TSUKUBA J. MATH.
Vol. 26 No. 1 (2002), 61–68

INTEGRAL GEOMETRY ON PRODUCT OF SPHERES

By

Hong Jae KANG

1. Introduction and Result

One of the oldest results in integral geometry is the Poincaré formula for the average of the intersection number of two curves. Many differential geometers have studied the Poincaré formula from various points of view. In particular, R. Howard [1] generalized this formula in Riemannian homogeneous spaces and obtained the following formula.

THEOREM 1.1 [1]. *Let G/K be a Riemannian homogeneous space with a G-invariant Riemannian metric, and let M and N be submanifolds of G/K with* $\dim M + \dim N = \dim(G/K)$. *Assume that G is unimodular and for almost all $g \in G$, M and gN intersect transversely. Then*

$$\int_G \sharp(M \cap gN)\, d\mu_G(g) = \int_{M\times N} \sigma_K(T_x^\perp M, T_y^\perp N)\, d\mu_{M\times N}(x, y),$$

where $\sharp(X)$ denotes the number of points in X and $\sigma_K(T_x^\perp M, T_y^\perp N)$ is defined by (2.1) *below.*

This theorem plays an important role in this paper. In the case that $G/K = \boldsymbol{R}^2$, this formula implies the classical Poincaré's one. In the case that G/K is a space of constant curvature, the isotropy group K acts transitively on the Grassmann manifolds consisting of subspaces in $T_o(G/K)$, so $\sigma_K(T_x^\perp M, T_y^\perp N)$ on the right side of the above integral in Theorem 1.1 is constant. Namely, $\sigma_K(V, W)$ is independent on V and W. Hence we can have clearly expressed $\sigma_K(T_x^\perp M, T_y^\perp N)$, that is,

$$\sigma_{SO(n)}(T_x^\perp M^p, T_y^\perp N^q) = \frac{\mathrm{vol}(S^0)\,\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\,\mathrm{vol}(S^q)}.$$

Received August 7, 2000.
Revised August 16, 2001.

In the case that G/K is a two-point homogeneous space of dimension n, Howard [1] showed that

$$\sigma_K(T_x^\perp M^1, T_y^\perp N^{n-1}) = \frac{\mathrm{vol}(K)\,\mathrm{vol}(S^0)\,\mathrm{vol}(S^n)}{\mathrm{vol}(S^1)\,\mathrm{vol}(S^{n-1})}.$$

Although Theorem 1.1 holds in a general situation, $\sigma_K(T_x^\perp M, T_y^\perp N)$ is complicated generally, and is not in a concrete enough form to be easily used. Moreover, unfortunately, there exist few results of the concrete calculation for $\sigma_K(T_x^\perp M, T_y^\perp N)$. For example in the case that G/K is an n-dimensional complex projective space $\boldsymbol{C}P^n$, for any real surfaces M and any complex hypersurfaces N, the author and Tasaki [2] gave

$$\sigma_{U(1)\times U(n)}(T_x^\perp M, T_y^\perp N) = \frac{\mathrm{vol}(U(n+1))}{2\,\mathrm{vol}(\boldsymbol{C}P^1)\,\mathrm{vol}(\boldsymbol{C}P^{n-1})}(1+\cos^2\theta_x)$$

using the Kähler angle θ_x of M at x. Recently, for any two real surfaces M and N in $\boldsymbol{C}P^2$, they evaluated the following in [3].

$$\sigma_{U(1)\times U(2)}(T_x^\perp M, T_y^\perp N) = \frac{\mathrm{vol}(U(3))}{\mathrm{vol}(\boldsymbol{R}P^2)^2}(2+2\cos^2\theta_x\cos^2\tau_y+\sin^2\theta_x\sin^2\tau_y)$$

As for other the concrete results for $\sigma_K(T_x^\perp M, T_y^\perp N)$, we know of no example yet. For details see [1] and [5].

In this paper, under this motivation, we shall explicitly describe $\sigma_K(\cdot\,,\cdot)$ and formulate a Poincaré formula for one-dimensional and 3-dimensional submanifolds in the product of unit sphere S^2. The purpose of this paper is to prove the following:

