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ON THE WELLPOSEDNESS IN THE ULTRADIFFERENTIABLE CLASSES OF THE CAUCHY PROBLEM FOR A WEAKLY HYPERBOLIC EQUATION OF SECOND ORDER

By

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§1. Introduction

We first consider the linear equation

$$(1)\quad \begin{cases} \partial_t^2 u - \sum_{l,h=1}^{n} \partial_{x_l}(a_{lh}(t,x)\partial_{x_h}u) + \sum_{l=1}^{n} b_l(t,x)\partial_{x_l}u + c(t,x)u = g(t,x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T]\times \boldsymbol{R}_x^n \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases}$$

where $A(t,x) = \{a_{lh}(t,x)\}_{1\le l,h\le n}$ is a real symmetric matrix whose components satisfy

$$(2)\qquad a_{lh}(t,x) \in Z([0,T]; B\{F_j\}_{R_1}(\boldsymbol{R}_x^n)),$$

and the weakly hyperbolic condition

$$(3)\qquad \sum_{l,h=1}^{n} a_{lh}(t,x)\xi_l\xi_h \ge 0 \qquad \text{for } {}^\forall t\in[0,T],\ {}^\forall x\in \boldsymbol{R}_x^n,\ {}^\forall \xi\in \boldsymbol{R}_\xi^n,$$

and $B(t,x) = \{b_l(t,x)\}_{1\le l\le n}$ is a vector whose components satisfy

$$(4)\qquad b_l(t,x),\quad c(t,x)\in C^0([0,T]; B\{F_j\}_{R_1}(\boldsymbol{R}_x^n)).$$

Here we used the following notations of the function classes.

i) the function classes $Z([0,T])$ is defined with the increasing function $\omega(t)$ on $[0,\infty]$ satisfying

$$(5)\qquad \omega(0)=0,\quad \omega(t)^{-1} = \omega(t^{-1}),\quad \omega(t)\ge t \qquad \text{for } {}^\forall t\in(0,\infty),$$

as follows

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$$Z([0,T]) = \{f(t) \in C^0([0,T]);\ |f(t) - f(s)| \le {}^{\exists}C\omega(t-s) \text{ for } {}^{\forall}t, {}^{\forall}s \in [0,T] \text{ such that } t-s \in [0,1]\}.$$

(e.g. if $\omega(t) = |t|^{\sigma}$ $(0 < \sigma \le 1)$, the the function space $Z([0,T])$ coincides Hölder space.)

ii) the function classes $B\{F_j\}_R(\boldsymbol{R}_x^n)$ is defined as follows

$$B\{F_j\}_R(\boldsymbol{R}_x^n) = \{f(x) \in C^{\infty}(\boldsymbol{R}_x^n); \max_{|\alpha|=j} |\partial_x^{\alpha} f(x)| \le MR^j F_j \text{ for } {}^{\forall}x \in \boldsymbol{R}_x^n, {}^{\forall}\alpha \in \boldsymbol{N}^n\}.$$

There are a lot of papers concerned with the relation between the Hölder continuous coefficients and the Gevrey wellposedness for weakly hyperbolic equations (see [CJS], [D], [Ki2], [Ki3], [Nt], [OT]). We know the fact that the combination of Hölder classes in t and the Gevrey classes in x is well suited for this kind of the study. In order to treat the problem in the ultradifferentiable classes we introduced the function classes $Z([0,T])$.

THEOREM 1. *Let $T > 0$, $R_1 > 0$, and $\{F_j\}_{j=0}^{\infty}$, $\{G_j\}_{j=0}^{\infty}$ be sequences of positive numbers. Assume that the coefficients satisfy* (2), (3), (4). *Then there exists the positive function $\rho(t) \in C^1(\boldsymbol{R}_t^1)$ such that for any $\tilde{\rho}(t) < \rho(t)$ and $u_0(x)$ and $u_1(x) \in D_{L_2}\{G_j\}_{\rho(0)^{-1}}(\boldsymbol{R}_x^n)$, $g(t,x) \in C^0([0,T]; D_{L_2}\{G_j\}_{\rho(t)^{-1}}(\boldsymbol{R}_x^n))$, the Cauchy problem* (1) *has a unique soltion $u \in C^2([0,T], D_{L_2}\{G_j\}_{\tilde{\rho}(t)^{-1}}(\boldsymbol{R}_x^n))$, provided $\{G_j\}_{j=0}^{\infty}$ satsfying the logarithmically convex condition i.e.,*

$$\frac{G_i}{iG_{i-1}} \le \frac{G_j}{jG_{j-1}} \qquad \text{for } 1 \le {}^{\forall}i \le {}^{\forall}j, \tag{6}$$

and

$$G_j \ge F_j, \tag{7}$$

and

$$\lim_{t \downarrow 0} \frac{\omega(t)^{1/2} G_{[1/t]}}{[1/t] G_{[1/t]-1}} = 0, \tag{8}$$

where

$$D_{L^2}\{G_j\}_R(\boldsymbol{R}_x^n) = \left\{ f(x) \in D_{L^2}^{\infty}(\boldsymbol{R}_x^n); \sum_{j=1}^{\infty} \left\{ \sum_{|\alpha|=j} \|\partial_x^{\alpha} f\|_{L^2}^2 \right\}^{1/2} / R^j G_j < \infty \right\}.$$

We remark that the logarithmically convex condition has many equivalent forms (see [M]). The condition (6) is one of them, and for example the following is also a logarithmically convex condition

$$G_j^2 \leq G_{j-1}G_{j+1} \qquad \text{for } j = 1, 2, \ldots.$$

When $\omega(t) = |t|^{\sigma}$ $(0 < \sigma \leq 1)$ and $G_j = j!^s$, the condition (8) is satisfied if $s < 1 + (\sigma/2)$. Hence Theorem 1 includes the Gevrey case (see [CJS], [D], [N]).

This result of the linear problem may be applied to some nonlinear problems. For example, the quasianalytic class which is one of the ultradifferentiable classes, is meaningful in the treatment of the Kirchhoff equations (see [Nh], [H]). In this paper we shall also consider the another type of the nonlinear equation

$$\text{(9)} \qquad \begin{cases} \partial_t^2 u - \partial_x(a(t,x)\partial_x u) + f(t,x,u,u_x) = 0 \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases} \quad \text{in } [0,T] \times P$$

where P is a fixed closed interval, and the coefficients satisfy

$$\text{(2)}' \qquad a(t,x) \in Z([0,T]; B\{F_j\}_{R_1}(P)),$$

$$\text{(3)}' \qquad a(t,x) \geq 0 \qquad \text{for } {}^{\forall}t \in [0,T],\ {}^{\forall}x \in P,$$

$$\text{(10)} \qquad f(t,x,u,v) \in C^0([0,T]; B\{F_j\}_{R_1}(P), B\{E_j\}_{R_2}(\boldsymbol{R}^1), B\{E_j\}_{R_2}(\boldsymbol{R}^1)).$$

When $F_j = j!^s$ and $E_j = j!$, K. Kajitani proved the wellposedness for the Leray-Volevich's systems (see [Ka1]). When $F_j = j!^s$ and $E_j = j!^{s'}$ with the exponents satisfying $s' < s$, P. D'ancona and R. Manfrin proved the wellposedness for the abstract n-dimensional equations of second order (see [DM]). For the simplicity we shall only treat the 1-dimensional and P-periodic case.

THEOREM 2. *Let $R_1 > 0$, $R_2 > 0$, and $\{E_j\}_{j=0}^{\infty}$, $\{F_j\}_{j=0}^{\infty}$, $\{G_j\}_{j=0}^{\infty}$ be sequences of positive numbers. Assume that the function $\omega(t)$ satisfyies $\omega(t) \geq t$ for ${}^{\forall}t \in [0,1]$ and the coefficients satisfy* (2)′, (3)′, (10). *Then there exist $T > 0$ and the positive function $\rho(t) \in C^1(\boldsymbol{R}_t^1)$ such that for any $\tilde{\rho}(t) < \rho(t)$, and $u_0(x)$ and $u_1(x) \in D_{L_2}\{G_j\}_{\rho(0)^{-1}}(P)$, $f(t,x,0,0) \in C^0([0,T]; D_{L_2}\{G_j\}_{\rho(t)^{-1}}(P))$, the Cauchy problem* (9) *has a unique soltion $u \in C^2([0,T], D_{L_2}\{G_j\}_{\tilde{\rho}(t)^{-1}}(P))$, provided $\{G_j\}_{j=0}^{\infty}$ satisfying the logarithmically convex condition* (6), *and* (7), *and*

$$\text{(11)} \qquad \sup_{t \in (0,1]} \frac{\omega(t)^{1/2} G_{[1/t]}}{[1/t] G_{[1/t]-1}} < \infty,$$

and moreover there exist $p \geq 1$, $C > 0$ *such that*

$$\frac{G_j}{j^p G_{j-1}} \leq C \frac{G_i}{i^p G_{i-1}} \qquad \text{for } 1 \leq {}^\forall i \leq {}^\forall j, \tag{12}$$

and for any fixed $R > 0$,

$$\sum_{j=1}^{\infty} R^j \frac{E_j}{G_j} < \infty. \tag{13}$$

The condition (11) is weaker than the condition (8), since the solution of Theorem 2 is local, while the solution of Theorem 1 is global. We remark that the condition (12) is not contrary to the condition (6). For example the Gevrey case $G_j = j!^s$ satisfies (6) obviously and also satisfies (12) with $p \geq s$, $C = 1$.

Since the ultradifferentiable classes of our theorems are included by the Gevrey class whose order is equal to two, Levi condition for the lower terms is not necessary. In the Gevrey classes whose order is greater than or equal to two, M. Reissig and K. Yagdjian also solved linear and nonlinear problems with the generalized Levi condition (see [RY1], [RY2]).

In the proofs of theorems we don't use the theory of the pseudo-differential operators. If one use the pseudo-differential operators in the Gevrey or ultra-differentiable classes, one may generalize the principal part of the equation (see [C], [Nt]). Concerned with the theory for the pseudo-differential operators in the Gevrey or ultradifferentiable classes, many useful methods are introduced in [Ka1], [Ka2], [M].

§ 2. Preliminaries

In this section we shall introduce some notations and inequalities.

The energy estimate in § 3 is derived as a approximation of the strictly hyperbolic equations with the smooth enough coefficients. Therefore we need to regularize the coefficients. With the function $\phi(t) \in C_0^\infty(\boldsymbol{R}_t^1)$ such that $supp\,\phi \subset [-1, 1]$, $\phi(t) \geq 0$ and $\int_{-\infty}^{\infty} \phi(t)\,dt = 1$, we shall put

$$\phi_{j+\nu}(t) = (j+\nu)\phi((j+\nu)t) \qquad \text{for } j \geq 2,\ \nu \geq 0$$

and

$$A_{j+\nu}(t) = A * \phi_{j+\nu}(t)(a_{lh,j+\nu}(t) = a_{lh} * \phi_{j+\nu}(t)).$$

Then we get the following lemma.

