

A REMARK ON WELL-POSEDNESS FOR HYPERBOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

By

Daniele DEL SANTO and Martino PRIZZI

Abstract. We prove some C^∞ and Gevrey well-posedness results for hyperbolic equations with singular coefficients.

1. Introduction

This work is devoted to the study of the well-posedness of the Cauchy problem for a linear hyperbolic operator whose coefficients depend only on time.

We consider the equation

$$(1.1) \quad u_{tt} - \sum_{i,j=1}^n a_{ij}(t)u_{x_i x_j} = 0$$

in $[0, T] \times \mathbf{R}^n$, with initial data

$$(1.2) \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x)$$

in \mathbf{R}^n . The matrix (a_{ij}) is supposed to be real and symmetric. Setting

$$(1.3) \quad a(t, \xi) := \sum_{i,j=1}^n a_{ij}(t)\xi_i \xi_j / |\xi|^2, \quad (t, \xi) \in [0, T] \times (\mathbf{R}^n \setminus \{0\}),$$

we assume that $a(\cdot, \xi) \in L^1(0, T)$ for all $\xi \in \mathbf{R}^n \setminus \{0\}$.

We suppose that the equation (1.1) is hyperbolic i.e.

$$(1.4) \quad a(t, \xi) \geq \lambda_0 \geq 0$$

for all $(t, \xi) \in [0, T] \times (\mathbf{R}^n \setminus \{0\})$.

In the strictly hyperbolic case (i.e. $\lambda_0 > 0$) it is well known that if the

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coefficients a_{ij} are Lipschitz-continuous then the Cauchy problem (1.1), (1.2) is well-posed in Sobolev spaces. In the same case if the a_{ij} 's are Log-Lipschitz-continuous or Hölder-continuous of index α , (1.1), (1.2) is well-posed in C^∞ or in the Gevrey space $\gamma^{(s)}$ for $s < 1/(1 - \alpha)$ respectively (see [1]). In the weakly hyperbolic case (i.e. $\lambda_0 = 0$) if the coefficients are $C^{k,\alpha}$ then the problem (1.1), (1.2) is $\gamma^{(s)}$ -well-posed for $s < 1 + (k + \alpha)/2$ (see [4]). Some counter examples show that all these results are sharp (see also [5]).

Recently Colombini, Del Santo and Kinoshita have considered the same problem for operators having coefficients which are C^1 on $[0, T] \setminus \{t_0\}$ with a singularity concentrated at t_0 . In this situation, under the main assumptions that

$$(1.5) \quad \begin{aligned} |t_0 - \cdot|^p a'(\cdot, \xi) &= \beta(\cdot, \xi) \in L^\infty(0, T) \quad \text{for all } \xi \in \mathbf{R}^n \setminus \{0\} \\ |t_0 - \cdot|^r a(\cdot, \xi) &= \alpha(\cdot, \xi) \in L^\infty(0, T) \quad \text{for all } \xi \in \mathbf{R}^n \setminus \{0\} \end{aligned}$$

it is possible to show that the Cauchy problem (1.1), (1.2) is $\gamma^{(s)}$ -well-posed, the value of s depending on p and r (see [2] and [3]) (here and in the following “ \cdot ” denotes the differentiation with respect to t).

The aim of the present work is to improve the results of [2] and [3] allowing the function β in (1.5) to be in a L^q space and removing the growth assumption on a . We make the following assumptions: let $1 \leq q \leq +\infty$ and $p \geq 0$ and let $t_0 \in [0, T]$; suppose that

$$\begin{aligned} \text{(H1)} \quad &a(\cdot, \xi) \in \bigcap_{\varepsilon > 0} W^{1,1}([0, t_0 - \varepsilon] \cup [t_0 + \varepsilon, T]) \quad \text{for all } \xi \in \mathbf{R}^n \setminus \{0\}; \\ \text{(H2)} \quad &|t_0 - \cdot|^p a'(\cdot, \xi) = \beta(\cdot, \xi) \in L^q(0, T) \quad \text{for all } \xi \in \mathbf{R}^n \setminus \{0\}. \end{aligned}$$

In the weakly hyperbolic case the results are the following.