THEOREM 1.2. *Let M be a submanifold of $S^2\times S^2$ of dimension* 1 *and N a submanifold of dimension* 3. *Assume that for almost all $g\in SO(3)\times SO(3)$, M and gN intersect transversely. For any point $x\in M$* (resp. $y\in N$), $\sin\theta_x$ *and* $\cos\theta_x$ (resp. $\sin\tau_y$ *and* $\cos\tau_y$) *denote the length of the first and second component of unit vector $u_x=(u_1,u_2)$* (resp. $v_y=(v_1,v_2)$) *of T_xM* (resp. $T_y^\perp N$), *respectively. Then we have*

$$\int_{SO(3)\times SO(3)} \sharp(M\cap gN)\,d\mu_{SO(3)\times SO(3)}(g) = \int_{M\times N}\sigma(\theta_x,\tau_y)\,d\mu_{M\times N}(x,y),$$

where

$$\sigma(\theta_x, \tau_y) = \begin{cases} 16c_{xy}\left\{2E\left(\dfrac{s_{xy}}{c_{xy}}\right) - \left(1 - \dfrac{s_{xy}^2}{c_{xy}^2}\right)K\left(\dfrac{s_{xy}}{c_{xy}}\right)\right\}, & \text{if } s_{xy} \leq c_{xy}, \\ 16s_{xy}\left\{2E\left(\dfrac{c_{xy}}{s_{xy}}\right) - \left(1 - \dfrac{c_{xy}^2}{s_{xy}^2}\right)K\left(\dfrac{c_{xy}}{s_{xy}}\right)\right\}, & \text{if } s_{xy} \geq c_{xy}. \end{cases}$$

Here K and E are the first and second kind of complete elliptic integral, and $s_{xy} = \sin\theta_x \sin\tau_y$, $c_{xy} = \cos\theta_x \cos\tau_y$.

The author would like to express his gratitude to Professor H. Tasaki for his valuable suggestions during the preparation of this paper. The author is indebted to the refree suggesting some revisions of this paper.

2. Preliminaries

In this section we shall review the Poincaré formula on Riemannian homogeneous spaces given by Howard [1] and the elliptic integrals.

Let E be a finite dimensional real vector space with an inner product. For vector subspaces V and W with orthonormal bases $v_1, \dots, v_p$ and $w_1, \dots, w_q$ respectively, we define $\sigma(V, W)$ by

$$\sigma(V, W) = |v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q|.$$

This definition is independent of the choice of orthonormal bases. Furthermore, if $p + q = \dim E$ then

$$\sigma(V, W) = \sigma(V^\perp, W^\perp).$$

Let G be a Lie group and K a closed subgroup of G. We assume that G has a left invariant Riemannian metric that is also invariant under the right actions of elements of K. This metric induces a G-invariant Riemannian metric on G/K. We denote by o the origin of G/K. If $x, y \in G/K$ and V is a vector subspace of $T_x(G/K)$ and W is a vector subspace of $T_y(G/K)$ then we define $\sigma_K(V, W)$ by

$$(2.1) \qquad \sigma_K(V, W) = \int_K \sigma((dg_x)_o^{-1} V, dk_o^{-1}(dg_y)_o^{-1} W)\, d\mu_K(k)$$

where g_x and g_y are elements of G such that $g_x o = x$ and $g_y o = y$. This definition is independent of the choice of g_x and g_y in G such that $g_x o = x$ and $g_y o = y$.

The action of $SO(2) \times SO(2)$ to $\boldsymbol{R}^4$ is defined by $A \cdot v = vA^{-1}$ for $v \in \boldsymbol{R}^4$ and $A \in SO(2) \times SO(2)$.

We now recall that *the incomplete elliptic integrals of the first* and *second kind* are defined by

$$F(\psi,k)=\int_0^{\psi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\quad 0<k<1,$$

$$E(\psi,k)=\int_0^{\psi}\sqrt{1-k^2\sin^2\theta}\,d\theta\quad 0<k<1,$$

respectively. If $\psi=\pi/2$ then the integrals are called *the complete elliptic integral of the first* and *second kind*, and are denoted by $K(k)$ and $E(k)$ or simply K and E respectively. It is trivial that $K(0)=E(0)=\pi/2$.

3. Proof of Theorem 1.2

Let S^2 be the standard sphere of dimension 2. The special orthogonal group $SO(3)$ acts transitively on S^2. The isotropy subgroup of $SO(3)$ at a point in S^2 is $SO(2)$. Thus $S^2\times S^2$ can be realized as a homogeneous space $(SO(3)\times SO(3))/(SO(2)\times SO(2))$. Let $\mathfrak{so}(3)\times\mathfrak{so}(3)$ be the Lie algebra of $SO(3)\times SO(3)$. Define an inner product on $\mathfrak{so}(3)\times\mathfrak{so}(3)$ by

$$(X,Y)=-\frac{1}{2}\operatorname{Trace}(XY)\quad (X,Y\in\mathfrak{so}(3)\times\mathfrak{so}(3)).$$

We extend this inner product $(\cdot,\cdot)$ on $\mathfrak{so}(3)\times\mathfrak{so}(3)$ to the left invariant Riemannian metric on $SO(3)\times SO(3)$. Then we obtain a biinvariant Riemannian metric on $SO(3)\times SO(3)$. This biinvariant Riemannian metric on $SO(3)\times SO(3)$ induces an $(SO(3)\times SO(3))$-invariant Riemannian metric on $(SO(3)\times SO(3))/(SO(2)\times SO(2))$.