LEMMA 1. *Let $A(t,x) = \{a_{lh}(t,x)\}_{1 \le l,h \le n}$ be a matrix whose components a_{lh} $(1 \le l,h \le n)$ satisfy* (2). *Then it holds that*

$$\begin{aligned} &\|A(t) - A_{j+\nu}(t)\|_{B^1(\boldsymbol{R}_x^n)} \le C_1\omega(j+\nu)^{-1} \\ &\|\partial_t A_{j+\nu}(t)\|_{B^0(\boldsymbol{R}_x^n)} \le C_2(j+\nu)\omega(j+\nu)^{-1}. \end{aligned} \tag{14}$$

PROOF. By (5) we can easily get for $\alpha \in \boldsymbol{N}^n$ satisfying $|\alpha| \le 1$

$$\begin{aligned} |\partial_x^\alpha(A(t) - A_{j+\nu}(t))| &\le \left|(j+\nu)\int_{-\infty}^{\infty} \phi((j+\nu)s)\partial_x^\alpha(A(t+s) - A(t))\,ds\right| \\ &\le C(j+\nu)\int_{-\infty}^{\infty} \phi((j+\nu)s)\omega(|s|)\,ds \\ &= C\int_{-\infty}^{\infty} \phi(s)\omega\left(\frac{|s|}{j+\nu}\right)ds \\ &\le C\omega\left(\frac{1}{j+\nu}\right)\int_{-1}^{1} \phi(s)\,ds \\ &= C\omega(j+\nu)^{-1}. \end{aligned}$$

Noting that $(j+\nu)\int_{-\infty}^{\infty} \phi'(s)A(t)\,ds = 0$, by (5) we also get

$$\begin{aligned} |\partial_t A_{j+\nu}(t)| &\le \left|(j+\nu)\int_{-\infty}^{\infty} -(j+\nu)\phi'((j+\nu)(s-t))A(s)\,ds\right| \\ &\le \left|(j+\nu)\int_{-\infty}^{\infty} \phi'(s)\left\{A(t) - A\left(t+\frac{s}{j+\nu}\right)\right\}ds\right| \\ &= C'(j+\nu)\int_{-1}^{1} |\phi'(s)|\omega\left(\frac{|s|}{j+\nu}\right)ds \\ &\le C'(j+\nu)\omega\left(\frac{1}{j+\nu}\right)\int_{-1}^{1} |\phi'(s)|\,ds \\ &= C''(j+\nu)\omega(j+\nu)^{-1}. \end{aligned}$$

The logarithmically convex condition is often used in the proofs of theorems. In particular it is required in order to show the following lemma.

LEMMA 2. *Let $\{G_j\}_{j=0}^{\infty}$ be sequences of positive numbers which satisfies the logarithmically convex condition*

(6) $$\frac{G_i}{iG_{i-1}} \le \frac{G_j}{jG_{j-1}} \qquad \textit{for } 1 \le {}^{\forall}i \le {}^{\forall}j.$$

Then it holds that

(15) $$\frac{G_{j-k+1}G_{k+1}j!}{G_j(j-k+1)!(k-1)!} \le Ck(k+1)$$

$$\textit{for } k = 1, 2, \dots, \quad j = k+1, k+2, \dots,$$

where C is independent of j and k.

This lemma plays important role to estimate the commutator part of the energy (see (22)). For the sequences of the Gevrey classes, i.e., $G_j = j!^s$ $(s > 1)$, this inequality (15) is also satisfied (see [D], [RY1]). We assert that for the sequences of the ultradifferrentiable classes, (15) still holds under the logarithmically convex condition (6).

PROOF. Noting the ranges of j and k, by (6) we get

$$\frac{G_{j-k+1}G_{k+1}j!}{G_j(j-k+1)!(k-1)!} = k(k+1)\frac{G_{j-k+1}G_{k+1}j!}{G_j(j-k+1)!(k+1)!}$$

$$= k(k+1)\frac{\frac{G_{j-k+1}}{(j-k+1)G_{j-k}}\cdots\frac{G_1}{1}\frac{G_{k+1}}{(k+1)G_k}\cdots\frac{G_1}{1}}{\frac{G_j}{jG_{j-1}}\frac{G_{j-1}}{(j-1)G_{j-2}}\cdots\frac{G_1}{1}}$$

if $j-k+1 \ge k+1$

$$= k(k+1)\frac{\frac{G_{j-k+1}}{(j-k+1)G_{j-k}}\cdots\frac{G_3}{3G_2}}{\frac{G_j}{jG_{j-1}}\cdots\frac{G_{k+2}}{(k+2)G_{k+1}}}$$

$$\times\frac{\frac{G_{k+1}}{(k+1)G_k}\cdots\frac{G_1}{1}}{\frac{G_{k+1}}{(k+1)G_k}\cdots\frac{G_1}{1}}\cdot\frac{\frac{G_2}{2G_1}\frac{G_1}{1}}{1}$$

$$\le k(k+1)\cdot 1\cdot 1\cdot C$$

$$= Ck(k+1) \qquad \text{for } k = 1, 2, \dots, \quad j = k+1, k+2, \dots.$$

if $k+1 \ge j-k+1$

$$= k(k+1)\,\frac{\dfrac{G_{k+1}}{(k+1)G_k}\cdots\dfrac{G_3}{3G_2}}{\dfrac{G_j}{jG_{j-1}}\cdots\dfrac{G_{j-k+2}}{(j-k+2)G_{j-k+1}}}$$

$$\cdot\frac{\dfrac{G_{j-k+1}}{(j-k+1)G_{j-k}}\cdots\dfrac{G_1}{1}}{\dfrac{G_{j-k+1}}{(j-k+1)G_{j-k}}\cdots\dfrac{G_1}{1}}\cdot\frac{\dfrac{G_2}{2G_1}\dfrac{G_1}{1}}{1}$$

$$\leq k(k+1)\cdot 1\cdot 1\cdot C$$

$$= Ck(k+1) \qquad \textit{for } k=1,2,\dots,\quad j=k+1,k+2,\dots,$$

where C depends on only G_1 and G_2. This implies (15).

In order to estimate the lower parts of the energy, we also need the another inequality which is similar to (15).

COROLLARY 3. *Let* $\{G_j\}_{j=0}^{\infty}$ *be a sequence of positive numbers which satisfies the logarithmically convex condition* (6). *Then it holds that*

$$\frac{G_{j-k-1}G_k(j-1)!}{G_{j-1}(j-k-1)!k!}\leq 1 \qquad \text{for } k=0,1,\dots,\quad j=k+1,k+2,\dots.$$

Naturally in the Gevrey case we can see easily that the above inequality holds. But in the ultradifferentiable case we still need the logarithmically convex condition (6) to show the above inequality.

PROOF. By (6) we get

$$\frac{G_{j-k-1}G_k(j-1)!}{G_{j-1}(j-k-1)!k!}=\frac{\dfrac{G_{j-k-1}}{(j-k-1)G_{j-k-2}}\cdots\dfrac{G_1}{1}\,\dfrac{G_k}{kG_{k-1}}\cdots\dfrac{G_1}{1}}{\dfrac{G_{j-1}}{(j-1)G_{j-2}}\,\dfrac{G_{j-2}}{(j-2)G_{j-3}}\cdots\dfrac{G_1}{1}}$$

if $j-k-1\geq k$

$$=\frac{\dfrac{G_{j-k-1}}{(j-k-1)G_{j-k-2}}\cdots\dfrac{G_1}{1}}{\dfrac{G_{j-1}}{(j-1)G_{j-2}}\cdots\dfrac{G_{k+1}}{(k+1)G_k}}\cdot\frac{\dfrac{G_k}{kG_{k-1}}\cdots\dfrac{G_1}{1}}{\dfrac{G_k}{kG_{k-1}}\cdots\dfrac{G_1}{1}}$$

$$\leq 1\cdot 1=1 \qquad \text{for } k=0,1,\dots,\quad j=k+1,k+2,\dots.$$

if $k+1 \geq j-k-1$

$$= \frac{\dfrac{G_k}{kG_{k-1}} \cdots \dfrac{G_1}{1}}{\dfrac{G_{j-1}}{(j-1)G_{j-2}} \cdots \dfrac{G_{j-k}}{(j-k)G_{j-k-1}}} \cdot \frac{\dfrac{G_{j-k-1}}{(j-k-1)G_{j-k-2}} \cdots \dfrac{G_1}{1}}{\dfrac{G_{j-k-1}}{(j-k-1)G_{j-k-2}} \cdots \dfrac{G_1}{1}}$$

$$\leq 1 \cdot 1 = 1 \qquad \text{for } k = 0, 1, \ldots, \quad j = k+1, k+2, \ldots.$$

This implies the corollary.

Furthermore, in order to estimate the nonlinear term in the proof of Theorem 2, we need the following lemma.

LEMMA 4. *Let* $\{G_j\}_{j=0}^{\infty}$ *be a sequence of positive numbers which satisfies the logarithmically convex condition* (6). *Then it holds that for* $h_i \geq 1$ $(1 \leq i \leq \mu)$ *satisfying* $h_1 + h_2 + \cdots + h_\mu = l$

$$\frac{l! G_{h_1} \cdots G_{h_\mu} G_\mu}{G_l h_1! \cdots h_\mu! \mu!} \leq \left(\frac{G_1}{G_0}\right)^\mu.$$

This inequality appear when we consider the compositions of functions. This can be also shown clearly in the Gevrey case. We shall prove this with the logarithmically convex condition (6).

PROOF. By (6) we get

$$\frac{l! G_{h_1} \cdots G_{h_\mu} G_\mu}{G_l h_1! \cdots h_\mu! \mu!}$$

$$= \frac{\dfrac{G_\mu}{\mu G_{\mu-1}} \dfrac{G_{h_1}}{h_1 G_{h_1-1}} \cdots \dfrac{G_1}{G_0}}{\dfrac{G_l}{l G_{l-1}} \cdots \dfrac{G_{l-h_1+1}}{(l-h_1+1)G_{l-h_1}}} \cdot \frac{\dfrac{G_{\mu-1}}{(\mu-1)G_{\mu-2}} \dfrac{G_{h_2}}{h_2 G_{h_2-1}} \cdots \dfrac{G_1}{G_0}}{\dfrac{G_{l-h_1}}{(l-h_1)G_{l-h_1-1}} \cdots \dfrac{G_{l-h_1-h_2+1}}{(l-h_1-h_2+1)G_{l-h_1-h_2}}}$$

$$\times \cdots \times \frac{\dfrac{G_1}{G_0} \dfrac{G_{h_\mu}}{h_\mu G_{h_\mu-1}} \cdots \dfrac{G_1}{G_0}}{\dfrac{G_{h_\mu}}{h_\mu G_{h_\mu-1}} \cdots \dfrac{G_1}{G_0}}$$

$$\leq \frac{G_1}{G_0} \frac{G_1}{G_0} \cdots \frac{G_1}{G_0} = \left(\frac{G_1}{G_0}\right)^\mu.$$

This implies the lemma.