THEOREM 1. *Assume that $3 \leq (p + 1/q)$. Then the Cauchy problem (1.1), (1.2) is $\gamma^{(\sigma)}$ -well-posed for $1 \leq \sigma < \frac{(p+1/q)-\frac{3}{2}}{(p+1/q)-2}$. If moreover*

$$(1.6) \quad |t_0 - \cdot|^r a(\cdot, \xi) = \alpha(\cdot, \xi) \in L^s(0, T) \quad \text{for all } \xi \in \mathbf{R}^n \setminus \{0\},$$

with $r \geq 0$, $1 \leq s \leq +\infty$ and $(r + 1/s) \leq 1$, then the Cauchy problem (1.1), (1.2) is $\gamma^{(\sigma)}$ -well-posed for $1 \leq \sigma < \frac{(p+1/q)-\frac{3}{2}(r+1/s)}{(p+1/q)-(r+1/s)-1}$.

THEOREM 2. *Assume that $(p + 1/q) < 3$. Then the Cauchy problem (1.1), (1.2) is $\gamma^{(\sigma)}$ -well-posed for all $1 \leq \sigma < \frac{3}{2}$.*

The result concerning the strictly hyperbolic case are contained in the following theorems.

THEOREM 3. *Assume that $1 < (p + 1/q) < 3$. Moreover, assume that $\lambda_0 > 0$. Then the Cauchy problem (1.1), (1.2) is $\gamma^{(\sigma)}$ -well-posed for all $1 \leq \sigma < \frac{(p+1/q)}{(p+1/q)-1}$.*

THEOREM 4. *Assume that $(p + 1/q) \leq 1$. Moreover, assume that $\lambda_0 > 0$. Then the Cauchy problem (1.1), (1.2) is C^∞ -well-posed.*

REMARK 1. Adapting to the present situation some counter examples contained in [4], [2], and [3] it is possible to see that the results of Theorems 1–4 are optimal. Let us show this in some detail in the case of Theorem 1. Suppose $p_0 + 1/q_0 = 3$. In this case $\frac{(p_0+1/q_0)-\frac{3}{2}}{(p_0+1/q_0)-2} = \frac{(p_0+1/q_0)-\frac{3}{2}(r_0+1/s_0)}{(p_0+1/q_0)-(r_0+1/s_0)-1} = \frac{3}{2}$; consequently Theorem 2 in [4] shows that this value of the Gevrey index cannot be improved. Consider next the case that $p_0 + 1/q_0 > 3$ and $(r_0 + 1/s_0) \leq 1$. Let $\bar{\sigma} > \sigma_0 = \frac{(p_0+1/q_0)-\frac{3}{2}(r_0+1/s_0)}{(p_0+1/q_0)-(r_0+1/s_0)-1}$. We fix $q_1 > q_0$ and $s_1 > s_0$ in such a way that $p_0 + 1/q_1 > 3$, $r_0 + 1/s_1 < 1$ and $\sigma_0 < \sigma_1 := \frac{(p_0+1/q_1)-\frac{3}{2}(r_0+1/s_1)}{(p_0+1/q_1)-(r_0+1/s_1)-1} < \bar{\sigma}$. From Theorem 4 in [3] we have that there exists a function $a : [0, 1[\rightarrow [1/2, +\infty[$ such that $a \in C^\infty([0, 1])$ and

$$(1 - t)^{p_0+1/q_1} a'(t) \in L^\infty, \quad (1 - t)^{r_0+1/s_1} a(t) \in L^\infty,$$

and there exist $u_0, u_1 \in \gamma^{(\sigma)}$ for all $\sigma > \sigma_1$ such that the Cauchy problem