Let M be a submanifold of $S^2\times S^2$ of dimension 1 and N a submanifold of dimension 3. By Theorem 1.1, we have

$$\begin{aligned}(3.1)\qquad &\int_{SO(3)\times SO(3)}\sharp(M\cap gN)\,d\mu_{SO(3)\times SO(3)}(g)\\ &=\int_{M\times N}\sigma_{SO(2)\times SO(2)}(T_xM,T_yN)\,d\mu_{M\times N}(x,y).\end{aligned}$$

Let $u_x=(u_1,u_2)$ and $v_y=(v_1,v_2)$ be unit vectors of T_xM and $T_y^{\perp}N$ respectively. By the action of $SO(2)\times SO(2)$, we can transport u_x and v_y to $((\sin\theta_x,0),(\cos\theta_x,0))$ and $((\sin\tau_y,0),(\cos\tau_y,0))$ respectively. Thus we can take

$$((-\cos\tau_y,0),(\sin\tau_y,0)),((0,1),(0,0)),((0,0),(0,1))$$

as an orthonormal basis of T_yN. We can simply write

$$\sigma(\theta_x, \tau_y) = \sigma_{SO(2)\times SO(2)}(T_xM, T_yN),$$

since $\sigma_{SO(2)\times SO(2)}(T_xM, T_yN)$ is dependent only on θ_x and τ_y.

Let $e_1, \dots, e_4$ be the standard orthonormal basis of $\boldsymbol{R}^4$. Then we have

$$\begin{aligned}
&\sigma(k^{-1}T_xM, T_yN) \\
&\qquad = |k^{-1}(\sin\theta e_1 + \cos\theta e_3) \wedge (-\cos\tau e_1 + \sin\tau e_3) \wedge e_2 \wedge e_4| \\
&\qquad = |(\sin\theta e_1 + \cos\theta e_3)k \wedge (-\cos\tau e_1 + \sin\tau e_3) \wedge e_2 \wedge e_4| \\
&\qquad = |\sin\theta \sin\tau \cos\alpha + \cos\theta \cos\tau \cos\beta|,
\end{aligned}$$

where

$$k = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{bmatrix} \in SO(2) \times SO(2).$$

Put $\sin\theta\sin\tau = s$ and $\cos\theta\cos\tau = c$ then we have

$$\begin{aligned}
\sigma(\theta, \tau) &= \int_{SO(2)\times SO(2)} |\sin\theta\sin\tau\cos\alpha + \cos\theta\cos\tau\cos\beta| \, d\mu_{SO(2)\times SO(2)}(k) \\
&= \int_0^{2\pi}\int_0^{2\pi} |s\cos\alpha + c\cos\beta| \, d\alpha d\beta.
\end{aligned}$$

We here give the following lemma to compute the above integral.

Lemma 3.1. *Let $S^1(r)$ be a circle with radius r. If $|a| \le 1$ then*

$$\int_{S^1(r)} |ra + x_1| \, d\mu_{S^1(r)}(x) = 2r^2(a(\pi - 2\arccos a) + 2\sqrt{1 - a^2}).$$

We can easily show this lemma and omit its proof.

It is sufficient to calculate the following:

$$\int_0^{2\pi}\int_0^{2\pi} |t\cos\alpha + \cos\beta| \, d\alpha d\beta, \quad (0 \le t \le 1).$$

Using Lemma 3.1 and the complete elliptic integral, we obtain

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} |t\cos\alpha + \cos\beta|\, d\alpha d\beta \\
&= \int_0^{2\pi} 2(t\cos\alpha(\pi - 2\arccos(t\cos\alpha)) + 2\sqrt{1 - t^2\cos^2\alpha})\, d\alpha \\
&= 16(1+t^2)\int_0^t \frac{1}{\sqrt{t^2 - x^2}\cdot\sqrt{1 - x^2}}\, dx - 32\int_0^t \frac{x^2}{\sqrt{t^2 - x^2}\cdot\sqrt{1 - x^2}}\, dx \\
&= 16(1+t^2)K(t) - 32(K(t) - E(t)) \\
&= 32E(t) - 16(1 - t^2)K(t).
\end{aligned}$$