§ 3. Proof of Theorem 1

In this section our main task is to invistigate the regularity in the space variable x of solutions. Therefore we shall derive the energy inequality which can be also applied to a certain extent in the proof of Theorem 2. The method of the finite or infinite order energy are used by many people. We shall also use this kind of energy for the proof. But the form of our energy is different slightly from others.

Using the notations which are prepared in § 2, we shall first define the partial energies

$$e_{j,\nu}(t) = \left\{ \int_{R_x^n} \sum_{|\alpha|=j-1} (A_{j+\nu}(t,x,\partial)\partial_x^\alpha u, \partial_x^\alpha u) + \omega(j+\nu)^{-1} \sum_{|\alpha|=j} |\partial_x^\alpha u|^2 \right.$$
$$\left. + (j+\nu)^2 \sum_{|\alpha|=j-1} |\partial_x^\alpha u|^2 + \sum_{|\alpha|=j-1} |\partial_t \partial_x^\alpha u|^2 \, dx \right\}^{1/2},$$

where $A_{j+\nu}(t,x,\partial) = \sum_{l,h=1}^n \partial_{x_l} a_{lhj+\nu}(t,x)\partial_{x_h}$.

Our energy includes the parameter ν which gives the various benefits to the proofs. In the proof of theorem 1, ν is taken large enough and play a role to show the global solution. In the proof of theorem 2, ν is taken zero and play a role as the weight for energies.

Hence we can easily see the following relations between the partial energies and the L_2-norms of the derivatives of u

$$(16)\qquad \begin{aligned} &\sum_{|\alpha|=j-1} \|\partial_x^\alpha u\|_{L_2}^2 \le (j+\nu)^{-2} e_{j,\nu}(t)^2, \quad \sum_{|\alpha|=j} \|\partial_x^\alpha u\|_{L_2}^2 \le \omega(j+\nu) e_{j,\nu}(t)^2, \\ &\sum_{|\alpha|=j-1} \|\partial_t \partial_x^\alpha u\|_{L_2}^2 \le e_{j,\nu}(t)^2, \quad \sum_{|\alpha|=j} \|\partial_t \partial_x^\alpha u\|_{L_2}^2 \le e_{j+1,\nu}(t)^2. \end{aligned}$$

Differentiating $\{e_{j,\nu}(t)\}^2$, by (14) we get

$$\frac{d}{dt}\{e_{j,\nu}(t)\}^2$$
$$= \int \sum_{|\alpha|=j-1} ((\partial_t A_{j+\nu}(t,x,\partial))\partial_x^\alpha u, \partial_x^\alpha u) + 2\Re \sum_{|\alpha|=j-1} (A_{j+\nu}(t,x,\partial)\partial_t \partial_x^\alpha u, \partial_x^\alpha u)$$
$$+ 2\Re\omega(j+\nu)^{-1} \sum_{|\alpha|=j} (\partial_t \partial_x^\alpha u, \partial_x^\alpha u) + 2\Re(j+\nu)^2 \sum_{|\alpha|=j-1} (\partial_t \partial_x^\alpha u, \partial_x^\alpha u)$$
$$+ 2\Re \sum_{|\alpha|=j-1} (\partial_t^2 \partial_x^\alpha u, \partial_t \partial_x^\alpha u) \, dx$$

$$
\begin{aligned}
&\leq \sum_{|\alpha|=j-1} ((\partial_t A_{j+\nu}(t,x,\partial))\partial_x^\alpha u, \partial_x^\alpha u)_{L_2} \\
&\quad + 2\Re \sum_{|\alpha|=j-1} ((A_{j+\nu}(t,x,\partial) - A(t,x,\partial))\partial_t \partial_x^\alpha u, \partial_x^\alpha u)_{L_2} \\
&\quad + 2\Re\omega(j+\nu)^{-1} \sum_{|\alpha|=j} (\partial_t \partial_x^\alpha u, \partial_x^\alpha u)_{L_2} + 2\Re(j+\nu)^2 \sum_{|\alpha|=j-1} (\partial_t \partial_x^\alpha u, \partial_x^\alpha u)_{L_2} \\
&\quad + 2\Re \sum_{|\alpha|=j-1} ([\partial_x^\alpha, A]u, \partial_t \partial_x^\alpha u)_{L_2} - 2\Re \sum_{|\alpha|=j-1} (\partial_x^\alpha B(t,x,\partial)u, \partial_t \partial_x^\alpha u)_{L_2} \\
&\quad - 2\Re \sum_{|\alpha|=j-1} (\partial_x^\alpha c(t,x)u, \partial_t \partial_x^\alpha u)_{L_2} + 2\Re \sum_{|\alpha|=j-1} (\partial_x^\alpha g(t,x), \partial_t \partial_x^\alpha u)_{L_2} \\
&\leq C_2(j+\nu)\omega(j+\nu)^{-1} \sum_{|\alpha|=j-1} \Big(\sum_{|\beta|=1} \|\partial_x^{\alpha+\beta} u\|\Big)^2 \\
&\quad + 2C_1\omega(j+\nu)^{-1} \sum_{|\alpha|=j-1} \Big(\sum_{|\beta|=1} \|\partial_x^{\alpha+\beta} u\|\Big)\Big(\sum_{|\beta|=1} \|\partial_t \partial_x^{\alpha+\beta} u\|\Big) \\
&\quad + 2\omega(j+\nu)^{-1} \sum_{|\alpha|=j} \|\partial_x^\alpha u\| \, \|\partial_t \partial_x^\alpha u\| + 2(j+\nu)^2 \sum_{|\alpha|=j-1} \|\partial_x^\alpha u\| \, \|\partial_t \partial_x^\alpha u\| \\
&\quad + 2 \sum_{|\alpha|=j-1} \|[\partial_x^\alpha, A(t,x,\partial)]u\| \, \|\partial_t \partial_x^\alpha u\| + 2 \sum_{|\alpha|=j-1} \|\partial_x^\alpha B(t,x,\partial)u\| \, \|\partial_t \partial_x^\alpha u\| \\
&\quad + 2 \sum_{|\alpha|=j-1} \|\partial_x^\alpha c(t,x)u\| \, \|\partial_t \partial_x^\alpha u\| + 2 \sum_{|\alpha|=j-1} \|\partial_x^\alpha g(t)\| \, \|\partial_t \partial_x^\alpha u\|,
\end{aligned}
$$

where $A(t,x,\partial) = \sum_{l,h=1}^n \partial_{x_l} a_{lh}(t,x) \partial_{x_h}$ and $B(t,x,\partial) = \sum_{l=1}^n b_l(t,x)\partial_{x_l}$.

Noting (16) and dividing by $e_{j,\nu}(t)$, we get

$$
\begin{aligned}
(17) \quad \frac{d}{dt} e_{j,\nu}(t) &\leq C_n(j+\nu)e_{j,\nu}(t) + C_n'\omega(j+\nu)^{-1/2} e_{j+1,\nu}(t) + C_n''(j+\nu)e_{j,\nu}(t) \\
&\quad + C_n'' \Big\{ \sum_{|\alpha|=j-1} \|[\partial_x^\alpha, A(t,x,\partial)]u\|^2 \Big\}^{1/2} + C_n'' \Big\{ \sum_{|\alpha|=j-1} \|\partial_x^\alpha B(t,x,\partial)u\|^2 \Big\}^{1/2} \\
&\quad + C_n'' \Big\{ \sum_{|\alpha|=j-1} \|\partial_x^\alpha c(t,x)u\|^2 \Big\}^{1/2} + C_n'' \Big\{ \sum_{|\alpha|=j-1} \|\partial_x^\alpha g(t)\|^2 \Big\}^{1/2}.
\end{aligned}
$$

In order to estimate the last four terms, we introduce the whole energy and investigate them more carefully. With the positive function $\rho(t)$ satisfying

$$(18) \qquad \rho(t)R_1 < 1,$$

we shall define the infinite order energy

$$E_\nu(t) = \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} e_{j,\nu}(t). \tag{19}$$

By (17), (19) we have

$$\begin{aligned} \frac{d}{dt}E_\nu(t) \le & \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-2}}{G_{j+\nu-2}} (j+\nu-2)\rho'(t) e_{j,\nu}(t) + \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_n (j+\nu) e_{j,\nu}(t) \\ & + \sum_{j=3}^{\infty} \frac{\rho(t)^{j+\nu-2}}{G_{j+\nu-3}} C_n' \omega (j+\nu-1)^{-1/2} e_{j,\nu}(t) + \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_n'' (j+\nu) e_{j,\nu}(t) \\ & + L(t), \end{aligned} \tag{20}$$

where

$$\begin{aligned} L(t) = & \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_n'' \left\{ \sum_{|\alpha|=j-1} \| [\partial_x^\alpha, A(t,x,\partial)] u \|^2 \right\}^{1/2} \\ & + \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_n'' \left\{ \sum_{|\alpha|=j-1} \| \partial_x^\alpha B(t,x,\partial) u \|^2 \right\}^{1/2} \\ & + \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_n'' \left\{ \sum_{|\alpha|=j-1} \| \partial_x^\alpha c(t,x) u \|^2 \right\}^{1/2} \\ & + \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_n'' \left\{ \sum_{|\alpha|=j-1} \| \partial_x^\alpha g(t) \|^2 \right\}^{1/2}. \end{aligned}$$

Picking up the last term, we preceed to estimate.

