TSUKUBA J. MATH.
Vol. 22 No. 3 (1998), 723-739

ON THE STRUCTURE OF TAKAHASHI MANIFOLDS

By

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Abstract. We study the topological structure of the closed orientable 3-manifolds obtained by Dehn surgeries along certain links, first considered by Takahashi in [23]. The interest about such manifolds arises from the fact that they include well-known families of 3-manifolds, previously studied by several authors, as the Fibonacci manifolds [7], [10], [11], the Fractional Fibonacci manifolds [14], and the Sieradski manifolds [5], [6], respectively. Our main result states that the Takahashi manifolds are 2-fold coverings of the 3-sphere branched along the closures of specified 3-string braids. We also describe many of the above-mentioned manifolds as n-fold cyclic branched coverings of the 3-sphere.

1. Introduction and main results

The goal of the paper is to study the topological structure of the closed connected orientable 3-manifolds obtained by Dehn surgeries along certain chains of unknotted oriented circles in the oriented 3-sphere. Our results complete in a sense the ones of a previous paper of Takahashi [23]. It turns out that the above manifolds contemporarily include well-known families of manifolds, treated in the literature (see references), as the (Fractional) Fibonacci manifolds and the Sieradski manifolds. So we can re-obtain several results of the quoted papers as simple corollaries of our main theorem. To state it we first consider the link L_{2n} resp. L'_n with $2n$ resp. n components, $n \geq 2$, each of which is unknotted oriented and linked with exactly two adjacent components as shown in Figure 1a resp. 1b.

1991 *Mathematics Subject Classification.* 57N10, 57R65, 57M12, 20F36.
Key words and phrases. 3-Manifolds, Dehn surgery, Group presentations, Branched coverings, Braids, Orbifolds, RR-systems.
Work performed under the auspices of the G.N.S.A.G.A. of the C.N.R. (National Research Council) of Italy and partially supported by the Ministero per la Ricerca Scientifica e Tecnologica of Italy within the project "Classificazione e Proprietà geometriche delle Varietà Reali e Complesse".
Received September 22, 1997
Revised May 7, 1998

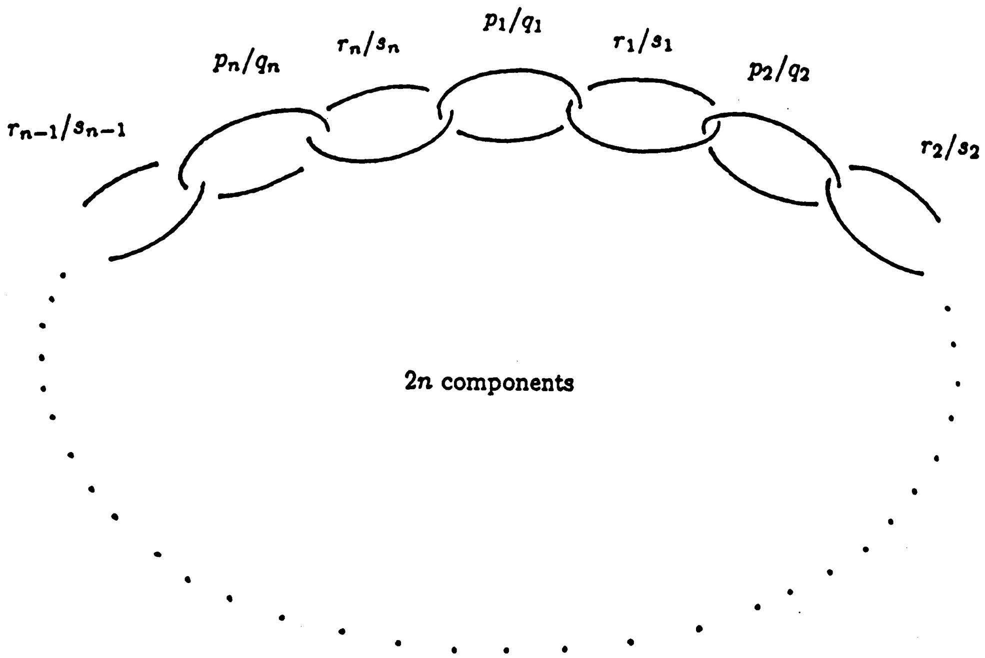


Figure 1a: the link L_{2n}.

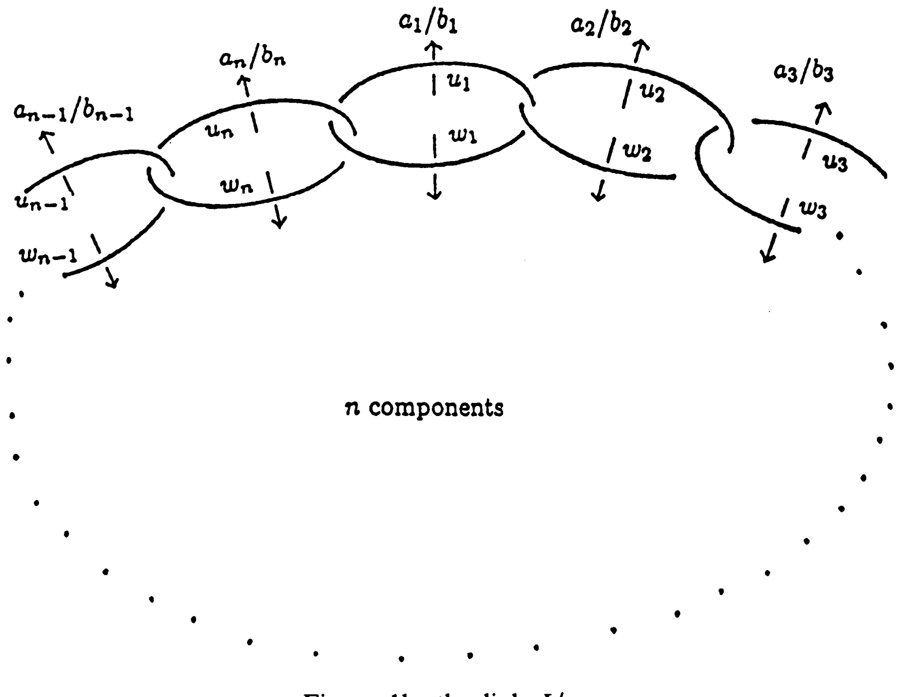


Figure 1b: the link L'_n.

