a_{i,n}} \\
&\le Q 2^{((K+\delta/3)/(K+\delta))(K+1+\delta)^t} 2^{(K+\delta)s^2}\left(\frac{2^{(\log_2 a_{1,t})^\gamma}}{a_{1,t}} + \frac{1}{a_{1,k}^B}\right) \\
&\le 2^{((K+\delta/2)/(K+\delta))(K+1+\delta)^t}\left(\frac{2^{(\log_2 a_{1,t})^\gamma}}{a_{1,t}} + \frac{1}{a_{1,k}^B}\right) \\
&\le \frac{2^{(\log_2 a_{1,t})^\gamma}}{a_{1,t}^{(\delta/2)/(K+\delta)}} + \frac{2^{((K+\delta/2)/(K+\delta))(K+1+\delta)^t}}{a_{1,k}^B}.
\end{aligned}$$

From this and (27) we obtain

$$1 \le \frac{2^{(\log_2 a_{1,t})^\gamma}}{a_{1,t}^{(\delta/2)/(K+\delta)}} + \frac{2^{((K+\delta/2)/(K+\delta))(K+1+\delta)^t}}{2^{B(\log_2 A)(K+1+\delta)^k}}$$

and this is a contradiction for sufficiently large $a_{1,t}$ and A.

4. Finally let us assume that for infinitely many n (25) holds and there is no $\delta > 0$ such that (21) holds. This implies that for every $\delta > 0$ and sufficiently large n (23) holds. Let δ be sufficiently small and A sufficiently large. From (1) we obtain

$$a_{1,n}^{1/(K+1)^n} > A \tag{36}$$

for infinitely many n. Let k be the least positive integer satisfying (36). Then

$$a_{1,k} > A^{(K+1)^k} = 2^{(\log_2 A)(K+1)^k}. \tag{37}$$

Let s be the greatest positive integer less than k such that (25) holds. The equation (1) and Borel's theorem imply (24) for infinitely many N. Let t be the least positive integer greater than s satisfying

$$a_{1,t}^{1/(K+1)^t} > \left(1+\frac{1}{t^2}\right) \max_{j=s,\dots,t-1} a_{1,j}^{1/(K+1)^j} \tag{38}$$

and

$$a_{1,r}^{1/(K+1)^r} \leq \left(1+\frac{1}{r^2}\right) \max_{j=s,\dots,r-1} a_{1,j}^{1/(K+1)^j} \tag{39}$$

for every $r = s+1, \dots, t-1$. From (39) we obtain

$$\begin{aligned} a_{1,r}^{1/(K+1)^r} &\leq \left(1+\frac{1}{r^2}\right) \max_{j=s,\dots,r-1} a_{1,j}^{1/(K+1)^j} \\ &\leq \left(1+\frac{1}{r^2}\right)\left(1+\frac{1}{(r-1)^2}\right) \max_{j=s,\dots,r-2} a_{1,j}^{1/(K+1)^j} \\ &\leq \cdots \leq \prod_{j=s+1}^{r} \left(1+\frac{1}{j^2}\right) a_{1,s}^{1/(K+1)^s} \leq D, \end{aligned}$$

where $D < \prod_{j=1}^{\infty}(1+1/j^2)$ is a positive real constant which does not depend on A and k. It follows that

$$a_{1,r} \leq D^{(K+1)^r} = 2^{(\log_2 D)(K+1)^r} \tag{40}$$

for every $r = s+1, \dots, t-1$. From this together with $a_{1,s} < 2^s$ and the fact that the sequence $\{a_{1,n}\}_{n=1}^{\infty}$ is nondecreasing, we obtain