We start to invistigate the first term of $L(t)$ which has the commutater. But moreover we need separate this term to three parts as follows (see [D]).

$$\begin{aligned} & \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_n'' \left\{ \sum_{|\alpha|=j-1} \| [\partial_x^\alpha, A(t,x,\partial)] u \|^2 \right\}^{1/2} \\ & = C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \left\{ \sum_{|\alpha|=j-1} \left\| \sum_{l,h} (\partial_x^{\alpha+e_l} a_{lh} \partial_x^{e_h} u - \partial_x^{e_l} a_{lh} \partial_x^{\alpha+e_h} u) \right\|^2 \right\}^{1/2} \\ & = C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \left\{ \sum_{|\alpha|=j-1} \left\| \sum_{l,h} \left(\sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\partial_x^{\alpha-\beta+e_l} a_{lh}) \partial_x^{\beta+e_h} u \right. \right. \right. \end{aligned} \tag{21}$$

$$+\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}a_{lh})\partial_x^{\beta+e_l+e_h}u-(\partial_x^{e_l}a_{lh})\partial_x^{\alpha+e_h}u-a_{lh}\partial_x^{\alpha+e_l+e_h}u\Bigg)\Bigg\|^2\Bigg\}^{1/2}$$

$$=C_n''\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}\Bigg\{\sum_{|\alpha|=j-1}\Bigg\|\sum_{l,h}\sum_{\beta\leq\alpha,\beta\neq\alpha}\binom{\alpha}{\beta}\Bigg((\partial_x^{\alpha-\beta+e_l}a_{lh})\partial_x^{\beta+e_h}u$$

$$+\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}a_{lh})\partial_x^{\beta+e_l+e_h}u\Bigg)\Bigg\|^2\Bigg\}^{1/2}$$

$$=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

where

$$\mathrm{I}=C_n''\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}\Bigg\{\sum_{|\alpha|=j-1}\Bigg\|\sum_{l,h}\sum_{\beta\leq\alpha,\beta\neq\alpha}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta+e_l}a_{lh})\partial_x^{\beta+e_h}u\Bigg\|^2\Bigg\}^{1/2}$$

$$\mathrm{II}=C_n''\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}\Bigg\{\sum_{|\alpha|=j-1}\Bigg\|\sum_{l,h}\sum_{\beta\leq\alpha,|\beta|<|\alpha|-1}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}a_{lh})\partial_x^{\beta+e_l+e_h}u\Bigg\|^2\Bigg\}^{1/2}$$

$$\mathrm{III}=C_n''\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}\Bigg\{\sum_{|\alpha|=j-1}\Bigg\|\sum_{l,h}\sum_{\beta\leq\alpha,|\beta|=|\alpha|-1}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}a_{lh})\partial_x^{\beta+e_l+e_h}u\Bigg\|^2\Bigg\}^{1/2}.$$

Now we introduce the useful lemma to estimate I and II.

LEMMA 5. *Let $H>1$, and $\{x_\beta\}$ a sequence of non-negative real numbers, indicized by $\beta\in N^n$, Then for every integer j*

$$\Bigg\{\sum_{|\alpha|=j-1}\Bigg(\sum_{\beta\leq\alpha,|\beta|\leq s}x_\beta\Bigg)^2\Bigg\}^{1/2}\leq C_{n,H}\sum_{k=0}^{s}H^{j-k-1}\Bigg(\sum_{|\beta|=k}x_\beta^2\Bigg)^{1/2}.$$

For the proof refer to [AS].

Thanks to this lemma, we can make desirable changes of the parameters of summations. Using Lemma 5 with

$$x_\beta=\sum_{l,h}\binom{j-1}{|\beta|}R_1^{j-|\beta|}F_{j-|\beta|}\|\partial_x^{\beta+e_h}u\|,\quad s=j-2,$$

we get

$$\mathrm{I} \le C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_a \left\{ \sum_{|\alpha|=j-1} \left(\sum_{l,h} \sum_{\beta\le\alpha,\beta\ne\alpha} \binom{j-1}{|\beta|} R_1^{j-|\beta|} F_{j-|\beta|} \|\partial_x^{\beta+e_h} u\| \right)^2 \right\}^{1/2}$$

$$\le C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_a C_{n,H} \sum_{k=0}^{j-2} H^{j-k-1}$$

$$\times \left\{ \sum_{|\beta|=k} \left(\sum_{l,h} \binom{j-1}{|\beta|} R_1^{j-|\beta|} F_{j-|\beta|} \|\partial_x^{\beta+e_h} u\| \right)^2 \right\}^{1/2}$$

Putting $\beta' = \beta + e_h,\ k' = k+1,$

$$= C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_a C_{n,H} \sum_{k'=1}^{j-1} H^{j-k'}$$

$$\times \left\{ \sum_{|\beta'|=k'} \left(\sum_{l,h} \binom{j-1}{k'-1} R_1^{j-k'+1} F_{j-k'+1} \|\partial_x^{\beta'} u\| \right)^2 \right\}^{1/2}$$

$$\le C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_a C_{n,H}' \sum_{k'=1}^{j-1} \binom{j-1}{k'-1} (HR_1)^{j-k'+1} F_{j-k'+1} \left\{ \sum_{|\beta'|=k'} \|\partial_x^{\beta'} u\|^2 \right\}^{1/2}.$$

Similarly using Lemma 5 with

$$x_\beta = \sum_{l,h} \binom{j-1}{|\beta|} R_1^{j-1-|\beta|} F_{j-1-|\beta|} \|\partial_x^{\beta+e_l+e_h} u\|, \quad s = j-3,$$

we get

$$\mathrm{II} \le C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_a$$

$$\times \left\{ \sum_{|\alpha|=j-1} \left(\sum_{l,h} \sum_{\beta\le\alpha,|\beta|<|\alpha|-1} \binom{j-1}{|\beta|} R_1^{j-1-|\beta|} F_{j-1-|\beta|} \|\partial_x^{\beta+e_l+e_h} u\| \right)^2 \right\}^{1/2}$$

$$\le C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_a C_{n,H} \sum_{k=0}^{j-3} H^{j-k-1}$$

$$\times \left\{ \sum_{|\beta|=k} \left(\sum_{l,h} \binom{j-1}{|\beta|} R_1^{j-1-|\beta|} F_{j-1-|\beta|} \|\partial_x^{\beta+e_l+e_h} u\| \right)^2 \right\}^{1/2}$$

Putting $\beta' = \beta + e_l + e_h$, $k' = k + 2$,

$$= C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_a C_{n,H} \sum_{k'=2}^{j-1} H^{j-k'+1}$$

$$\times \left\{ \sum_{|\beta'|=k'} \left(\sum_{l,h} \binom{j-1}{k'-2} R_1^{j-k'+1} F_{j-k'+1} \|\partial_x^{\beta'} u\| \right)^2 \right\}^{1/2}$$

$$\le C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_a C_{n,H}' \sum_{k'=2}^{j-1} \binom{j-1}{k'-2} (HR_1)^{j-k'+1} F_{j-k'+1} \left\{ \sum_{|\beta'|=k'} \|\partial_x^{\beta'} u\|^2 \right\}^{1/2}.$$

Noting that $\binom{j-1}{k'-1} + \binom{j-1}{k'-2} = \binom{j}{k'-1}$, we obtain

$$\text{(22)} \qquad \mathrm{I} + \mathrm{II} \le C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_a C_{n,H}' \sum_{k=1}^{j-1} \binom{j}{k-1}$$

$$\times (HR_1)^{j-k+1} F_{j-k+1} \left\{ \sum_{|\beta|=k} \|\partial_x^{\beta} u\|^2 \right\}^{1/2}$$

$$\le C_n'' M_a (HR_1)^2 \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} (\rho(t) HR_1)^{j-k-1} \left\{ \frac{G_{k+\nu-1}}{G_{k+1}} \frac{G_j}{G_{j+\nu-2}} \right\}$$

$$\times \left\{ \frac{F_{j-k+1}}{G_{j-k+1}} \right\} \left\{ \frac{G_{j-k+1} G_{k+1} j!}{G_j (j-k+1)!(k-1)!} \right\} \frac{\rho(t)^{k+\nu}}{G_{k+\nu-1}} \left\{ \sum_{|\beta|=k} \|\partial_x^{\beta} u\|^2 \right\}^{1/2}.$$

In the next step we use the following inequalities.

i) By (7) it holds that

$$\frac{F_{j-k+1}}{G_{j-k+1}} \le 1 \qquad \text{for } k = 1, 2, \ldots, \quad j = k+1, k+2, \ldots.$$

ii) It holds that

when $\nu \ge 3$

$$\frac{G_{k+\nu-1}}{G_{k+1}} \frac{G_j}{G_{j+\nu-2}} = \frac{\dfrac{G_{k+\nu-1}}{G_{k+\nu-2}} \dfrac{G_{k+\nu-2}}{G_{k+\nu-3}} \cdots \dfrac{G_{k+2}}{G_{k+1}}}{\dfrac{G_{j+\nu-2}}{G_{j+\nu-3}} \dfrac{G_{j+\nu-3}}{G_{j+\nu-4}} \cdots \dfrac{G_{j+1}}{G_j}}$$

$$\text{by (6)} \le \frac{k+\nu-1}{j+\nu-2} \frac{k+\nu-2}{j+\nu-3} \cdots \frac{k+2}{j+1}$$

$$\le 1 \qquad \text{for } k = 1, 2, \ldots, \quad j = k+1, k+2, \ldots.$$

when $v = 2$

$$\frac{G_{k+v-1}}{G_{k+1}}\frac{G_j}{G_{j+v-2}} = 1 \qquad \text{for } k = 1, 2, \dots, \quad j = k+1, k+2, \dots.$$

when $v = 1$

$$\begin{aligned}\frac{G_{k+v-1}}{G_{k+1}}\frac{G_j}{G_{j+v-2}} &= \frac{G_k}{G_{k+1}}\frac{G_j}{G_{j-1}}\\ \text{by (12)} &\leq C\left(\frac{j}{k+1}\right)^p\\ &\leq C(j-k)^p \qquad \text{for } k = 1, 2, \dots, \quad j = k+1, k+2, \dots.\end{aligned}$$

when $v = 0$

$$\begin{aligned}\frac{G_{k+v-1}}{G_{k+1}}\frac{G_j}{G_{j+v-2}} &= \frac{\dfrac{G_j}{G_{j-1}}\dfrac{G_{j-1}}{G_{j-2}}}{\dfrac{G_{k+1}}{G_k}\dfrac{G_k}{G_{k-1}}}\\ \text{by (12)} &\leq C\left(\frac{j}{k+1}\right)^p C\left(\frac{j-1}{k}\right)^p\\ &\leq C^2(j-k)^{2p} \qquad \text{for } k = 1, 2, \dots, \quad j = k+1, k+2, \dots.\end{aligned}$$

Then by (16), (18), (22), i), ii) and Lemma 2, we get

$$\begin{aligned}\mathrm{I} + \mathrm{II} &\leq C_n''' M_a \sum_{k=1}^{\infty} k(k+1)(k+1+v)^{-1}\frac{\rho(t)^{k+v}}{G_{k+v-1}} e_{k+1,v}(t)\\ &\leq C_n''' M_a \sum_{j=2}^{\infty} \frac{\rho(t)^{j+v-1}}{G_{j+v-2}}(j-1)e_{j,v}(t).\end{aligned}$$

As for III, we introduce the another useful lemma.