Let us denote by $M(p_1/q_1,\dots,p_n/q_n;r_1/s_1,\dots,r_n/s_n)$ resp. $M'(a_1/b_1,\dots,a_n/b_n)$ the closed connected orientable 3-manifold obtained by Dehn surgery along L_{2n} resp. L'_n with surgery coefficients p_i/q_i and r_i/s_i resp. a_i/b_i, $i=1,2,\dots,n$, according to Figure 1. In [23] Takahashi gave finite presentations of the fundamental group of the manifolds $M(p_i/q_i;r_i/s_i)$, so for convenience we refer to such manifolds as the *Takahashi manifolds*.

These presentations actually coincide with the standard ones of the Fibonacci groups

$$F(2,2n)=\langle x_1,x_2,\dots,x_{2n} : x_ix_{i+1}=x_{i+2}\ (\text{indices mod } 2n)\rangle$$

resp. the Fractional Fibonacci groups

$$F^{k/l}(2,2n)=\langle x_1,x_2,\dots,x_{2n} : x_i^lx_{i+1}^k=x_{i+2}^l\ (\text{indices mod } 2n)\rangle$$

when $p_i/q_i=1$ and $r_i/s_i=-1$ resp. $p_i/q_i=k/l$ and $r_i/s_i=-k/l$ for every $i=1,2,\dots,n$. It is well-known that the above presentations correspond to spines of closed orientable 3-manifolds, called the Fibonacci manifolds and the Fractional Fibonacci manifolds, respectively. It was also proved that the Fibonacci manifolds resp. the Fractional Fibonacci manifolds are two-fold cyclic coverings of the 3-sphere branched over the Turk's head links Th_n resp. the links $Th_n^{k/l}$, that are the closures of the 3-string braids $(\sigma_1\sigma_2^{-1})^n$ resp. $(\sigma_1^{k/l}\sigma_2^{-k/l})^n$ (see [7], [10], [11] and [14]). Our main theorem extends these results to the case of Takahashi manifolds.

THEOREM 1. *For any coprime integers p_i and q_i resp. r_i and s_i, $i=1,2,\dots,n$, and for any integer $n\ge 2$, the Takahashi manifold $M(p_1/q_1,\dots,p_n/q_n;r_1/s_1,\dots,r_n/s_n)$ is the two-fold cyclic covering of the 3-sphere branched along the closure of the rational 3-string braid*

$$\sigma_1^{p_1/q_1}\sigma_2^{r_1/s_1}\cdots\sigma_1^{p_n/q_n}\sigma_2^{r_n/s_n}.$$

We also obtain finite presentations of the fundamental group of the manifolds $M'(a_1/b_1,\dots,a_n/b_n)$, and further prove that these manifolds are still examples of Takahashi manifolds. Our presentations coincide with the standard ones of the Sieradski groups

$$S(n)=\langle x_1,x_2,\dots,x_n : x_ix_{i+2}=x_{i+1}\ (\text{indices mod } n)\rangle$$

when $a_i/b_i=-1$, for every $i=1,2,\dots,n$. It was proved that $S(n)$ corresponds to a spine of the n-fold cyclic covering of the 3-sphere branched over the trefoil knot (see [5], also for other types of generalizations).

The following extends this result to the case of manifolds $M'(a_i/b_i)$.

Theorem 2. *For any coprime integers a_i and b_i, $i = 1, 2, \dots, n$, and for any integer $n \geq 2$, the manifold $M'(a_1/b_1, \dots, a_n/b_n)$ is homeomorphic to the Takahashi manifold $M(p_1/q_1, \dots, p_n/q_n; r_1/s_1, \dots, r_n/s_n)$, where $r_i/s_i = 1$ and $p_i/q_i = a_i/b_i + 2$, and so it is the two-fold cyclic covering of the 3-sphere branched along the closure of the rational 3-string braid*

$$\sigma_1^{a_1/b_1+2}\sigma_2 \cdots \sigma_1^{a_n/b_n+2}\sigma_2.$$

Finally we remark that the link L_{2n} is hyperbolic (see [1], p. 222) so according to the Thurston-Jorgensen theory of hyperbolic surgery (see [24]) we get the following result:

Theorem 3. *For any integer $n \geq 2$, and for all but a finite number of pairs (p_i, q_i) and (r_i, s_i), the Takahashi manifolds $M(p_1/q_1, \dots, p_n/q_n; r_1/s_1, \dots, r_n/s_n)$ are hyperbolic.*

2. The Takahashi manifolds

The following was proved by Takahashi in [23].

Theorem 4. *The fundamental group of the 3-manifold $M(p_i/q_i; r_i/s_i)$ obtained by Dehn surgery along the oriented link L_{2n} with surgery coefficients p_i/q_i and r_i/s_i, $i = 1, 2, \dots, n$, admits the finite presentation*

$$\Pi_1(M(p_i/q_i; r_i/s_i)) = \langle x_1, x_2, \dots, x_{2n} : x_{2i}^{s_i} x_{2i+1}^{p_{i+1}} = x_{2(i+1)}^{s_{i+1}}, x_{2i-1}^{q_i} x_{2i}^{-r_i} = x_{2i+1}^{q_{i+1}} \ (\textit{indices mod } n) \rangle.$$

Generalizing an example given in [23] (case $n = 3$) yields the following

Theorem 5. *The fundamental group of the 3-manifold $M'(a_1/b_1, \dots, a_n/b_n)$ obtained by Dehn surgery along the oriented link L'_n with surgery coefficients a_i/b_i, $i = 1, 2, \dots, n$, admits the finite presentation*

$$\Pi_1(M'(a_1/b_1, \dots, a_n/b_n)) = \langle x_1, x_2, \dots, x_n : x_i^{a_i+b_i} x_{i+1}^{b_{i+1}} x_i^{-b_i} x_{i-1}^{b_{i-1}} = 1 \ (\textit{indices mod } n) \rangle.$$

PROOF. Let

$$\begin{aligned}\Pi_1(S^3\backslash L'_n) = \langle u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_n : w_i u_{i-1} &= u_{i-1} u_i \ (R_i)\\ u_i w_{i+1} &= w_{i+1} w_i \ (Q_i)\\ &\text{(indices mod } n)\rangle\end{aligned}$$

be the Wirtinger presentation of the link group of L'_n where the generators u_i, w_i are taken as shown in Figure 1b. If m_i and l_i denote the meridian and the longitude, respectively, of the i-th component of L'_n, then we have

$$m_i = u_i, \quad l_i = w_{i+1}u_{i-1}, \quad [m_i, l_i] = 1.$$

The presentation of $\Pi_1(M'(a_1/b_1, \dots, a_n/b_n))$ comes from the one of $\Pi_1(S^3\backslash L'_n)$ by adding the relations $m_i^{a_i} l_i^{b_i} = 1$, for any $i = 1, 2, \dots, n$.