$$(1.7) \quad u_{tt} - a(t)u_{xx} = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),$$

has no solution in $W^{2,1}([0, 1], \mathcal{D}'^{(\sigma)}(\mathbf{R}))$ for all $\sigma > \sigma_1$. Consequently

$$(1 - t)^{p_0} a'(t) \in L^{q_0}, \quad (1 - t)^{r_0} a(t) \in L^{s_0},$$

$u_0, u_1 \in \gamma^{(\bar{\sigma})}$ and the Cauchy problem (1.7) does not have a solution in $W^{2,1}([0, 1], \mathcal{D}'^{(\bar{\sigma})}(\mathbf{R}))$.

REMARK 2. Let us remark that Theorem 1 is a nontrivial improvement of Theorem 2 in [3] also in the case of $q = \infty$. In fact the growth condition on a is removed and the result is sharp (see [3, Th. 4]).

2. Proof of Theorems 1–4

As a preliminary step, let us observe that, since the coefficients a_{ij} are real integrable functions, the Cauchy problem (1.1), (1.2) is well posed in $\mathcal{A}'(\mathbf{R}^n)$, the space of real analytic functionals. Moreover, if the initial data vanish in a ball, then the solution vanishes in a cone, whose slope depends on the coefficients a_{ij} .

Therefore it will be sufficient to show that, under the hypotheses of each theorem, if u_0 and u_1 have compact support then the corresponding solution u is not only in $W^{2,1}([0, T], \mathcal{A}'(\mathbf{R}^n))$, but it belongs to a Gevrey space in the x variable. Our main tools in doing this will be the Paley-Wiener theorem (in the version of [1, p. 517], to which we refer here and throughout) and some energy estimates.

Denoting by v the Fourier transform of u with respect to x , equation (1.1) reads

$$(2.1) \quad v''(t, \xi) + a(t, \xi)|\xi|^2 v(t, \xi) = 0.$$

Let ε be a positive parameter and for each ε let $a_\varepsilon : [0, T] \times (\mathbf{R}^n \setminus \{0\}) \rightarrow \mathbf{R}$ be a strictly positive real function such that $a_\varepsilon(\cdot, \xi) \in W^{1,1}(0, T)$ for all $\xi \in \mathbf{R}^n \setminus \{0\}$. We define the *approximate energy* of v by

$$(2.2) \quad E_\varepsilon(t, \xi) := a_\varepsilon(t, \xi)|\xi|^2 |v(t, \xi)|^2 + |v'(t, \xi)|^2, \quad (t, \xi) \in [0, T] \times (\mathbf{R}^n \setminus \{0\}).$$

Differentiating E_ε with respect to t and using (2.1) we get

$$\begin{aligned} E'_\varepsilon(t, \xi) &= a'_\varepsilon(t, \xi)|\xi|^2 |v(t, \xi)|^2 + 2a_\varepsilon(t, \xi)|\xi|^2 \operatorname{Re}(v'(t, \xi)\bar{v}(t, \xi)) \\ &\quad + 2 \operatorname{Re}(v''(t, \xi)\bar{v}'(t, \xi)) \\ &\leq \left(\frac{|a'_\varepsilon(t, \xi)|}{a_\varepsilon(t, \xi)} + \frac{|a_\varepsilon(t, \xi) - a(t, \xi)|}{a_\varepsilon(t, \xi)^{1/2}} |\xi| \right) E_\varepsilon(t, \xi). \end{aligned}$$

By Gronwall's lemma we obtain

$$(2.3) \quad E_\varepsilon(t, \xi) \leq E_\varepsilon(0, \xi) \exp \left(\int_0^T \frac{|a'_\varepsilon(t, \xi)|}{a_\varepsilon(t, \xi)} dt + |\xi| \int_0^T \frac{|a_\varepsilon(t, \xi) - a(t, \xi)|}{a_\varepsilon(t, \xi)^{1/2}} dt \right)$$

for all $t \in [0, T]$ and for all $\xi \in \mathbf{R}^n$, $|\xi| \geq 1$.