Lemma 6. *Let $(T_{lh}(x))$ be a Hermitian non-negative matrix of functions in $C^2(\boldsymbol{R}^n)$. Then for every $n \times n$ symmetric matrix (η_{lh}), for $j - 1, \dots, n$*

$$\left(\sum_{lh} \partial_{x_m} T_{lh}(x)\eta_{lh}\right)^2 \leq C_{n,T} \sum_{l,h,q} T_{lh}(x)\eta_{lq}\eta_{hq}.$$

For the proof refer to [O]. We remark that this lemma is needed to derive the hyperbolicity from n-dimensional equations of second order (see [D]). While Glaeser's inquality is also used in the 1-dimensional case (see [RY1], [RY2]).

Nothing that $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$ for $a \geq 0$, $b \geq 0$, it holds that

$$\mathrm{III} \leq C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \left\{ \sum_{|\alpha|=j-1} \left\| \sum_{l,h} \sum_{\beta \leq \alpha, |\beta|=|\alpha|-1} \binom{\alpha}{\beta} (\partial_x^{\alpha-\beta} a_{lh,j+\nu}) \partial_x^{\beta+e_l+e_h} u \right\|^2 \right\}^{1/2}$$

$$+ C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}$$

$$\times \left\{ \sum_{|\alpha|=j-1} \left\| \sum_{l,h} \sum_{\beta \leq \alpha, |\beta|=|\alpha|-1} \binom{\alpha}{\beta} (\partial_x^{\alpha-\beta} \{a_{lh} - a_{lh,j+\nu}\}) \partial_x^{\beta+e_l+e_h} u \right\|^2 \right\}^{1/2}.$$

Putting $\gamma = \alpha - \beta$ satisfying $|\gamma| = 1$, by Lemma 6 and (14) we get

$$\mathrm{III} \leq C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} (j-1) \left\{ \sum_{|\alpha|=j-1} \left\| \sum_{l,h} \sum_{|\gamma|=1} (\partial_x^{\gamma} a_{lh,j+\nu}) \partial_x^{\alpha-\gamma+e_l+e_h} u \right\|^2 \right\}^{1/2}$$

$$+ C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} (j-1) \left\{ \sum_{|\alpha|=j-1} \left\| \sum_{l,h} \sum_{|\gamma|=1} (\partial_x^{\gamma} \{a_{lh} - a_{lh,j+\nu}\}) \partial_x^{\alpha-\gamma+e_l+e_h} u \right\|^2 \right\}^{1/2}$$

$$\leq C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_{n,a}(j-1)$$

$$\times \left\{ \sum_{|\alpha|=j-1, |\gamma|=1, |q|=1} \left(\sum_{l,h} a_{lh,j+\nu} \partial_x^{\alpha-\gamma+q+e_l} u, \partial_x^{\alpha-\gamma+q+e_l} u \right) \right\}^{1/2}$$

$$+ C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_{n,a}'(j-1)$$

$$\times \left\{ \sum_{|\alpha|=j-1, |\gamma|=1, |e_l|=1, |e_h|=1} \omega(j+\nu)^{-1} \|\partial_x^{\alpha-\gamma+e_l+e_h} u\|^2 \right\}^{1/2}$$

$$\leq C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_{n,a}(j-1) \left\{ \sum_{|\alpha|=j-1} (A_{j+\nu}(t,x,\partial) \partial_x^{\alpha} u, \partial_x^{\alpha} u) \right\}^{1/2}$$

$$+ C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_{n,a}'(j-1) \omega^{-1}(j+\nu) \left\{ \sum_{|\alpha|=j} \|\partial_x^{\alpha} u\|^2 \right\}^{1/2}$$

$$\leq C_{n,a}'''(j-1) e_{j,\nu}(t).$$

Thus finally we obtain

$$\text{(23)}\quad \text{the first term of } L(t) = \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} C_n'' \Big\{ \sum_{|\alpha|=j-1} \|[\partial_x^\alpha, A(t,x,\partial)]u\|^2 \Big\}^{1/2}$$

$$\le \mathrm{I} + \mathrm{II} + \mathrm{III}$$

$$\le C_3 \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} (j-1) e_{j,\nu}(t).$$

Secondly we shall estimate the lower terms of $L(t)$. Lemma 5 and i) are also used again. But we must modify ii) and use Corollary 3 instead of Lemma 2 in §2 to estimate them. It is not necessary to change the inhomogeneous term of $L(t)$.

Since the third term of $L(t)$ can be treated quite similarly as the second term of $L(t)$, we only estimate the second term of $L(t)$. Using Lemma 5 with

$$x_\beta = \sum_l \binom{j-1}{|\beta|} R_1^{j-1-|\beta|} F_{j-1-|\beta|} \|\partial_x^{\beta+e_l} u\|, \quad s = j-1,$$

we get

the second term of $L(t)$

$$= C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \Big\{ \sum_{|\alpha|=j-1} \Big\| \sum_l \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial_x^{\alpha-\beta} b_l \partial_x^\beta \partial_x^{e_l} u \Big\|^2 \Big\}^{1/2}$$

$$\le C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_b \Big\{ \sum_{|\alpha|=j-1} \Big(\sum_l \sum_{\beta\le\alpha} \binom{j-1}{|\beta|} R_1^{j-1-|\beta|} F_{j-1-|\beta|} \|\partial_x^{\beta+e_l} u\| \Big)^2 \Big\}^{1/2}$$

$$\le C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} M_b C_{n,H} \sum_{k=0}^{j-1} H^{j-k-1}$$

$$\times \Big\{ \sum_{|\beta|=k} \Big(\sum_l \binom{j-1}{|\beta|} R_1^{j-1-|\beta|} F_{j-1-|\beta|} \|\partial_x^{\beta+e_l} u\| \Big)^2 \Big\}^{1/2}$$

$$\le C_{n,H}'' M_b \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \sum_{k=0}^{j-2} \binom{j-1}{k} M_b (HR_1)^{j-1-k} F_{j-1-k} \Big\{ \sum_{|\beta|=k+1} \|\partial_x^\beta u\|^2 \Big\}^{1/2}$$

$$+ C_{n,H}'' M_b \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} F_0 \Big\{ \sum_{|\beta|=j} \|\partial_x^\beta u\|^2 \Big\}^{1/2}$$

$$= C''_{n,H} M_b \sum_{k=2}^{\infty} \sum_{j=k+2}^{\infty} (\rho(t) H R_1)^{j-k-1} \left\{ \frac{F_{j-1-k}}{G_{j-1-k}} \right\} \left\{ \frac{G_{k+\nu-1}}{G_k} \frac{G_{j-1}}{G_{j+\nu-2}} \right\}$$

$$\times \left\{ \frac{G_{j-k-1} G_k (j-1)!}{G_{j-1} (j-k-1)! k!} \right\} \frac{\rho(t)^{k+\nu}}{G_{k+\nu-1}} \left\{ \sum_{|\beta|=k+1} \|\partial_x^\beta u\|^2 \right\}^{1/2}$$

$$+ C''_{n,H} M_b F_0 \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \left\{ \sum_{|\beta|=j} \|\partial_x^\beta u\|^2 \right\}^{1/2}.$$

In the next step we use the following which is modifications of ii).
ii)$'$ It holds that when $\nu \geq 2$

$$\frac{G_{k+\nu-1}}{G_k} \frac{G_{j-1}}{G_{j+\nu-2}} \leq 1 \qquad \text{for } k = 2, 3, \dots, \quad j = k+2, k+3, \dots.$$

when $\nu = 1$

$$\frac{G_{k+\nu-1}}{G_k} \frac{G_{j-1}}{G_{j+\nu-2}} = 1 \qquad \text{for } k = 2, 3, \dots, \quad j = k+2, k+3, \dots.$$

when $\nu = 0$

$$\frac{G_{k+\nu-1}}{G_k} \frac{G_{j-1}}{G_{j+\nu-2}} \leq C(j-k)^p \qquad \text{for } k = 2, 3, \dots, \quad j = k+2, k+3, \dots.$$

Then by (16), (18), i), ii)$'$, and Corollary 3 we get

$$\text{(24)} \quad \text{the second term of } L(t) \leq C'''_{n,H} M_b \sum_{k=2}^{\infty} \frac{\rho(t)^{k+\nu}}{G_{k+\nu-1}} \omega(k+\nu+1)^{1/2} e_{k+1,\nu}(t)$$

$$+ C''_{n,H} M_b F_0 \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \omega(j+\nu)^{1/2} e_{j,\nu}(t)$$

$$\leq C_4 \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \omega(j+\nu)^{1/2} e_{j,\nu}(t).$$

Similarly we get

$$\text{(25)} \quad \text{the third term of } L(t) \leq C_5 \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \omega(j+\nu)^{1/2} e_{j,\nu}(t).$$

At last summing up (23), (24), (25), we can get

$$(26)\ \sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}C_n''L(t)\le\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}\{C_3(j-1)+(C_4+C_5)\omega(j+\nu)^{1/2}\}e_{j,\nu}(t)$$
$$+C_n''\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}\left\{\sum_{|\alpha|=j-1}\|\partial_x^\alpha g(t)\|^2\right\}^{1/2}.$$

Consequently by (20), (26) we have

(27)

$$\frac{d}{dt}E_\nu(t)\le\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-2}}{G_{j+\nu-2}}(j+\nu-1)e_{j,\nu}(t)$$
$$\times\left\{\rho'(t)+\rho(t)\left(C_n\frac{j+\nu}{j+\nu-1}+C_n''\frac{j+\nu}{j+\nu-1}+C_3\frac{j-1}{j+\nu-1}\right.\right.$$
$$\left.\left.+C_4\frac{\omega(j+\nu)^{1/2}}{j+\nu-1}+C_5\frac{\omega(j+\nu)^{1/2}}{j+\nu-1}\right)+C_n'\frac{\omega(j+\nu-1)^{-1/2}G_{j+\nu-2}}{(j+\nu-1)G_{j+\nu-3}}\right\}$$
$$+\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}C_n''\left\{\sum_{|\alpha|=j-1}\|\partial_x^\alpha g(t)\|^2\right\}^{1/2}$$
$$\le\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-2}}{G_{j+\nu-2}}(j+\nu-1)e_{j,\nu}(t)\left\{\rho'(t)+\rho(t)\right.$$
$$\left.\times(2C_n+2C_n''+C_3+\sqrt{2}C_4+\sqrt{2}C_5)+C_n'\frac{\omega(j+\nu-1)^{-1/2}G_{j+\nu-2}}{(j+\nu-1)G_{j+\nu-3}}\right\}$$
$$+C_n''\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}\left\{\sum_{|\alpha|=j-1}\|\partial_x^\alpha g(t)\|^2\right\}^{1/2},$$

here we used

$$\frac{j+\nu}{j+\nu-1}\le 2,\ \frac{j-1}{j+\nu-1}\le 1,\ \frac{\omega(j+\nu)^{1/2}}{j+\nu-1}\le\frac{(j+\nu)^{1/2}}{j+\nu-1}\le\sqrt{2}\quad\text{for }{}^\forall j\ge 1,\ {}^\forall\nu\ge 0.$$