Since a_i and b_i are coprime integers, there exist two integers c_i and d_i such that

$$b_i c_i - a_i d_i = 1$$

for every $i = 1, 2, \dots, n$.

Setting

$$x_i = m_i^{c_i} l_i^{d_i},$$

it follows that

$$\begin{aligned} m_i &= x_i^{b_i}\\ l_i &= x_i^{-a_i}\\ u_i &= x_i^{b_i},\end{aligned}$$

and hence

$$w_i = l_{i-1} u_{i-2}^{-1} = x_{i-1}^{-a_{i-1}} x_{i-2}^{-b_{i-2}} \ (S_i).$$

Now relations R_i and S_i directly imply

$$x_i^{a_i+b_i} x_{i+1}^{b_{i+1}} x_i^{-b_i} x_{i-1}^{b_{i-1}} = 1,$$

where the indices are taken mod n as usual. Finally, using these relations and S_i, one can verify that relations Q_i become identities for every $i = 1, 2, \dots, n$. Thus the proof is completed. □

Now we are going to prove that the finite group presentations of Theorems 4 and 5 correspond to spines of the represented manifolds. For that, we first recall some definitions about *RR*-systems (see [20]). Let D be a regular hexagon in the plane E^2. For each pair of opposite faces construct a finite set (possibly empty), *station* say, of parallel line segments, called *tracks*, through D with endpoints on these opposite faces. Let $\{D_i : i = 1, 2, \dots, s\}$ be a set of disjoint regular hexagons in E^2. A *route* is an arc whose interior lies in $E^2 \backslash U_{i=1}^{s} D_i$ connecting endpoints of tracks. A *RR-system* is the union in E^2 of a finite set of hexagons with stations and a finite set of disjoint routes in $S^2 \backslash U_{i=1}^{s} D_i$ such that each endpoint of every track intersects exactly one route in one of its endpoints. A *RR*-system gives rise to a family of group presentations whose generators x_i $(i = 1, 2, \dots, s)$ are in one-to-one correspondence with the hexagons D_i. In each hexagon we start from some vertex of the boundary and proceed clockwise (according to an orientation of S^2) along an edge which corresponds to a station m_i of D_i. Orient the tracks of this station so that the positive direction is toward this edge. Label the stations corresponding to the second and third edges encountered by $m_i + n_i$ and n_i respectively, and orient the tracks of these stations toward the respective edges. By walking along each closed arc (made by tracks and routes) we write a word on generators x_i $(i = 1, 2, \dots, s)$ in the following way: as we enter in each hexagon D_i we give the name of the station as exponent of x_i with sign $+1$ resp. -1 if our direction of travel concords resp. opposes the orientation of the tracks (see [20] for more details). Osborne and Stevens proved in [20] that a finite group presentation with the same number of generators and relations corresponds to a spine of a closed connected orientable 3-manifold if and only if it arises from an *RR*-system. Since the group presentation of Theorem 4 resp. 5 is induced by the *RR*-system depicted in Figure 2 (as communicated us by Hog-Angeloni [12]) resp. 3, we get the following

THEOREM 6. *The finite group presentation*

$$\langle x_1, x_2, \dots, x_{2n} : x_{2i}^{s_i} x_{2i+1}^{p_{i+1}} = x_{2(i+1)}^{s_{i+1}},\ x_{2i-1}^{q_i} x_{2i}^{-r_i} = x_{2i+1}^{q_{i+1}} \rangle$$

resp.

$$\langle x_1, x_2, \dots, x_n : x_i^{a_i+b_i} x_{i+1}^{b_{i+1}} x_i^{-b_i} x_{i-1}^{b_{i-1}} = 1 \rangle$$

corresponds to a spine of the 3*-manifold* $M(p_1/q_1, \dots, p_n/q_n; r_1/s_1, \dots, r_n/s_n)$ *resp.* $M'(a_1/b_1, \dots, a_n/b_n)$.

We observe that if $p_i = q_i = s_i = 1$ and $r_i = -1$, for any $i = 1, 2, \dots, n$, then $\Pi_1(M(1, \dots, 1; -1, \dots, -1)) = \langle x_1, x_2, \dots, x_{2n} : x_i x_{i+1} = x_{i+2}$ (indices mod $2n$)$\rangle$ is the Fibonacci group $F(2, 2n)$, first introduced by Conway in [8].