Now we are able to give the

PROOF OF THEOREM 1. First of all, observe that condition (1.6) is always satisfied at least with $r = 0$ and $s = 1$ (recall that $a_{ij} \in L^1(0, T)$).

Since $u_0, u_1 \in \gamma^{(\sigma)} \cap C_0^\infty$, the Paley-Wiener theorem ensures that there exist $M, \delta > 0$ such that

$$(2.4) \quad |v(0, \xi)|^2 + |v'(0, \xi)|^2 \leq M \exp(-\delta|\xi|^{1/\sigma})$$

for all $\xi \in \mathbf{R}^n$, $|\xi| \geq 1$. To verify that $u \in W^{2,1}([0, T], \gamma^{(\sigma)})$ it is sufficient to show that there exist $M', \delta' > 0$ such that

$$(2.5) \quad |v(t, \xi)|^2 + |v'(t, \xi)|^2 \leq M' \exp(-\delta' |\xi|^{1/\sigma})$$

for all $t \in [0, T]$ and for all $\xi \in \mathbf{R}^n$, $|\xi| \geq 1$. We consider first the case $t_0 = T$. For $\varepsilon \in]0, T]$, we set

$$(2.6) \quad a_\varepsilon(t, \xi) := \begin{cases} a(t, \xi) + \varepsilon^{2-(r+1/s)}(T-t)^{-2} & \text{for } 0 \leq t \leq T - \varepsilon \\ \varepsilon^{-(z+r)}a(t, \xi)(T-t)^{z+r} + \varepsilon^{-(r+1/s)} & \text{for } T - \varepsilon \leq t \leq T \end{cases}$$

where z is any positive number such that

$$(2.7) \quad z > \max\{1/s, (p + 1/q) - r - 1\}.$$

Then

$$(2.8) \quad a_\varepsilon(t, \xi) = \begin{cases} \alpha(t, \xi)(T-t)^{-r} + \varepsilon^{2-(r+1/s)}(T-t)^{-2} & \text{for } 0 \leq t \leq T - \varepsilon \\ \varepsilon^{-(z+r)}\alpha(t, \xi)(T-t)^z + \varepsilon^{-(r+1/s)} & \text{for } T - \varepsilon \leq t \leq T \end{cases}$$

and

$$(2.9) \quad a'_\varepsilon(t, \xi) = \begin{cases} \beta(t, \xi)(T-t)^{-p} - 2\varepsilon^{2-(r+1/s)}(T-t)^{-3} & \text{for } 0 \leq t \leq T - \varepsilon \\ \varepsilon^{-(z+r)}(\beta(t, \xi)(T-t)^{z+r-p} - (z+r)\alpha(t, \xi)(T-t)^{z-1}) & \text{for } T - \varepsilon \leq t \leq T \end{cases}$$

Our choice of z implies that $a_\varepsilon(\cdot, \xi) \in W^{1,1}(0, T)$ for all $\xi \in \mathbf{R}^n \setminus \{0\}$. By (2.8) and (2.9) we get

$$\begin{aligned} \int_0^T \frac{|a'_\varepsilon(t, \xi)|}{a_\varepsilon(t, \xi)} dt &\leq \int_0^{T-\varepsilon} \frac{|\beta(t, \xi)|(T-t)^{-p}}{\varepsilon^{2-(r+1/s)}(T-t)^{-2}} dt \\ &\quad + \int_0^{T-\varepsilon} \frac{2\varepsilon^{2-(r+1/s)}(T-t)^{-3}}{\varepsilon^{2-(r+1/s)}(T-t)^{-2}} dt \\ &\quad + \int_{T-\varepsilon}^T \frac{\varepsilon^{-(z+r)}|\beta(t, \xi)|(T-t)^{z+r-p}}{\varepsilon^{-(r+1/s)}} dt \\ &\quad + \int_{T-\varepsilon}^T \frac{\varepsilon^{-(z+r)}(z+r)|\alpha(t, \xi)|(T-t)^{z-1}}{\varepsilon^{-(r+1/s)}} dt \end{aligned}$$