POPOSITION 7. *Let u be a solution to the Cauchy problem* (1). *Then there exist the positive function $\rho(t)$ and ν_0 such that for any $T>0$ and any $\nu\ge\nu_0$*

(28)

$$E_\nu(t)\le E_\nu(0)+C_n''\int_0^T\sum_{j=2}^{\infty}\frac{\rho(s)^{j+\nu-1}}{G_{j+\nu-2}}\left\{\sum_{|\alpha|=j-1}\|\partial_x^\alpha g(s)\|^2\right\}^{1/2}ds\qquad\textit{for } 0\le t\le T.$$

PROOF. Noting that from the logarithmically convex condition (6)

$$G_j^2 \leq G_{j-1}G_{j+1} \qquad \text{for } j = 1, 2, \ldots,$$

we get

$$\frac{\omega(j+\nu-1)^{-1/2}G_{j+\nu-2}}{(j+\nu-1)G_{j+\nu-3}} \leq \frac{\omega\left(\dfrac{1}{j+\nu-1}\right)^{1/2} G_{j+\nu-1}}{\left(\dfrac{1}{j+\nu-1}\right) G_{j+\nu-2}}.$$

Hence by (8) we can see that there exists a large enough $\nu_0 > 0$ such that for arbitrary $\varepsilon > 0$

$$\frac{\omega(j+\nu-1)^{-1/2}G_{j+\nu-2}}{(j+\nu-1)G_{j+\nu-3}} \leq \varepsilon \qquad \text{for } {}^\forall\nu \geq \nu_0 \ ({}^\forall j \geq 2).$$

Now we shall determine $\rho(t)$ such that

$$\begin{cases} \rho'(t) + \rho(t)(2C_n + 2C_n'' + C_3 + \sqrt{2}C_4 + \sqrt{2}C_5) + C_n'\varepsilon = 0 \\ \rho(0) = \rho_0. \end{cases}$$

Hence we get the monotone decreasing function

$$\rho(t) = e^{-(2C_n+2C_n''+C_3+\sqrt{2}C_4+\sqrt{2}C_5)t}\left\{\rho_0 - \frac{C_n'\varepsilon}{2C_n + 2C_n'' + C_3 + \sqrt{2}C_4 + \sqrt{2}C_5}\right.$$
$$\left. \times \left(e^{(2C_n+2C_n''+C_3+\sqrt{2}C_4+\sqrt{2}C_5)t} - 1\right)\right\}.$$

Here we remark that for any given $T > 0$, by taking small enough $\varepsilon > 0$, we can make $\rho(T)$ positive.

Thus by (27) we have

$$\frac{d}{dt}E_\nu(t) \leq C_n'' \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \left\{ \sum_{|\alpha|=j-1} \|\partial_x^\alpha g(t)\|^2 \right\}^{1/2} \qquad \text{for } 0 \leq t \leq T.$$

Therefore we obtain (28). This implies the proposition.

In order to conclude Theorem 1, we must modify the energy inequality (28). Since the index number of the sequence G_j is slided by ν in this energy inequality, we shall pull back the index number to the standard one.

From the definition of $E_\nu(t)$ and $e_{j,\nu}(t)$, we can see that

$$\text{(29)} \qquad \sum_{j=2}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \omega(j+\nu)^{-1/2} \left\{ \sum_{|\alpha|=j} \|\partial_x^\alpha u\|^2 + \sum_{|\alpha|=j-1} \|\partial_t \partial_x^\alpha u\|^2 \right\}^{1/2}$$

$$\leq E_\nu(t) \leq C_6 \sum_{j=1}^{\infty} \frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}} \left\{ \sum_{|\alpha|=j} \|\partial_x^\alpha u\|^2 + \sum_{|\alpha|=j-1} \|\partial_t \partial_x^\alpha u\|^2 \right\}^{1/2}.$$

Moreover noting that

$$G_{j+\nu-2} = G_j \frac{G_{j+1}}{G_j} \cdot \frac{G_{j+2}}{G_{j+1}} \cdots \frac{G_{j+\nu-2}}{G_{j+\nu-3}}$$

$$\text{by (8)} \leq G_j \{C(j+1)\omega(j+1)^{1/2}\}\{C(j+2)\omega(j+2)^{1/2}\}$$

$$\cdots \{C(j+\nu-2)\omega(j+\nu-2)^{1/2}\}$$

$$\leq \{C(j+1)(j+1)^{1/2}\}\{C(j+2)(j+2)^{1/2}\} \cdots \{C(j+\nu-2)(j+\nu-2)^{1/2}\}$$

$$\leq C^{\nu-2} G_j (j+\nu-2)^{3/2(\nu-2)},$$

and while

$$G_{j+\nu-2} = G_j \frac{G_{j+1}}{G_j} \cdot \frac{G_{j+2}}{G_{j+1}} \cdots \frac{G_{j+\nu-2}}{G_{j+\nu-3}}$$

$$\text{by (6)} \geq G_j \left\{ (j+1) \frac{G_1}{1 G_0} \right\} \left\{ (j+2) \frac{G_1}{1 G_0} \right\} \cdots \left\{ (j+\nu-2) \frac{G_1}{1 G_0} \right\}$$

$$\geq C G_j \quad (C > 0),$$

we can see that

$$\text{(30)} \qquad C_7 (j+\nu-2)^{(-3/2)(\nu-2)} \frac{1}{G_j} \leq \frac{1}{G_{j+\nu-2}} \leq C_8 \frac{1}{G_j}.$$

Thus by (29), (30) it holds that

$$\text{(31)} \qquad E_\nu(t) \geq \sum_{j=2}^{\infty} \frac{\rho(t)^j}{G_j} \{C_7 \rho(t)^{\nu-1} (j+\nu-2)^{(-3/2)(\nu-2)} \omega(j+\nu)^{-1/2}\}$$

$$\times \left\{ \sum_{|\alpha|=j} \|\partial_x^\alpha u\|^2 + \sum_{|\alpha|=j-1} \|\partial_t \partial_x^\alpha u\|^2 \right\}^{1/2}$$

$$\geq C_7\rho(t)^{\nu-1}\sum_{j=2}^{\infty}\frac{\tilde{\rho}(t)^j}{G_j}\left\{j^{(-3/2)(\nu-2)}\omega(j)^{-1/2}\left(\frac{\rho(t)}{\tilde{\rho}(t)}\right)^j\right\}$$

$$\times\left\{\sum_{|\alpha|=j}\|\partial_x^\alpha u\|^2+\sum_{|\alpha|=j-1}\|\partial_t\partial_x^\alpha u\|^2\right\}^{1/2}$$

$$\geq C_7C_9\rho(T)^{\nu-1}\sum_{j=2}^{\infty}\frac{\tilde{\rho}(t)^j}{G_j}\left\{\sum_{|\alpha|=j}\|\partial_x^\alpha u\|^2+\sum_{|\alpha|=j-1}\|\partial_t\partial_x^\alpha u\|^2\right\}^{1/2}.$$

By (29), (30) we also easily obtain the followings.

(32) $$E_\nu(0)\leq C_6C_8\rho(0)^{\nu-1}\sum_{j=1}^{\infty}\frac{\rho(0)^j}{G_j}\left\{\sum_{|\alpha|=j}\|\partial_x^\alpha u(0)\|^2+\sum_{|\alpha|=j-1}\|\partial_t\partial_x^\alpha u(0)\|^2\right\}^{1/2}.$$

(33)
$$\sum_{j=2}^{\infty}\frac{\rho(t)^{j+\nu-1}}{G_{j+\nu-2}}\left\{\sum_{|\alpha|=j-1}\|\partial_x^\alpha g(t)\|^2\right\}^{1/2}\leq C_8\rho(0)^{\nu-1}\sum_{j=2}^{\infty}\frac{\rho(t)^j}{G_j}\left\{\sum_{|\alpha|=j-1}\|\partial_x^\alpha g(t)\|^2\right\}^{1/2}.$$

Finally by (28), (31), (32), (33) we have a priori estimate

$$\sum_{j=2}^{\infty}\frac{\tilde{\rho}(t)^j}{G_j}\left\{\sum_{|\alpha|=j}\|\partial_x^\alpha u(t)\|^2+\sum_{|\alpha|=j-1}\|\partial_t\partial_x^\alpha u(t)\|^2\right\}^{1/2}$$

$$\leq C_T\left\{\sum_{j=1}^{\infty}\frac{\rho(0)^j}{G_j}\left\{\left(\sum_{|\alpha|=j}\|\partial_x^\alpha u_0\|^2\right)^{1/2}+\left(\sum_{|\alpha|=j-1}\|\partial_x^\alpha u_1\|^2\right)^{1/2}\right\}\right.$$

$$\left.+\int_0^T\sum_{j=2}^{\infty}\frac{\rho(s)^j}{G_j}\left\{\sum_{|\alpha|=j-1}\|\partial_x^\alpha g(s)\|^2\right\}^{1/2}ds\right\}\quad\text{for } 0\leq t\leq T.$$

where $C_T=C_7^{-1}C_9^{-1}C_6C_8((\rho(0))/(\rho(T)))^{\nu-1}$ $(\nu\geq\nu_0)$.

In our ultradifferentiable case the existence and uniqueness of solutions are also shown by the same argument as Gevrey case. (see [CJS], [D], [DR], [J], [RY1]) We shall comment briefly. The above a priori estimate with the initial data $\equiv 0$ gives the uniqueness. In order to show the existence, we may approximate the coefficients and the inhomogeneous term and the initial data by the sequences such that the Cauchy problems have a solution. Using the above a priori estimate, by compactness argument we can get a solution to (1) as a limit of solutions to the auxiliary Cauchy problems. Hence we find that u belongs to $C^2([0,T],D_{L_2}\{G_j\}_{\tilde{\rho}(t)^{-1}}(\boldsymbol{R}_x^n))$ for any $u_0(x)$ and $u_1(x)\in D_{L_2}\{G_j\}_{\rho(0)^{-1}}(\boldsymbol{R}_x^n)$, $g(t,x)\in C^0([0,T];D_{L_2}\{G_j\}_{\rho(t)^{-1}}(\boldsymbol{R}_x^n))$. This concludes the proof of Theorem 1.