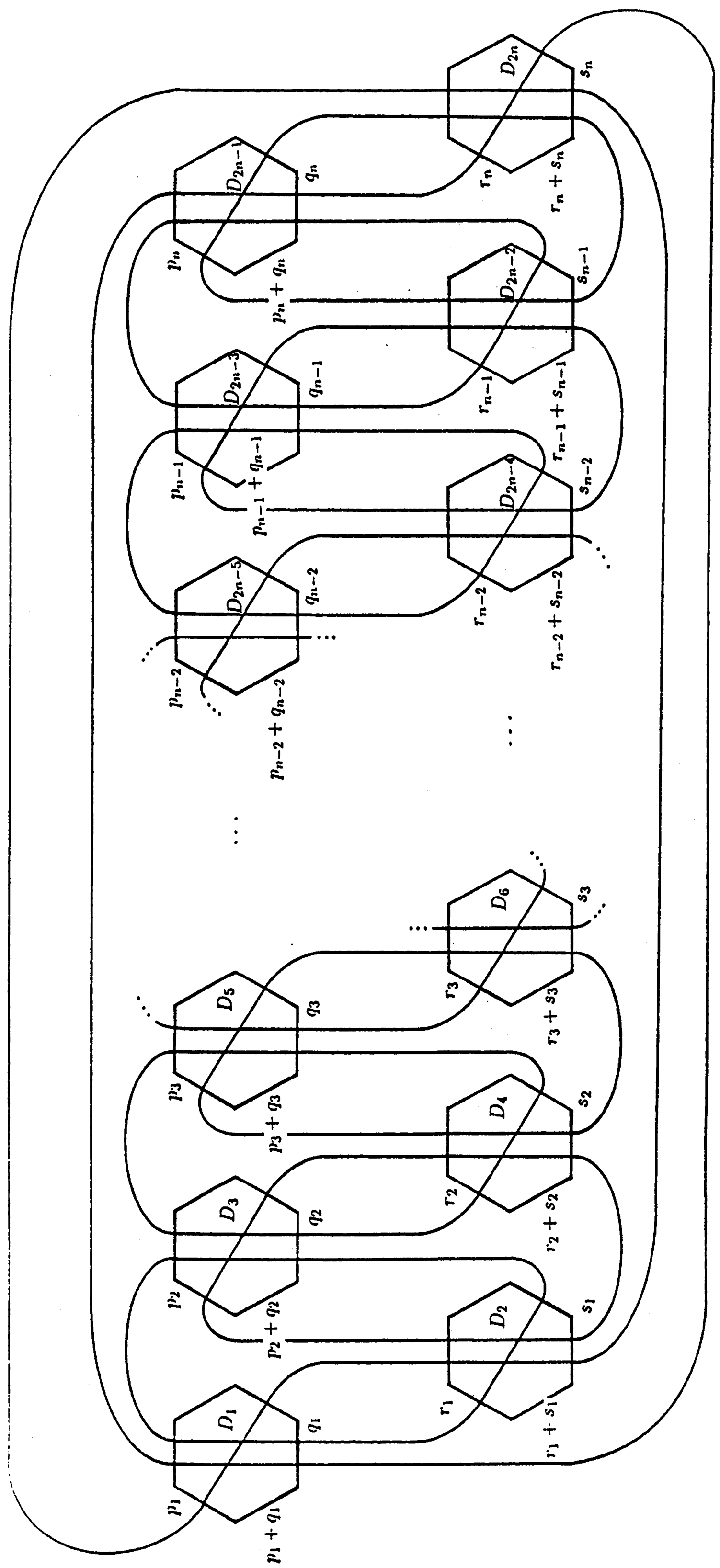
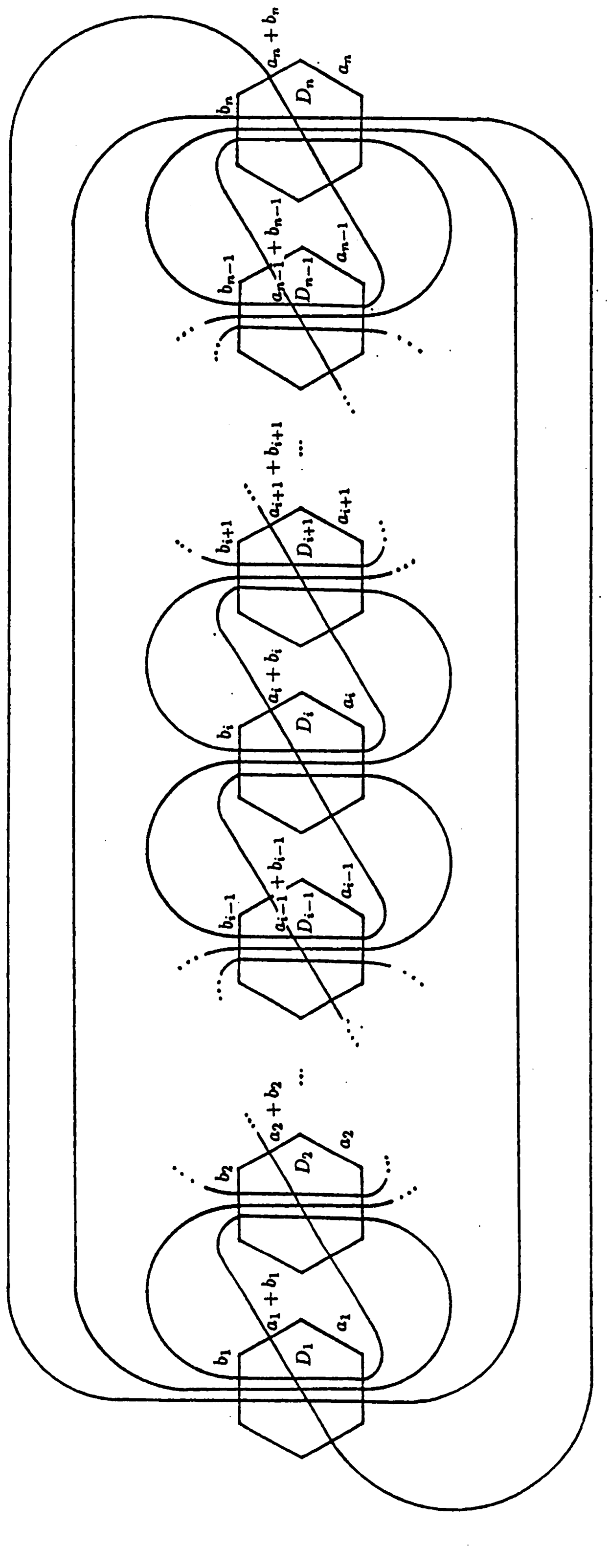

D_{2n}
s_n
r_n
$r_n + s_n$
D_{2n-1}
q_n
p_n
$p_n + q_n$
D_{2n-2}
s_{n-1}
r_{n-1}
$r_{n-1} + s_{n-1}$
D_{2n-3}
q_{n-1}
p_{n-1}
$p_{n-1} + q_{n-1}$
D_{2n-4}
s_{n-2}
r_{n-2}
$r_{n-2} + s_{n-2}$
D_{2n-5}
q_{n-2}
p_{n-2}
$p_{n-2} + q_{n-2}$
D_6
s_3
r_3
$r_3 + s_3$
D_5
q_3
p_3
$p_3 + q_3$
D_4
s_2
r_2
$r_2 + s_2$
D_3
q_2
p_2
$p_2 + q_2$
D_2
s_1
r_1
$r_1 + s_1$
D_1
q_1
p_1
$p_1 + q_1$

Figure 2.



$a_n + b_n$
a_n
b_n
D_n
b_{n-1}
$a_{n-1} + b_{n-1}$
a_{n-1}
D_{n-1}
$a_{i+1} + b_{i+1}$
b_{i+1}
a_{i+1}
D_{i+1}
$a_i + b_i$
b_i
a_i
D_i
$a_{i-1} + b_{i-1}$
b_{i-1}
a_{i-1}
D_{i-1}
$a_2 + b_2$
b_2
a_2
D_2
$a_1 + b_1$
b_1
a_1
D_1

Figure 3.