The choice of z allows us to use Hölder inequality; an easy computation shows that

$$(2.10) \quad \int_0^T \frac{|a'_\varepsilon(t, \xi)|}{a_\varepsilon(t, \xi)} dt \leq C'(1 + |\log \varepsilon|)\varepsilon^{-(p+1/q)+(r+1/s)+1},$$

where C' is a constant depending only on C, r, s, p, q and z . On the other hand,

$$\begin{aligned} \int_0^T \frac{|a_\varepsilon(t, \xi) - a(t, \xi)|}{a_\varepsilon(t, \xi)^{1/2}} dt &= \int_0^{T-\varepsilon} \frac{\varepsilon^{2-(r+1/s)}(T-t)^{-2}}{\varepsilon^{1-(1/2)(r+1/s)}(T-t)^{-1}} dt \\ &+ \int_{T-\varepsilon}^T \frac{\varepsilon^{-(z+r)}\alpha(t, \xi)(T-t)^z}{\varepsilon^{-(1/2)(r+1/s)}} dt \\ &+ \int_{T-\varepsilon}^T \frac{\varepsilon^{-(r+1/s)}}{\varepsilon^{-(1/2)(r+1/s)}} dt + \int_{T-\varepsilon}^T \frac{\alpha(t, \xi)(T-t)^{-r}}{\varepsilon^{-(1/2)(r+1/s)}} dt. \end{aligned}$$

The first three summands on the right hand side can be estimated again by using Hölder inequality. In order to estimate the fourth summand, we shall distinguish the case $(r+1/s) < 1$ and $(r+1/s) = 1$. In the first case, we use once more Hölder inequality; in the second case, we use the fact that $\alpha(t, \xi)(T-t)^{-r} = a(t, \xi) \in L^1(0, T)$. At the end, we get

$$(2.11) \quad \int_0^T \frac{|a_\varepsilon(t, \xi) - a(t, \xi)|}{a_\varepsilon(t, \xi)^{1/2}} dt \leq C''(1 + |\log \varepsilon|)\varepsilon^{-(1/2)(r+1/s)+1},$$

where C'' is a constant depending only on C, r, s, p, q and z . By (2.3), (2.10) and (2.11) we obtain

$$(2.12) \quad E(t, \xi) \leq E(0, \xi) \exp(\tilde{C}(1 + |\log \varepsilon|)(\varepsilon^{-(p+1/q)+(r+1/s)+1} + |\xi|\varepsilon^{-(1/2)(r+1/s)+1}))$$

for all $t \in [0, T]$ and for all $\xi \in \mathbf{R}^n$, $\xi \geq 1$, where \tilde{C} is a positive constant depending only on C, r, s, p, q and z .

Now, by (2.2) and (2.6), we have

$$(2.13) \quad E_\varepsilon(0, \xi) \leq (a(0, \xi) + T^{-(r+1/s)})|\xi|^2|v(0, \xi)|^2 + |v'(0, \xi)|^2$$

and

$$(2.14) \quad E_\varepsilon(t, \xi) \geq T^{-2}\varepsilon^{2-(r+1/s)}|\xi|^2|v(t, \xi)|^2 + |v'(t, \xi)|^2.$$

Then choosing $\varepsilon := |\xi|^{-[(p+1/q)-(3/2)(r+1/s)]^{-1}}$ we deduce

$$\begin{aligned} &T^{-2}|\xi|^{2-(2-(r+1/s))/((p+1/q)-(3/2)(r+1/s))}|v(t, \xi)|^2 + |v'(t, \xi)|^2 \\ &\leq (\tilde{K}|\xi|^2|v(0, \xi)|^2 + |v'(0, \xi)|^2) \\ &\quad \times \exp(\tilde{C}(1 + |\log|\xi||)|\xi|^{((p+1/q)-(r+1/s)-1)/((p+1/q)-(3/2)(r+1/s))}). \end{aligned}$$

Using the Paley-Wiener theorem, the well-posedness follows for all $1 \leq \sigma < \frac{(p+1/q)-\frac{3}{2}(r+1/s)}{(p+1/q)-(r+1/s)-1}$.