§4. Proof of Theorem 2

We shall first show the semilinear equation (9) is equivalent to the following more convenient form of semilinear 2×2 system.

$$(34)\begin{cases} U_{tt} - (a(t,x)U_x)_x + \{f_q(t,x,U_1,U_2) - a_x(t,x)\}U_x + g(U) = 0 \\ U(0,x) = (u_0(x), u_{0x}(x)), \quad \partial_t U(0,x) = (u_1(x), u_{1x}(x)), \end{cases} \quad \text{in } [0,T] \times P$$

where

$$g(U) = \begin{pmatrix} f(t,x,U_1,U_2) - f_q(t,x,U_1,U_2)U_2 + a_x(t,x)U_2 \\ f_p(t,x,U_1,U_2)U_2 + f_x(t,x,U_1,U_2) - a_{xx}(t,x)U_2 \end{pmatrix},$$

and $f_p = \partial_{U_1} f$, $f_q = \partial_{U_2} f$.

If u is a solution of (9), by differentiating (9) we can easily see that $U = (u, u_x)$ is a solution of (34).

Conversely if $U = (U_1, U_2)$ is a solution of (34), $V = U_2 - U_{1x}$ is a solution of the linear equation

$$\begin{cases} V_{tt} - (a(t,x)V_x)_x + \{f_q(t,x,U_1,U_2) - 2a_x(t,x)\}V_x \\ \qquad + \{\partial_x(f_q(t,x,U_1,U_2) - a_x(t,x)) + f_p(t,x,U_1,U_2) - a_{xx}(t,x)\}V = 0 \\ V(0,x) = 0, \quad \partial_t V(0,x) = 0, \end{cases}$$

Noting that the initial data and ihnomogeneous term are zero, we get $V \equiv 0$, i.e., $U_2 = U_{1x}$. Hence returning to (34), we find that U_1 satisfies

$$\begin{cases} U_{1tt} - (a(t,x)U_{1x})_x + f(t,x,U_1,U_2) = 0 \\ U_1(0,x) = u_0(x), \quad \partial_t U_1(0,x) = u_1(x). \end{cases}$$

Thus $u = U_1$ is a solution of (9).

We remark that the principle part of the system can be written with the particular form of the matrix as follows.

$$U_{tt} - \left(\begin{pmatrix} a(t,x) & 0 \\ 0 & a(t,x) \end{pmatrix} U_x \right)_x$$

Hence we can also calculate the energy quite similarly for the two dimensions of the system (34), which can be treated as the 1-dimensional equation. Therefore we may consider the following equation instead of (9).

$$
(35)\qquad \begin{cases} \partial_t^2 u - \partial_x(a(t,x)\partial_x u) + f_b(t,x,u)\partial_x u \\ \quad + f_c(t,x,u)u + f(t,x,0) = 0 \qquad \text{in } [0,T]\times P \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases}
$$

where

$$
(36)\qquad f_b(t,x,u), \quad f_c(t,x,u) \in C^0([0,T]; B\{F_j\}_{R_1}(P), B\{E_j\}_{R_2}(\boldsymbol{R}^1)).
$$

In this section our aim is to derive the energy inequality of the linearized equation such that u in the coefficients of (35) is replased by v. Here we must pay attention to the lower term and bounded term whose coefficients are composite functions. In order to further investigate the composite functions, we shall use the energy defined with the partial sum (see [RY1], [RY2]). Moreover since the solution of Theorem 2 is local, $\nu \geq \nu_0$ is not necessary. In particular by taking $\nu = 0$, we can get the same effect as the weight for the partial enagies, whose form is j^{ks} (where k depends on the dimension and s is Gevrey order). This weight j^{ks} often apears in the Gevrey case to treat the nonlinear problem (see [DS], [DM]). For the proof of theorem 2 we can not use the weight j^{ks}, since there exists no exponent corresponding to s in the case of the ultradifferentiable classes.

Thus we put $\nu = 0$ and define the energy

$$
E_0^{(N)}(u)(t) = \sum_{j=2}^{N} \frac{\rho(t)^{j-1}}{G_{j-2}} e_{j,0}(u)(t) \qquad \text{for } N \geq 2,
$$

where

$$
e_{j,0}(u)(t) = \left\{ \int_P a * \phi_j(t) |\partial_x^j u|^2 + \omega(j)^{-1} |\partial_x^j u|^2 + j^2 |\partial_x^{j-1} u|^2 + |\partial_t \partial_x^{j-1} u|^2 \, dx \right\}^{1/2}.
$$

The nonlinear lower term is changed into the three term as follows.

$$
\begin{aligned}
(37)\qquad & \sum_{j=2}^{N} \frac{\rho(t)^{j-1}}{G_{j-2}} \|\partial_x^{j-1}(f_b \cdot \partial_x u)\| \\
&= \sum_{j=2}^{N} \frac{\rho(t)^{j-1}}{G_{j-2}} \|\partial_x^{j-2}(f_b \cdot \partial_x^2 u)\| + \sum_{j=2}^{N} \frac{\rho(t)^{j-1}}{G_{j-2}} \|\partial_x^{j-2}(f_b^{(1,0)} \cdot \partial_x u)\| \\
&\quad + \sum_{j=2}^{N} \frac{\rho(t)^{j-1}}{G_{j-2}} \|\partial_x^{j-2}(f_b^{(0,1)} \cdot \partial_x v \cdot \partial_x u)\|,
\end{aligned}
$$

where $f_b^{(i,\mu)}$ denotes $\partial_x^i \partial_v^\mu f_b(t,x,v)$.

We shall investigate this term separately. By Corollary 3 we can get

$$\text{(38)} \qquad \text{the first term} \le \sum_{j=2}^{N} \frac{\rho(t)^{j-1}}{G_{j-2}} \sum_{k=0}^{j-2} \binom{j-2}{k} \|\partial_x^{j-2-k} f_b\|_\infty \|\partial_x^{k+2} u\|$$

$$\le \sum_{j=k+2}^{N} \frac{\rho(t)^{j-2-k}}{G_{j-2-k}} \|\partial_x^{j-2-k} f_b\|_\infty \sum_{k=0}^{N-2} \frac{\rho(t)^{k+1}}{G_k} \|\partial_x^{k+2} u\|$$

$$\le \left\{ \sum_{j=2}^{N} \frac{\rho(t)^{j-2}}{G_{j-2}} \|\partial_x^{j-2} f_b\|_\infty \right\} \left\{ \sum_{k=2}^{N} \frac{\rho(t)^{k-1}}{G_{k-2}} \|\partial_x^{k} u\| \right\},$$

where $\|\cdot\|_\infty$ denotes L_∞-norm. In particular picking up the first factor of the first term, we proceed to estimate. By Leibniz formula, we obtain

$$\text{(39)} \quad \text{the first factor}$$

$$\le \sum_{j=2}^{N} \frac{\rho(t)^{j-2}}{G_{j-2}} \sum_{i+l=j-2} \frac{(j-2)!}{i!} \sum_{\mu=0}^{l} \frac{1}{\mu!} \|f_b^{(i,\mu)}(x,v)\|_\infty$$

$$\times \sum_{h_1+\cdots+h_\mu=l,\, 1\le h_i} \frac{\|\partial_x^{h_1} v\|_\infty \cdots \|\partial_x^{h_\mu} v\|_\infty}{h_1! \cdots h_\mu!}$$

$$\le \sum_{j=2}^{N} \sum_{i+l=j-2} \sum_{\mu=0}^{l} M_{f_b} R_1^i R_2^\mu \frac{E_\mu}{G_\mu} \frac{F_i}{G_i} \rho^{i-\mu} \left\{ \frac{G_i G_l (j-2)!}{G_{j-2} i! l!} \right\}$$

$$\times \sum_{|h|=l,\, 1\le h_i} \frac{\rho^{h_1+1} \|\partial_x^{h_1} v\|_\infty \cdots \rho^{h_\mu+1} \|\partial_x^{h_\mu} v\|_\infty}{G_{h_1} \cdots G_{h_\mu}} \left\{ \frac{l! G_{h_1} \cdots G_{h_\mu} G_\mu}{G_l h_1! \cdots h_\mu! \mu!} \right\}.$$

Noting that by Sobolev embedding theorem for periodical functions and (14), it holds that

$$\|\partial_x^{h_i} v\|_\infty \le C_0 \|\partial_x^{h_i+1} v\|_\infty \le C_0 (h_i+2)^{-1} e_{h_i+2,0}(t)$$

$$\le \frac{C_0}{3} e_{h_i+2,0}(v)(t) \qquad \text{for } i = 1, 2, \ldots, v,$$

by (7), (39), Corollary 3, and Lemma 4 we get

$$\text{(40)} \quad \text{the first factor}$$

$$\le M_b \sum_{j=2}^{N} \sum_{i+l=j-2} (\rho(t) R_1)^i \sum_{\mu=0}^{l} \left(\frac{R_2}{\rho(T)} \frac{C_0}{3} \frac{G_1}{G_0} \right)^\mu \left(\frac{E_v}{G_\mu} \right)$$

$$\times \sum_{|h|=l,\, 1\le h_i} \frac{\rho^{h_1+1} e_{h_1+2,0}(v)(t) \cdots \rho^{h_\mu+1} e_{h_\mu+2,0}(v)(t)}{G_{h_1} \cdots G_{h_\mu}}$$

$$\leq M_b\left\{\sum_{i=0}^{N-2}(\rho(t)R_1)^i\right\}\left\{\sum_{\mu=0}^{N-2}\left(\frac{R_2}{\rho(T)}\frac{C_0}{3}\frac{G_1}{G_0}\right)^{\mu}\left(\frac{E_\mu}{G_\mu}\right)\left(\sum_{m=1}^{N-2}\frac{\rho^{m+1}}{G_m}e_{m+2,0}(v)(t)\right)^{\mu}\right\}$$

$$\leq C_{10}\sum_{\mu=0}^{N-2}(C_{11}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{E_\mu}{G_\mu}\right),$$

where $C_{10}=M_b\sum_{i=0}^{\infty}(\rho(t)R_1)^i$, $C_{11}=R_3(C_3/3)(G_1/G_0)$.

Thus by (38), (40) we get

$$\text{(41)}\quad \text{the first term} \leq C_{10}\left\{\sum_{\mu=0}^{N-2}(C_{11}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{E_\mu}{G_\mu}\right)\right\}\left\{\sum_{k=2}^{N}\frac{\rho^{k-1}}{G_{k-2}}\|\partial_x^k u\|\right\}.$$

For the second term and the third term, with some small changes we can get almost similarly

$$\text{(42)}\qquad \text{the second term} \leq C_{12}\left\{\sum_{\mu=0}^{N-2}(C_{11}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{E_\mu}{G_\mu}\right)\right\}$$

$$\times\left\{\sum_{k=2}^{N}\frac{\rho^{k-1}}{G_{k-2}}\|\partial_x^{k-1}u\|\right\},$$

$$\text{(43)}\quad \text{the third term} \leq C_{10}\left\{\sum_{\mu=0}^{N-2}R_2(C_{11}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{G_{\mu+1}}{G_\mu}\right)\left(\frac{E_{\mu+1}}{G_{\mu+1}}\right)\right\}$$

$$\times\left\{\sum_{k=2}^{N}\frac{\rho(t)^{k-1}}{G_{k-2}}\|\partial_x^{k-2}(\partial_x v\cdot\partial_x u)\|\right\},$$

where $C_{12}=M_bR_1\sum_{i=0}^{\infty}(\rho(t)R_1)^i(G_{i+1}/G_i)\leq M_bR_1C\sum_{i=0}^{\infty}(\rho(t)R_1)^i(i+1)^{3/2}(<\infty)$.