$$\begin{aligned} \left(\prod_{r=1}^{t-1} a_{1,r}\right)^K &= \left(\prod_{r=1}^{s} a_{1,r}\right)^K \left(\prod_{r=s+1}^{t-1} a_{1,r}\right)^K \\ &\leq \left(\prod_{r=1}^{s} 2^s\right)^K \left(\prod_{r=s+1}^{t-1} 2^{(\log_2 D)(K+1)^r}\right)^K \\ &= 2^{Ks^2} 2^{(\log_2 D)((K+1)^t-(K+1)^{s+1})} \leq 2^{(\log_2 D)(K+1)^t} \end{aligned} \tag{41}$$

and

$$2^{(K-1)\sum_{r=1}^{t-1}(\log_2 a_{1,r})^\alpha} = 2^{(K-1)\sum_{r=1}^{s}(\log_2 a_{1,r})^\alpha} 2^{(K-1)\sum_{r=s+1}^{t-1}(\log_2 a_{1,r})^\alpha}$$

$$\leq 2^{(K-1)\sum_{r=1}^{s}(\log_2 2^s)^\alpha} 2^{(K-1)\sum_{r=s+1}^{t-1}(\log_2 2^{(\log_2 D)(K+1)^r})^\alpha}$$

$$= 2^{(K-1)s^{\alpha+1}} 2^{(K-1)\sum_{r=s+1}^{t-1}((\log_2 D)^\alpha (K+1)^{\alpha r})}$$

$$= 2^{(K-1)s^{\alpha+1}} 2^{(K-1)(\log_2 D)^\alpha ((K+1)^{\alpha t} - (K+1)^{\alpha(s+1)})/((K+1)^\alpha - 1)}$$

$$\leq 2^{E(K+1)^{\alpha t}}, \tag{42}$$

where $E = (K-1)(\log_2 D)^\alpha/((K+1)^\alpha - 1)$ is a positive real constant which does not depend on t, k and A. Then (41) and (42) imply

$$\left(\prod_{r=1}^{t-1} a_{1,r}\right)^K 2^{(K-1)\sum_{r=1}^{t-1}(\log_2 a_{1,r})^\alpha} \leq 2^{(\log_2 D)(K+1)^t} 2^{E(K+1)^{\alpha t}} \leq 2^{D(K+1)^t}. \tag{43}$$

Notice that (37) and (40) also imply $t \leq k$. Now from (38) with $a_{1,s} \leq 2^s$ and the fact that the sequence $\{a_{1,n}\}_{n=1}^\infty$ is nondecreasing, we obtain

$$a_{1,t} > \left(1 + \frac{1}{t^2}\right)^{(K+1)^t} \left(\max_{j=s,\dots,t-1} a_{1,j}^{1/(K+1)^j}\right)^{(K+1)^t}$$

$$> \left(1 + \frac{1}{t^2}\right)^{(K+1)^t} \left(\max_{j=s,\dots,t-1} a_{1,j}^{1/(K+1)^j}\right)^{K((K+1)^{t-1} + (K+1)^{t-2} + \dots + (K+1)^s)}$$

$$> \left(1 + \frac{1}{t^2}\right)^{(K+1)^t} \left(\prod_{j=1}^{t-1} a_{1,j}\right)^K \left(\prod_{j=1}^{s-1} a_{1,j}\right)^{-K}$$

$$> \left(1 + \frac{1}{t^2}\right)^{(K+1)^t} \left(\prod_{j=1}^{t-1} a_{1,j}\right)^K 2^{-Kt^2}. \tag{44}$$

As in the third case Lemma 1 and Lemma 2 imply (35) for our definition of the number t.

Finally from (20), (23), (35), (37), (42), (43), (44) we obtain

$$1 \leq Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^K 2^{(K-1)\sum_{n=1}^{t-1}(\log_2 a_{1,n})^\alpha} \sum_{i=1}^{K} \sum_{n=t}^{\infty} \frac{b_{i,n}}{a_{i,n}}$$

$$\leq Q\left(\prod_{n=1}^{t-1} a_{1,n}\right)^K 2^{(K-1)\sum_{n=1}^{t-1}(\log_2 a_{1,n})^\alpha} \left(\frac{2^{(\log_2 a_{1,t})^\gamma}}{a_{1,t}} + \frac{1}{a_{1,k}^B}\right)$$