If $p_i = k$, $r_i = -k$ and $q_i = s_i = l$, for any $i = 1, 2, \dots, n$, then

$$\Pi_1(M(k/l, \dots, k/l; -k/l, \dots, -k/l)) = \langle x_1, x_2, \dots, x_{2n} : x_i^l x_{i+1}^k = x_{i+2}^l \\ \text{(indices mod } 2n)\rangle$$

is the Fractional Fibonacci group $F^{k/l}(2, 2n)$, studied by Kim and Vesnin in [14]. If $a_i = -1$ and $b_i = 1$, for any $i = 1, 2, \dots, n$, then

$$\Pi_1(M'(-1, \dots, -1)) = \langle x_1, x_2, \dots, x_n : x_i x_{i+2} = x_{i+1} \text{ (indices mod } n)\rangle$$

is the Sieradski group (see [22] and [5]).

Now we apply the Kirby-Rolfsen calculus on links with coefficients (see [15], [16] and [21]) to prove the following result.

THEOREM 7. *The manifold $M'(a_1/b_1, \dots, a_n/b_n)$ is homeomorphic to the Takahashi manifold $M(p_1/q_1, \dots, p_n/q_n; r_1/s_1, \dots, r_n/s_n)$ if $r_i/s_i = 1$ and $p_i/q_i = a_i/b_i + 2$ for any $i = 1, 2, \dots, n$.*

PROOF. Let us consider the link L_{2n} of Figure 1a with surgery coefficients $r_i/s_i = 1$, for any $i = 1, 2, \dots, n$ and twist about each component of L_{2n} with coefficient $r_i/s_i = 1$ in the left-hand sense ($\tau = -1$). We obtain the link L_n' with n components of Figure 1b and surgery coefficients $a_i/b_i = p_i/q_i - 2$, for any $i = 1, 2, \dots, n$. The sequence of surgery moves is illustrated in Figure 4. □

3. Branched coverings

In this section we are going to prove Theorem 1. For this we use a well-known result of Montesinos (Theorem 1 of [19]) which states that a closed orientable 3-manifold, obtained by Dehn surgery along a strongly-invertible link L of n components, is a 2-fold cyclic covering of S^3 branched over a link of at most $n + 1$ components. Following [4] and [9], let $\sigma_i^{t/h}$ denote the rational t/h-tangle, whose incoming arcs are i-th and $(i + 1)$-th strings (Here t and h are coprime integers). If t/h is written as a continued fraction

$$t/h = \cfrac{1}{c_1 + \cfrac{1}{\ddots + \cfrac{1}{c_z}}}$$

and $t, h, c_1, \dots, c_z$ are positive integers with $c_z \geq 2$, then the rational t/h-tangle is defined as in Figure 5.