The estimate of the second term ends by (42), but we continue to estimate the third term. Picking up the last factor of the third term, by (16) and Sobolev embedding theorem we obtain.

$$\text{(44)}\quad \text{the last factor} \leq \sum_{k=2}^{N}\frac{\rho(t)^{k-1}}{G_{k-2}}\sum_{m=0}^{k-2}\binom{k-2}{m}\|\partial_x^{k-1-m}v\cdot\partial_x^{m+1}u\|$$

$$\leq \rho(t)^{-1}\sum_{k=2}^{N}\frac{\rho(t)^{k-1-m}}{G_{k-2-m}}\|\partial_x^{k-1-m}v\|\sum_{m=0}^{k-2}\frac{\rho(t)^{m+1}}{G_m}\|\partial_x^{m+1}u\|_\infty$$

$$\leq \rho(T)^{-1}\sum_{k=2}^{N}\frac{\rho(t)^{k-1}}{G_{k-2}}\|\partial_x^{k-1}v\|\sum_{m-2}^{N}\frac{\rho(t)^{m-1}}{G_{m-2}}\|\partial_x^{m-1}u\|_\infty$$

$$\leq \rho(T)^{-1}\frac{C_0}{6}\sum_{k=2}^{N}\frac{\rho(t)^{k-1}}{G_{k-2}}\,e_{k,0}(v)(t)\sum_{m=2}^{N}\frac{\rho(t)^{m-1}}{G_{m-2}}\|\partial_x^m u\|$$

$$= C_{13}E_0^{(N)}(v)(t)\sum_{m=2}^{N}\frac{\rho(t)^{m-1}}{G_{m-2}}\|\partial_x^m u\|.$$

$E_0^{(N)}(v)(t)$ in the last factor enters into the power of $E_0^{(N)}(v)(t)$ in the first factor. Hence the summation $\sum_{\mu=0}^{N-2}$ in the first factor is changed to $\sum_{\mu=1}^{N-1}$.

Thus by (43), (44) we get the estimate of the third term

$$\text{(45)}\quad \text{the third term} \leq C_{10}C_{13}R_2\left\{\sum_{\mu=1}^{N-1}(C_{11}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{G_{\mu+1}}{G_\mu}\right)\left(\frac{E_{\mu+1}}{G_{\mu+1}}\right)\right\}$$

$$\times\left\{\sum_{m=2}^{N}\frac{\rho(t)^{m-1}}{G_{m-2}}\|\partial_x^m u\|\right\}$$

$$\leq C_{14}\left\{\sum_{\mu=0}^{N}(C_{11}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{E_\mu}{G_\mu}\right)\right\}$$

$$\times\left\{\sum_{m=2}^{N}\frac{\rho(t)^{m-1}}{G_{m-2}}\|\partial_x^m u\|\right\}.$$

At last by (41), (42), (45) we have the estimate of the nonlinear lower term

$$\text{(46)}\quad \sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-1}(f_b\cdot\partial_x u)\|$$

$$\leq (C_{10}+C_{14})\left\{\sum_{\mu=0}^{N}(C_{11}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{E_\mu}{G_\mu}\right)\right\}\left\{\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^j u\|\right\}$$

$$+\,C_{12}\left\{\sum_{\mu=0}^{N-2}(C_{11}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{E_\mu}{G_\mu}\right)\right\}\left\{\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-1}u\|\right\}.$$

Concerned with the nonlinear bounded term, we also separate this to the three terms and similarly we can get

$$\text{(47)}\quad \sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-1}(f_c\cdot u)\|$$

$$=\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-2}(f_c^{(1,0)}\cdot u)\|+\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-2}(f_c^{(0,1)}\cdot\partial_x v\cdot u)\|$$

$$+\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-2}(f_c\cdot\partial_x u)\|$$

$$\le C_{15}\left\{\sum_{\mu=0}^{N-2}(C_{16}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{E_\mu}{G_\mu}\right)\right\}\left\{\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-2}u\|\right\}$$

$$+C_{17}\left\{\sum_{\mu=0}^{N}(C_{16}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{E_\mu}{G_\mu}\right)\right\}\left\{\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-1}u\|\right\}.$$

Here the constants C_{15} and C_{17} correspond to the constant C_{12} in (46), and the constant C_{16} corresponds to the constant C_{11} in (46).

By (14), (16), (46), (47) we have the estimate of the nonlinear term

$$\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-1}(f_b\cdot\partial_x u)\|+\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-1}(f_c\cdot u)\| \tag{48}$$

$$\le C_{18}\left\{\sum_{\mu=0}^{N}(C_{19}\rho(T)^{-1}E_0^{(N)}(v)(t))^{\mu}\left(\frac{E_\mu}{G_\mu}\right)\right\}\left\{\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\,\omega(j)^{1/2}e_{j,0}(t)\right\}$$

$$\le C_{20}\sum_{j=2}^{N}\frac{\rho(t)^{j-1}}{G_{j-2}}\,\omega(j)^{1/2}e_{j,0}(t),$$

where C_{20} depends on $E_0^{(N)}(v)(t)$ and $\rho(T)^{-1}$.

Hence we can reduce to (24), (25) and get the energy inequality (28) of proposition 7 with $v_0=0$.

PROPOSITION 8. *There exist* $T^*>0$, $D>0$ *such that if* $E_0^{(N)}(v)(t)=\sum_{j=0}^{N}(\rho(t)^{j-1}/G_{j-2})e_{j,0}(v)(t)\le D$, *then it holds that*

$$E_0^{(N)}(u)(t)\le D \qquad \textit{for } t\in[0,T^*]. \tag{49}$$

PROOF. We first suppose $D=2E_0^{(N)}(u)(0)$ and $\rho(T)^{-1}\le(\rho_0/2)^{-1}$ for some $T>0$. Hence the constant C_{20} of (48) depends on D and ρ_0.

We can get the following instead of (27).

$$\frac{d}{dt}E_0^{(N)}(u)(t)\le\sum_{j=2}^{\infty}\frac{\rho(t)^{j-2}}{G_{j-2}}(j-1)e_{j,0}(t)$$

$$\times\{\rho'(t)+\rho(t)(C_2+2+C_3+\sqrt{2}C_{20})+(C_1+1)C_{21}\}$$

$$+\sum_{j=2}^{\infty}\frac{\rho(t)^{j-1}}{G_{j-2}}\|\partial_x^{j-1}f(t,x,0)\|,$$

here we used by (11)

$$\frac{\omega(j-1)^{-1/2}G_{j-2}}{(j-1)G_{j-3}} \le \frac{\omega\left(\frac{1}{j-1}\right)^{1/2}G_{j-1}}{\left(\frac{1}{j-1}\right)G_{j-2}} \le \sup_{t\in(0,1]} \frac{\omega(t)^{1/2}G_{[1/t]}}{\left[\frac{1}{t}\right]G_{[1/t]-1}} \le C_{21}.$$

Now we take $\rho(t)$ such that

$$\begin{cases} \rho'(t)+\rho(t)(C_2+2+C_3+\sqrt{2}C_{20})+(C_1+1)C_{21}=0 \\ \rho(0)=\rho_0. \end{cases}$$

Hence we get the monotone decreasing function

$$\rho(t)=e^{-(C_2+2+C_3+\sqrt{2}C_{20})t}\left\{\rho_0-\frac{(C_1+1)C_{21}}{C_2+2+C_3+\sqrt{2}C_{20}}(e^{(C_2+2+C_3+\sqrt{2}C_{20})t}-1)\right\},$$

and we can find small enough $T>0$ such that $\rho(T)^{-1}\le(\rho_0/2)^{-1}$. Then we have the energy inequality

$$(50)\quad E_0^{(N)}(u)(t)\le E_0^{(N)}(u)(0)+\int_0^T\sum_{j=2}^{\infty}\frac{\rho(s)^{j-1}}{G_{j-2}}\|\partial_x^{j-1}f(s,x,0)\|\,ds \qquad \text{for } t\in[0,T].$$

Moreover we can find small enough $T^*\in(0,T]$ such that

$$(51)\qquad \int_0^{T^*}\sum_{j=2}^{\infty}\frac{\rho(s)^{j-1}}{G_{j-2}}\|\partial_x^{j-1}f(s,x,0)\|\,ds\le E_0^{(N)}(u)(0).$$

Thus by (50), (51) we have

$$E_0^{(N)}(u)(t)\le E_0^{(N)}(u)(0)+E_0^{(N)}(u)(0)\le D \qquad \text{for } t\in[0,T^*].$$

Based on a priori estimates (49), the local existence and uniqueness is shown similarly as the Gevrey case. (see [DM], [RY2]) We shall give a brief statement. Defining the bounded set

$$X_D=\{v\in C^1([0,T],D_{L_2}\{G_j\}_{\rho(t)^{-1}}(P);E_0^{(N)}(v)(t)\le D\}$$

in the locally convex space $C^1([0,T],D_{L_2}\{G_j\}_{\rho(t)^{-1}}(P))$, and the operator $Q:v\longrightarrow u$, from Proposition 8 we can see that Q maps continuously X_D into itself. Using Tichonoff Fixed Point Theorem, we can get a solution $u\in C^2([0,T],D_{L_2}\{G_j\}_{\tilde\rho(t)^{-1}}(P))$. While assuming that $u_1,u_2\in C^2([0,T],D_{L_2}\{G_j\}_{\tilde\rho(t)^{-1}}(P))$ are solutions of (35), and putting $w=u_1-u_2$, we get

(52)
$$\begin{cases} \partial_t^2 w - \partial_x(a(t,x)\partial_x w) + f_b(u_1)\partial_x w + \{f_c(u_1) + g_b(u_1,u_2)\partial_x u_2 + g_c(u_1,u_2)u_2\}w = 0 \\ w(0,x) = 0, \quad \partial_t w(0,x) = 0, \end{cases}$$

where $g_b(u_1,u_2) = \int_0^1 f_b^{(0,1)}(t,x,\tau u_1 + (1-\tau)u_2)\,d\tau$, $g_c(u_1,u_2) = \int_0^1 f_c^{(0,1)}(t,x,\tau u_1 + (1-\tau)u_2)\,d\tau$.

Using the same methods, we can derive the energy inequality of (52), and obtain $u_1 \equiv u_2$. This concludes the proof of Theorem 2.

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