If $t_0 = 0$, for $\varepsilon \in]0, T]$ we set

$$(2.15) \quad a_\varepsilon(t, \xi) := \begin{cases} \varepsilon^{-(z+r)} a(t, \xi) t^{z+r} + \varepsilon^{-(r+1/s)} & \text{for } 0 \leq t \leq \varepsilon \\ a(t, \xi) + \varepsilon^{2-(r+1/s)} t^{-2} & \text{for } \varepsilon \leq t \leq T \end{cases}$$

where z satisfies (2.7). Our choice of z implies that $a_\varepsilon(\cdot, \xi) \in W^{1,1}(0, T)$ for all $\xi \in \mathbf{R}^n \setminus \{0\}$. So, in particular, $a_\varepsilon(\cdot, \xi)$ is continuous on $[0, T]$. Arguing as before, we obtain (2.12). An easy computation shows that $|a(t, \xi)| \leq \tilde{K} t^{1-(p+1/q)}$ for all $\xi \in \mathbf{R}^n \setminus \{0\}$. It follows that

$$\begin{aligned} a_\varepsilon(0, \xi) &= \lim_{\tau \rightarrow 0} a_\varepsilon(\tau, \xi) = \lim_{\tau \rightarrow 0} (\varepsilon^{-(z+r)} a(\tau, \xi) \tau^{z+r} + \varepsilon^{-(r+1/s)}) \\ &\leq \tilde{K} \limsup_{\tau \rightarrow 0} (\varepsilon^{-(z+r)} \tau^{z+r+1-(p+1/q)} + \varepsilon^{-(r+1/s)}). \end{aligned}$$

By (2.7) we deduce that $a_\varepsilon(0, \xi) \leq \tilde{K} \varepsilon^{-(r+1/s)}$. It follows that

$$(2.16) \quad E_\varepsilon(0, \xi) \leq \tilde{K} \varepsilon^{-(r+1/s)} |\xi|^2 |v(0, \xi)|^2 + |v'(0, \xi)|^2.$$

Moreover, we have also

$$(2.17) \quad E_\varepsilon(t, \xi) \geq T^{-2} \varepsilon^{2-(r+1/s)} |\xi|^2 |v(t, \xi)|^2 + |v'(t, \xi)|^2.$$

Then, choosing again $\varepsilon := |\xi|^{-[(p+1/q)-(3/2)(r+1/s)]^{-1}}$, we deduce

$$\begin{aligned} &|\xi|^{2-(2-(r+1/s))/((p+1/q)-(3/2)(r+1/s))} |v(t, \xi)|^2 + |v'(t, \xi)|^2 \\ &\leq (\tilde{K} |\xi|^{2+(r+1/s)/((p+1/q)-(3/2)(r+1/s))} |v(0, \xi)|^2 \\ &\quad + |v'(0, \xi)|^2) \exp(\tilde{C}(1 + |\log|\xi||) |\xi|^{((p+1/q)-(r+1/s)-1)/((p+1/q)-(3/2)(r+1/s))}). \end{aligned}$$

Using the Paley-Wiener theorem, the well-posedness follows again for all $1 \leq \sigma < \frac{(p+1/q)-\frac{3}{2}(r+1/s)}{(p+1/q)-(r+1/s)-1}$.