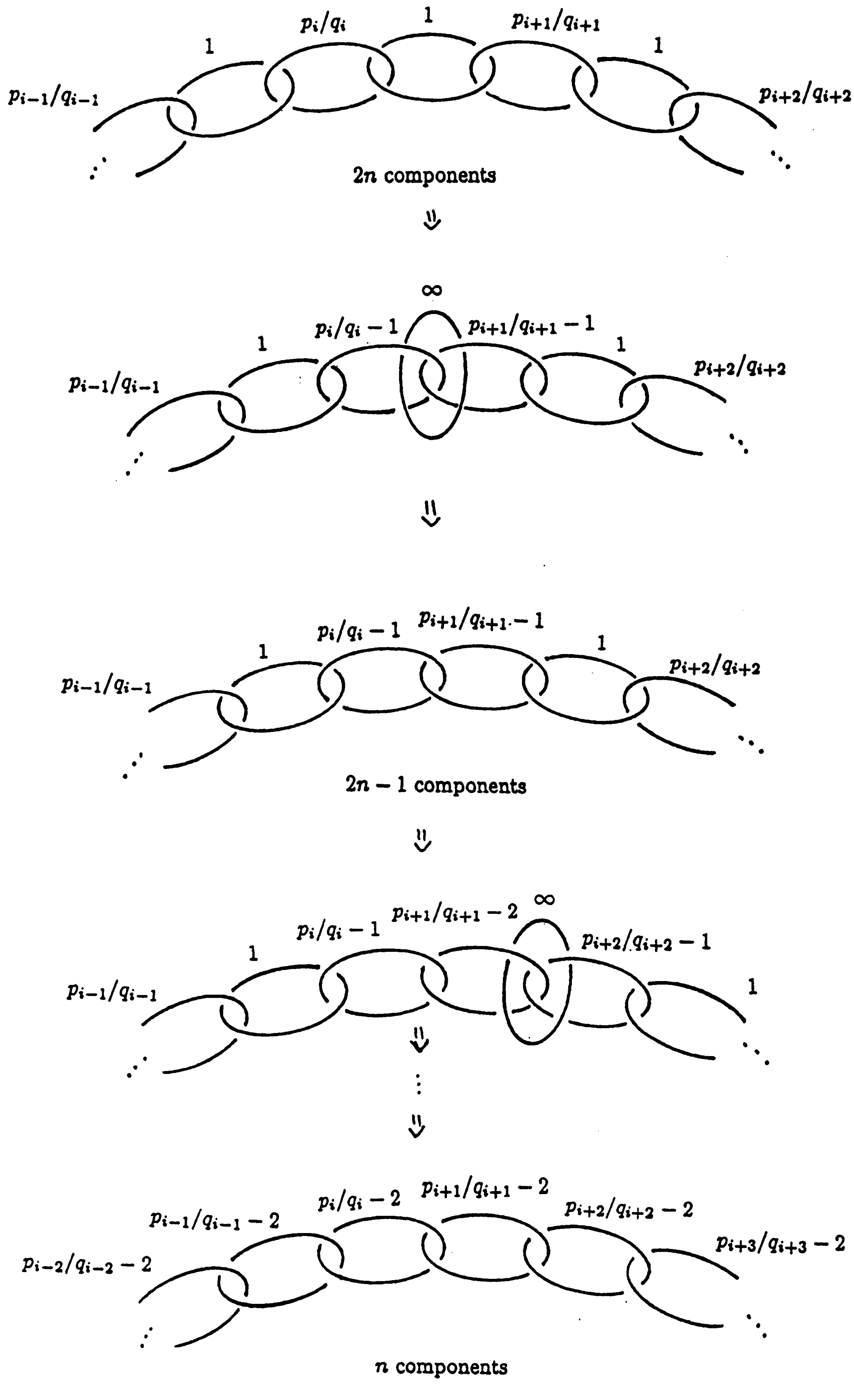

p_i/q_i
1
p_{i+1}/q_{i+1}
1
1
p_{i-1}/q_{i-1}
p_{i+2}/q_{i+2}
2n components
∞
$p_i/q_i - 1$
$p_{i+1}/q_{i+1} - 1$
1
1
p_{i-1}/q_{i-1}
p_{i+2}/q_{i+2}
$p_i/q_i - 1$
$p_{i+1}/q_{i+1} - 1$
1
1
p_{i-1}/q_{i-1}
p_{i+2}/q_{i+2}
2n − 1 components
$p_{i+1}/q_{i+1} - 2$
∞
$p_i/q_i - 1$
$p_{i+2}/q_{i+2} - 1$
1
p_{i-1}/q_{i-1}
1
$p_i/q_i - 2$
$p_{i+1}/q_{i+1} - 2$
$p_{i+2}/q_{i+2} - 2$
$p_{i-1}/q_{i-1} - 2$
$p_{i-2}/q_{i-2} - 2$
$p_{i+3}/q_{i+3} - 2$
n components

Figure 4.

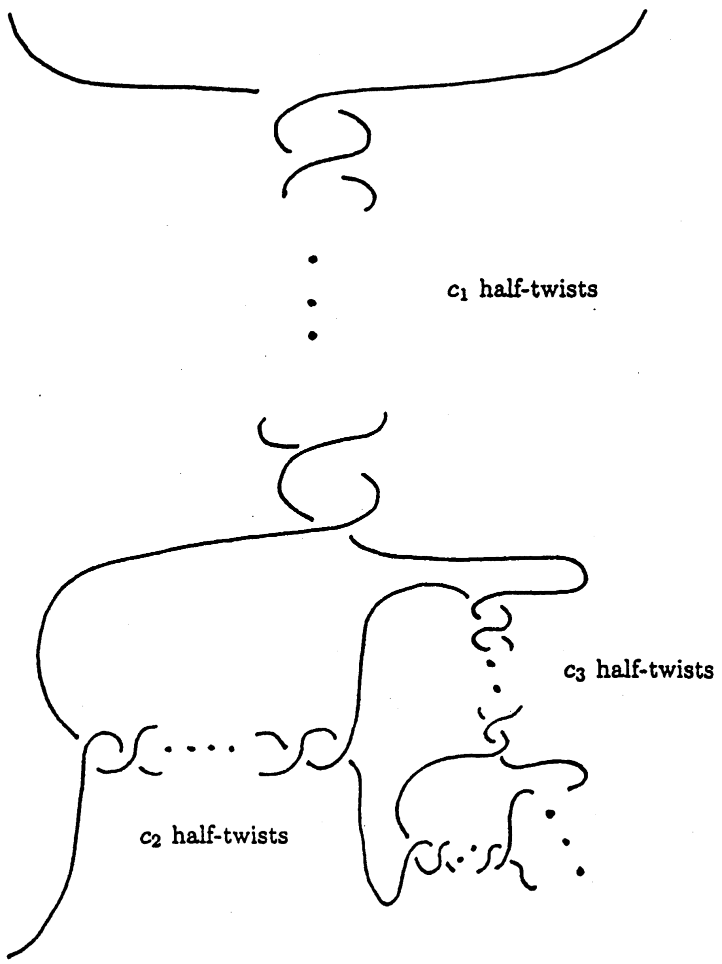


Figure 5: the rational t/h-tangle with $t, h > 0$.

Proof of Theorem 1. The link L_{2n} is strongly-invertible. In fact there exists an involution $p : S^3 \to S^3$ whose axis r intersects each component of the link L_{2n} in two points (see Figure 6a).

The Montesinos theorem assures that $M(p_1/q_1, \ldots, p_n/q_n; r_1/s_1, \ldots, r_n/s_n)$ is a two-fold covering of the 3-sphere branched along the link constructed as follows.

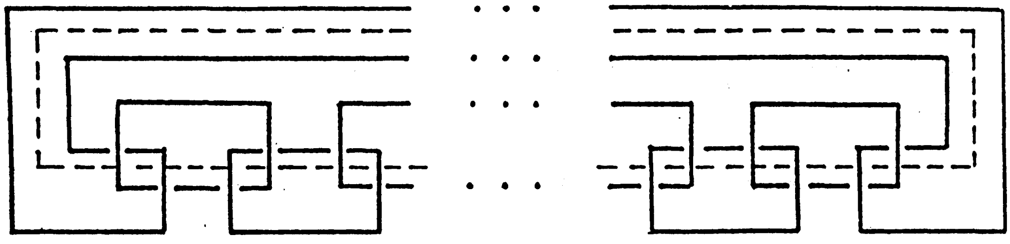


Figure 6a: the strongly-invertible link L_{2n}.

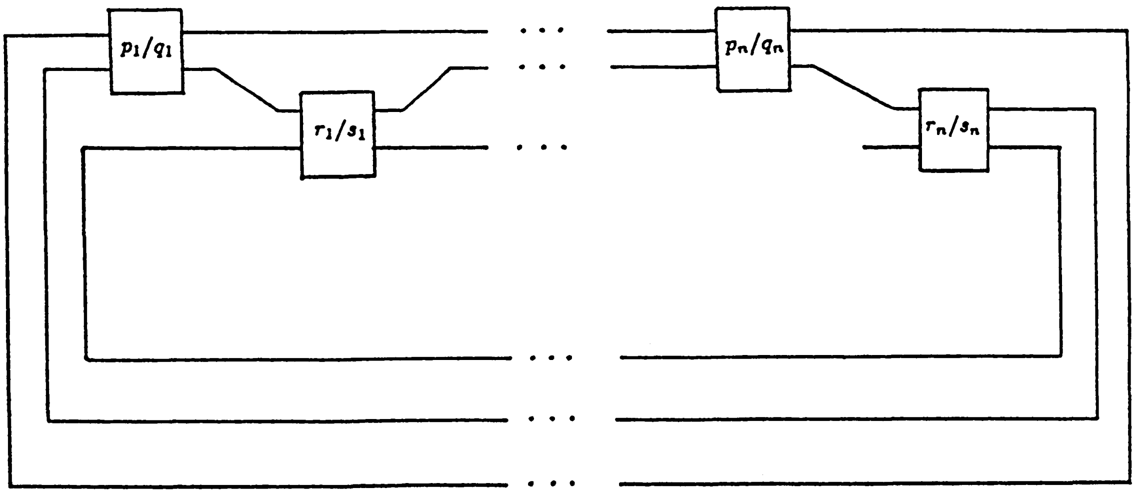


Figure 6b: the closure of the 3-string braid $\sigma_1^{p_1/q_1}\sigma_2^{r_1/s_1}\cdots\sigma_1^{p_n/q_n}\sigma_2^{r_n/s_n}$.