$$= \frac{Q(\prod_{n=1}^{t-1} a_{1,n})^K 2^{(K-1)\sum_{n=1}^{t-1}(\log_2 a_{1,n})^\alpha} 2^{(\log_2 a_{1,t})^\gamma}}{a_{1,t}}$$

$$+ \frac{Q(\prod_{n=1}^{t-1} a_{1,n})^K 2^{(K-1)\sum_{n=1}^{t-1}(\log_2 a_{1,n})^\alpha}}{a_{1,k}^B}$$

$$\leq \frac{Q 2^{E(K+1)^{\alpha t}} 2^{(\log_2 a_{1,t})^\gamma}}{(1+1/t^2)^{(K+1)^t} 2^{-Kt^2}} + \frac{Q 2^{D(K+1)^t}}{a_{1,k}^B}$$

$$\leq \frac{Q 2^{E(K+1)^{\alpha t}} 2^{(\log_2 2^{(K+1+\delta)^t})^\gamma}}{(1+1/t^2)^{(K+1)^t} 2^{-Kt^2}} + \frac{Q 2^{D(K+1)^t}}{2^{(\log_2 A) B (K+1)^k}}$$

$$\leq \frac{2^{2E(K+1)^{\alpha t}} 2^{(K+1+\delta)^{\gamma t}}}{(1+1/t^2)^{(K+1)^t}} + \frac{2^{2D(K+1)^t}}{2^{(\log_2 A) B (K+1)^k}}.$$

Thus

$$1 \leq 2^{2E(K+1)^{\alpha t} + (K+1+\delta)^{\gamma t} - (\log_2(1+1/t^2))(K+1)^t} + 2^{2D(K+1)^t - (\log_2 A) B (K+1)^k}.$$

This is a contradiction for a sufficiently large t, k and A and for sufficiently small δ. Now the proof of Theorem 5 is complete. □

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References

[1] R. Apery: Irrationality of $\zeta(2)$ and $\zeta(3)$, Soc. Math. France, Asterisque **61** (1979), 11–13.
[2] C. Badea: The irrationality of certain infinite series, Glasgow Math. J. **29** (1987), 221–228.
[3] C. Badea: A theorem on irrationality of infinite series and applications, Acta arithmetica, LXIII. 4 (1993), 313–323.
[4] D. Duverney: A criterion of irrationality, Portugaliae Math., vol. 53, Farc. 2 (1996), 229–237.
[5] P. Erdős: Some problems and results on the irrationality of the sum of infinite series, Journal of Mathematical Sciences, **10** (1975), 1–7.
[6] P. Erdős and E. G. Straus: On the irrationality of certain Ahmes series, J. Indian Math. Soc. **27** (1968), 129–133.
[7] P. Erdős and E. G. Straus: On the irrationality of certain series, Pacific journal of mathematics, Vol. 55, No. 1 (1974), 85–92.
[8] J. Hančl: Expression of real numbers with the help of infinite series, Acta Arithmetica LIX. 2 (1991), 97–104.
[9] J. Hančl: Criterion for Irrational Sequences, Journal of Number Theory, Vol. 43, No. 1 (1993), 88–92.
[10] J. Hančl: Linearly unrelated sequences, Pacific Journal of Mathematics, Vol. 190, No. 2 (1999), 299–310.

[11] M. Huttner: Irrationalite de certaines integrales hypergeometriques, J. Num. Theory **26** (1987), 166–178.

[12] K. Nishioka: Mahler functions and transcendence, Lecture notes in Mathematics 1631, Springer (1996).

[13] A. Oppenheim: Criteria for irrationality of certain classes of numbers, Amer. Math. Monthly 61, No. 4 (1954), 235–241.

Department of Mathematics
University of Ostrava
Dvořákova 7, 701 03, Ostrava 1, Czech republic
e-mail: hancl@osu.cz, sobkova@osu.cz