Finally, if $t_0 \in]0, T[$, it will be sufficient to solve first the Cauchy problem in $[0, t_0]$, then to solve the problem in $[t_0, T]$ with the initial data obtained from the previous one and finally to glue together the two solutions. \square

In order to prove Theorem 2, we proceed exactly like in the proof of Theorem 1. In this case the role of condition (1.6) is played by the estimate

$$(2.18) \quad a(t, \xi) \leq C' |t - t_0|^{-(p+1/q)+1} \quad \text{for all } \xi \in \mathbf{R}^n \setminus \{0\},$$

which is a direct consequence of condition (H2). The function $a_\varepsilon(\cdot, \xi)$ is defined by

$$(2.19) \quad a_\varepsilon(t, \xi) := \begin{cases} a(t, \xi) + \varepsilon^{3-(p+1/q)} (T - t)^{-2} & \text{for } 0 \leq t \leq T - \varepsilon \\ a(T - \varepsilon, \xi) + \varepsilon^{1-(p+1/q)} & \text{for } T - \varepsilon \leq t \leq T \end{cases}$$

if $t_0 = T$ and by

$$(2.20) \quad a_\varepsilon(t, \xi) := \begin{cases} a(\varepsilon, \xi) + \varepsilon^{1-(p+1/q)} & \text{for } 0 \leq t \leq \varepsilon \\ a(t, \xi) + \varepsilon^{3-(p+1/q)}t^{-2} & \text{for } \varepsilon \leq t \leq T \end{cases}$$

if $t_0 = 0$. Arguing like in the proof of Theorem 1, we get

$$(2.21) \quad \int_0^T \frac{|a'_\varepsilon(t, \xi)|}{a_\varepsilon(t, \xi)} dt \leq C''(1 + |\log \varepsilon|)\varepsilon^{(p+1/q)-3},$$

and

$$(2.22) \quad \int_0^T \frac{|a_\varepsilon(t, \xi) - a(t, \xi)|}{a_\varepsilon(t, \xi)^{1/2}} dt \leq C''(1 + |\log \varepsilon|)\varepsilon^{-(1/2)(p+1/q)+3/2}$$

and the conclusion follows by choosing $\varepsilon := |\xi|^{-(2/3)[3-(p+1/q)]^{-1}}$.

Theorem 3 is the strictly hyperbolic version of Theorem 2. We define again a_ε by (2.19) and (2.20), but in this case the positive lower bound for $a(t, \xi)$ allows us to obtain better estimates for $\int_0^T \frac{|a'_\varepsilon(t, \xi)|}{a_\varepsilon(t, \xi)} dt$. Let us consider, for example, the case $t_0 = T$. First observe that, by rescaling the x variable if necessary, we can always assume that $\lambda_0 = 1$. Then we can minorize $a_\varepsilon(t, \xi)$ by the constant 1 on $[0, T - \varepsilon^{(1/2)[3-(p+1/q)}]$ and by $\varepsilon^{3-(p+1/q)}(T - t)^{-2}$ on $[T - \varepsilon^{(1/2)[3-(p+1/q)}], T - \varepsilon]$. So we obtain that

$$(2.23) \quad \int_0^T \frac{|a'_\varepsilon(t, \xi)|}{a_\varepsilon(t, \xi)} dt \leq C''(1 + |\log \varepsilon|)\varepsilon^{(1/2)((p+1/q)-1)((p+1/q)-3)}.$$

The conclusion follows by choosing $\varepsilon := |\xi|^{-2[p+1/q]^{-1}[3-(p+1/q)]^{-1}}$.