Let N_i be a tubular neighbourhood of the i-th component of the link L_{2n}, for each $i = 1, 2, \ldots, 2n$. If $\pi : S^3 \to S^3/p$ is the canonical projection, then $\pi(N_i)$ is the trivial tangle which consists of a 3-ball B_i where $\pi(r \cap N_i)$ are two arcs. Let us denote by B'_{2j-1} resp. B'_{2j} the (p_j/q_j)-tangle resp. (r_j/s_j)-tangle for $j = 1, 2, \ldots, n$ with the underlying 3-ball B_i. The manifold $M(p_1/q_1, \ldots, p_n/q_n; r_1/s_1, \ldots, r_n/s_n)$ is the 2-fold branched covering of

$$\left(\bigcup_{i=1}^{2n} B'_i\right) \cup_\pi \left(S^3 \setminus \bigcup_{i=1}^{2n} N_i\right)$$

where the branch set is a link formed by arcs of tangles B'_i and $\pi(r \cap (S^3 \setminus U_{i=1}^{2n} N_i))$. Using Reidemeister moves, one can easily see that the branch set is

$$\sigma_1^{p_1/q_1}\sigma_2^{r_1/s_1}\cdots\sigma_1^{p_n/q_n}\sigma_2^{r_n/s_n},$$

as shown in Figure 6b. □

COROLLARY 8. *If $p_i/q_i = p/q$ and $r_i/s_i = r/s$, for every $i = 1, 2\ldots, n$, then the Takahashi manifold $M(p/q, r/s) = M(p/q, \ldots, p/q; r/s, \ldots, r/s)$ is the two-fold covering of the 3-sphere branched over the link $(\sigma_1^{p/q}\sigma_2^{r/s})^n$, and then n-fold cyclic covering of the 3-sphere branched over the link $(\sigma_1^{p/q}\sigma_2^{r/s})^2$.*

In particular we obtain as corollaries some results proved in [14], [11], [10], and [7].

Corollary 9. *If* $p_i/q_i = k/l$ *and* $r_i/s_i = -k/l$, *for every* $i = 1, 2, \ldots, n$, *then the Takahashi manifold* $M(k/l, -k/l) = M(k/l, \ldots, k/l; -k/l, \ldots, -k/l)$ *is the Fractional Fibonacci manifold defined in* [14], *and so it is the two-fold covering of the* 3*-sphere branched over the link* $(\sigma_1^{k/l}\sigma_2^{-k/l})^n$ *and the n-fold cyclic covering of the* 3*-sphere* S^3 *branched over the link* $(\sigma_1^{k/l}\sigma_2^{-k/l})^2$.

Some particular case of Corollary 9 was also treated in [17] and [18].

Corollary 10. *If* $p_i/q_i = 1$ *and* $r_i/s_i = -1$, *for every* $i = 1, 2, \ldots, n$, *then the Takahashi manifold* $M(1, -1) = M(1, \ldots, 1; -1, \ldots, -1)$ *is the Fibonacci manifold considered in* [10], [7], [11], *and so it is the two-fold covering of the* 3*-sphere branched over the link* $(\sigma_1\sigma_2^{-1})^n$ *and the n-fold cyclic covering of the* 3*-sphere branched over the figure-eight knot* $(\sigma_1\sigma_2^{-1})^2$.

Now Theorems 1 and 7 directly imply Theorem 2, and the following corollaries (compare also with [5]).

Corollary 11. *If* $a_i/b_i = k/l$, *for any* $i = 1, 2, \ldots, n$, *then the Takahashi manifold* $M'(k/l, \ldots, k/l)$ *is the* 2*-fold covering of* S^3 *branched over the closed* 3*-string braid* $(\sigma_1^{k/l+2}\sigma_2)^n$, *and the n-fold cyclic covering of* S^3 *branched over the link* $(\sigma_1^{k/l+2}\sigma_2)^2$.

Corollary 12. *If* $a_i/b_i = -1$, *for any* $i = 1, 2, \ldots, n$, *then we have the Sieradski manifold* $M'(-1, \ldots, -1)$ *which is the* 2*-fold covering of* S^3 *branched over the torus link* $(\sigma_1\sigma_2)^n = K(n, 3)$ *and the n-fold cyclic covering of* S^3 *branched over the trefoil knot* $(\sigma_1\sigma_2)^2$.

We note that the 3-string braid $\sigma_1^{p_1}\sigma_2^{r_1}\cdots\sigma_1^{p_n}\sigma_2^{r_n}$ is a 6-plat (see [2]) so it may be represented as a 3-bridge link. By Theorem 5 of [3] we obtain the following

Corollary 13. *The manifold* $M(p_1, \ldots, p_n; r_1, \ldots, r_n)$ *has Heegaard genus* ≤ 2. *In particular, the Fibonacci manifolds and the Sieradski manifolds have Heegaard genus* ≤ 2.