Hence we have the following:

$$\begin{aligned}
\sigma(\theta, \tau) &= \int_0^{2\pi}\int_0^{2\pi} |s\cos\alpha + c\cos\beta|\, d\alpha d\beta \\
&= \begin{cases} 32cE\left(\dfrac{s}{c}\right) - 16c\left(1 - \left(\dfrac{s}{c}\right)^2\right)K\left(\dfrac{s}{c}\right), & \text{if } s \le c \\[2ex] 32sE\left(\dfrac{c}{s}\right) - 16s\left(1 - \left(\dfrac{c}{s}\right)^2\right)K\left(\dfrac{c}{s}\right), & \text{if } s \ge c. \end{cases}
\end{aligned}$$

Thus (3.1) implies Theorem 1.2.

REMARK 3.2. Let $M = S^1$ and $N = S^1 \times S^2$ in Theorem 1.2. Then, for almost all $g \in SO(3) \times SO(3)$, we have $\sharp(M \cap gN) = 2$. Thus we have

$$\int_{SO(3)\times SO(3)} \sharp(M \cap gN)\, d\mu_{SO(3)\times SO(3)}(g) = 2\operatorname{vol}(SO(3))\operatorname{vol}(SO(3)).$$

Finally we can give the following corollary as an application of the integral formula in Theorem 1.2.

COROLLARY 3.3. *Under the hypothesis of Theorem* 1.2:
(1) *If* $N = S^1 \times S^2$ *then we have*

$$\frac{1}{\operatorname{vol}(SO(3)\times SO(3))}\int_{SO(3)\times SO(3)} \sharp(M \cap gN)\, d\mu_{SO(3)\times SO(3)}(g) \le \frac{\operatorname{vol}(M)}{\pi}.$$

The inequality becomes an equality if and only if M *is a curve in* S^2.

(2) *If* $M = S^1$ $(\subset S^2)$ *then we have*

$$\frac{1}{\mathrm{vol}(SO(3) \times SO(3))} \int_{SO(3)\times SO(3)} \sharp(M \cap gN)\, d\mu_{SO(3)\times SO(3)}(g) \leq \frac{\mathrm{vol}(N)}{4\pi^2}.$$

The equality holds if and only if N *is a submanifold of* $L \times S^2$. *Here* L *is a curve in* S^2.

PROOF. (1) In this case we can take $\sin\theta_x e_1 + \cos\theta_x e_3$ and e_2, e_3, e_4 as an orthonormal basis of T_xM and T_yN respectively. Here e_1, e_2, e_3, e_4 is the standard orthonormal basis of $\boldsymbol{R}^4$. Thus we obtain

$$\sigma(\theta_x, \tau_y) = 32 \sin\theta_x E(0) - 16 \sin\theta_x K(0) = 8\pi \sin\theta_x.$$

We therefore have

$$\begin{aligned}\int_{SO(3)\times SO(3)} \sharp(M \cap gN)\, d\mu_{SO(3)\times SO(3)}(g) &= \int_{M\times N} 8\pi \sin\theta_x\, d\mu_{M\times N}(x,y) \\ &= 8\pi\, \mathrm{vol}(N) \int_M \sin\theta_x\, d\mu_M(x) \\ &\leq 8\pi\, \mathrm{vol}(N)\, \mathrm{vol}(M).\end{aligned}$$

Using known facts that

$$\mathrm{vol}(N) = \mathrm{vol}(S^1 \times S^2) = \mathrm{vol}(SO(3)) = 8\pi^2,$$

completes the proof.

(2) In this case we can obtain

$$\sigma(\theta_x, \tau_y) = 8\pi \sin\tau_y.$$

This, by a computation similar to that in (1), completes the proof. □

References

[1] R. Howard, The kinematic formula in Riemannian homogeneous spaces, Mem. Amer. Math. Soc., No. 509, **106** (1993).

[2] H. J. Kang and H. Tasaki, Integral geometry of real surfaces in complex projective spaces, Tsukuba J. Math., **25** (2001), 155–164.

[3] H. J. Kang and H. Tasaki, Integral geometry of real surfaces in the complex projective plane, Geom. Dedicata, **90** (2002), 99–106.

[4] L. A. Santaló, Integral Geometry and Geometric Probability, Addison-Wesley, London, 1977.

[5] H. Tasaki, Generalization of Kähler angle and integral geometry in complex projective spaces, "Steps in Differential Geometry", Proceedings of Colloquium on Differential Geometry, 349–361, Inst. Math. Inform., Debrecen, 2001.

Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki
305-8571
Japan
kang@math.tsukuba.ac.jp