Finally, we give the

PROOF OF THEOREM 4. Since $u_0, u_1 \in C_0^\infty$, the Paley-Wiener theorem ensures that for all $\zeta > 0$ there exists $M_\zeta > 0$ such that

$$(2.24) \quad |v(0, \xi)|^2 + |v'(0, \xi)|^2 \leq M_\zeta |\xi|^{-\zeta}$$

for all $\xi \in \mathbf{R}^n$, $|\xi| \geq 1$. To verify that $u \in W^{2,1}([0, T], C_0^\infty)$ it is sufficient to show that for all $\eta > 0$ there exists $M_\eta > 0$ such that

$$(2.25) \quad |v(t, \xi)|^2 + |v'(t, \xi)|^2 \leq M_\eta |\xi|^{-\eta}$$

for all $t \in [0, T]$ and for all $\xi \in \mathbf{R}^n$, $|\xi| \geq 1$. We give the details only in the case $t_0 = T$. If $q = 1$, then necessarily $p = 0$. This means that $a(\cdot, \xi) \in W^{1,1}(0, T)$ and it is well known that this is enough to detect C^∞ -well-posedness of the Cauchy problem (1.1), (1.2). If $q > 1$, for $\varepsilon \in]0, T]$, we set

$$(2.26) \quad a_\varepsilon(t, \xi) := \begin{cases} a(t, \xi) & \text{for } 0 \leq t \leq T - \varepsilon \\ a(T - \varepsilon, \xi) & \text{for } T - \varepsilon \leq t \leq T \end{cases}$$

Now observe that

$$\begin{aligned} |a(t, \xi)| &\leq |a(0, \xi)| + \int_0^t |a'(\tau, \xi)| \, d\tau \leq |a(0, \xi)| + \int_0^t \beta(\tau, \xi)(T - \tau)^{-p} \, d\tau \\ &\leq |a(0, \xi)| + \|\beta(\cdot, \xi)\|_{L^q} \left(\int_0^t (T - \tau)^{-pq'} \, d\tau \right)^{1/q'} \leq C(1 + |\log(T - t)|^{1/q'}) \end{aligned}$$

An easy computation shows that

$$(2.27) \quad \int_0^T |a'_\varepsilon(t, \xi)| \, dt \leq C' |\log \varepsilon|^{1/q'}$$

and

$$(2.28) \quad \int_0^T |a_\varepsilon(t, \xi) - a(t, \xi)| \, dt \leq C' \varepsilon |\log \varepsilon|^{1/q'}$$

Then we deduce by (2.3) that

$$(2.29) \quad \begin{aligned} &|\xi|^2 |v(t, \xi)|^2 + |v'(t, \xi)|^2 \\ &\leq (a(0, \xi) |\xi|^2 |v(0, \xi)|^2 + |v'(0, \xi)|^2) \\ &\quad \times \exp(C' |\log \varepsilon|^{1/q'} + C' |\xi| \varepsilon |\log \varepsilon|^{1/q'}). \end{aligned}$$

Here, for simplicity, we have assumed that $\lambda_0 = 1$. Choosing $\varepsilon := |\xi|^{-1}$, we obtain

$$(2.30) \quad \begin{aligned} &|\xi|^2 |v(t, \xi)|^2 + |v'(t, \xi)|^2 \\ &\leq (a(0, \xi) |\xi|^2 |v(0, \xi)|^2 + |v'(0, \xi)|^2) \exp(C' |\log |\xi||^{1/q'}). \end{aligned}$$

Now, for $|\xi| \geq e$, we have $|\log |\xi||^{1/q'} \leq |\log |\xi||$, and hence

$$(2.31) \quad |\xi|^2 |v(t, \xi)|^2 + |v'(t, \xi)|^2 \leq (a(0, \xi) |\xi|^2 |v(0, \xi)|^2 + |v'(0, \xi)|^2) |\xi|^{C'}$$

By the Paley-Wiener theorem, the well-posedness in C_0^∞ follows. □

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Daniele Del Santo
Università di Trieste
Dipartimento di Scienze Matematiche
Via Valerio 12/1, 34127 Trieste
Italy
E-mail address: delsanto@univ.trieste.it

Martino Prizzi
Università di Trieste
Dipartimento di Scienze Matematiche
Via Valerio 12/1, 34127 Trieste
Italy
E-mail address: prizzi@mathsun1.univ.trieste.it