4. Orbifolds

Let $L(1/q, 1/s, n)(2)$ resp. $L(1/q, 1/s, 2)(n)$ be the orbifold whose underlying space is S^3 and whose singular set is the link $L(1/q, 1/s, n) := \sigma_1^{1/q}\sigma_2^{1/s}\cdots\sigma_1^{1/q}\sigma_2^{1/s}$

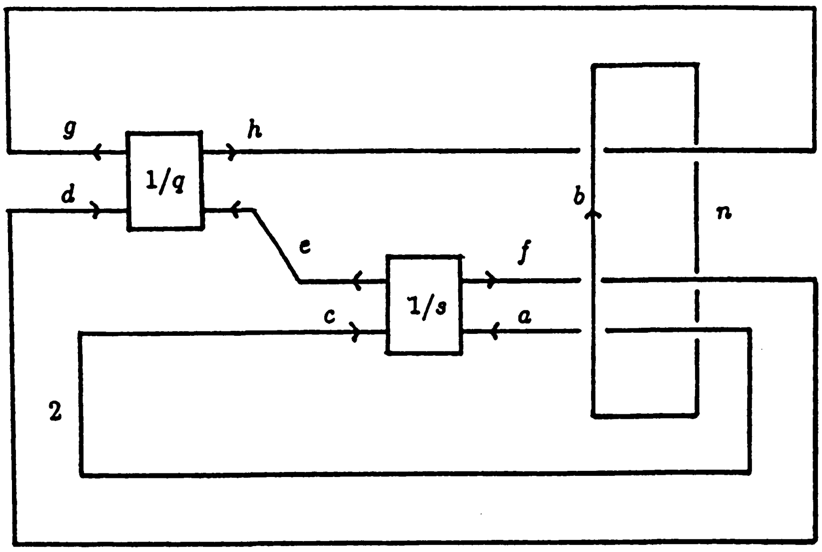


Figure 7a: the link $\mathscr{L}(1/q, 1/s)$.

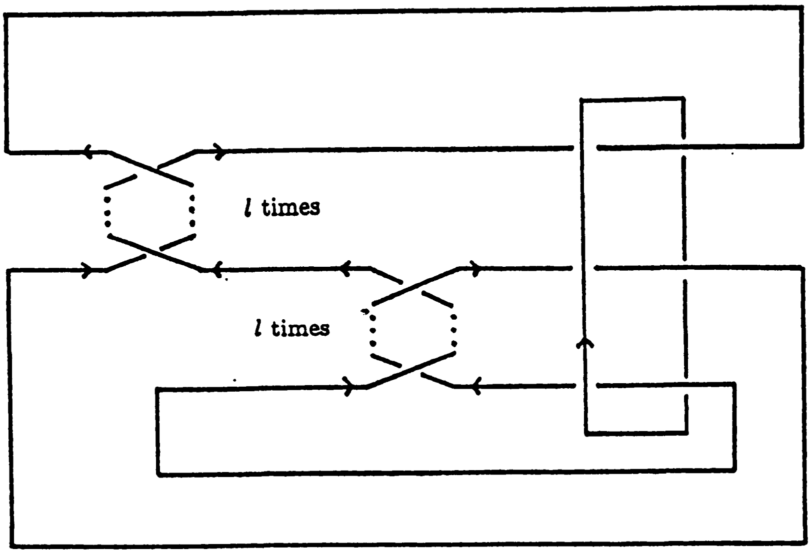


Figure 7b: $\mathscr{L}(1/l, -1/l) = \mathscr{L}^{1/l}$.

(n times) resp. $L(1/q, 1/s, 2) := \sigma_1^{1/q}\sigma_2^{1/s}\sigma_1^{1/q}\sigma_2^{1/s}$ with branch index 2 resp. n. Let $\mathscr{L}(1/q, 1/s)(2, n)$ be the orbifold whose underlying space is the 3-sphere and whose singular set is the two-component link $\mathscr{L}(1/q, 1/s)$ shown in Figure 7a, with branch indices 2 and n on its components (which are equivalent).

The following extends Theorem 3.2 of [14].

THEOREM 14. *The following diagram of cyclic branched coverings holds*:

$$\begin{tikzcd}
M(1/q,1/s) \arrow[r, equal] \arrow[d, "2"'] & M(1/q,1/s) \arrow[d, "n"] \\
L(1/q,1/s,n)(2) \arrow[d, "n"'] & L(1/q,1/s,2)(n) \arrow[d, "2"] \\
\mathscr{L}(1/q,1/s)(2,n) \arrow[r, equal] & \mathscr{L}(1/q,1/s)(2,n).
\end{tikzcd}$$

PROOF. The statement follows from the following easily verifiable facts:

1) The manifold $M(1/q,1/s)$ admits a $(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_n)$-action which is induced by the natural $(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_n)$-symmetry of the link L_{2n};

2) The quotient orbifolds $M(1/q,1/s)/(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_n)$, $M(1/q,1/s)/\boldsymbol{Z}_2$, and $M(1/q,1/s)/\boldsymbol{Z}_n$ are equivalent to $\mathscr{L}(1/q,1/s)(2,n)$, $L(1/q,1/s,n)(2)$ and $L(1/q,1/s,2)(n)$, respectively.

Hence we have the following sequences of maps

$$M(1/q,1/s) \xrightarrow{2} L(1/q,1/s,n)(2) \xrightarrow{n} \mathscr{L}(1/q,1/s)(2,n)$$

and

$$M(1/q,1/s) \xrightarrow{n} L(1/q,1/s,2)(n) \xrightarrow{2} \mathscr{L}(1/q,1/s)(2,n)$$

which induce the subgroup embeddings

$$\Pi_1(M(1/q,1/s)) \lhd \Pi_1(L(1/q,1/s,n)(2)) \lhd \Pi_1(\mathscr{L}(1/q,1/s)(2,n))$$

and

$$\Pi_1(M(1/q,1/s)) \lhd \Pi_1(L(1/q,1/s,2)(n)) \lhd \Pi_1(\mathscr{L}(1/q,1/s)(2,n)),$$

where

$$\begin{aligned}
&[\Pi_1(\mathscr{L}(1/q,1/s)(2,n)) : \Pi_1(L(1/q,1/s,n)(2))] \\
&\qquad = [\Pi_1(L(1/q,1/s,2)(n)) : \Pi_1(M(1/q,1/s))] = n
\end{aligned}$$

and

$$\begin{aligned}
&[\Pi_1(L(1/q,1/s,n)(2)) : \Pi_1(M(1/q,1/s))] \\
&\qquad = [\Pi_1(\mathscr{L}(1/q,1/s)(2,n)) : \Pi_1(L(1/q,1/s,2)(n))] = 2.
\end{aligned}$$

This completes the proof. $\square$

For $q = l$ and $s = -l$ we re-obtain Theorem 3.2 of [14] since $\mathscr{L}(1/l, -1/l)$ coincides with the link $\mathscr{L}^{1/l}$ defined in [14], and shown in Figure 7b for convenience.

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