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EXISTENCE OF GLOBAL SOLUTIONS TO NONLINEAR MASSLESS DIRAC SYSTEM AND WAVE EQUATION WITH SMALL DATA

By

Nickolay TZVETKOV*

Abstract. We prove existence of global solutions to a semilinear massless Dirac system with small initial data. We study solutions in generalised Sobolev spaces suggested by S. Klainerman. Our approach is based on using conservation law of charge together with Sobolev type weighted estimates for the spinor field. Our result seems to be sharp in a view of blowing-up results obtained by F. John (see [7]). We also study decay properties of the spinor field.

With similar methods we prove global existence for a nonlinear wave equation in three space dimension. The same equation was studied by T. Sideris [14] and H. Takamura [15]. They proved global existence for spherically symmetrical initial data. In this work we remove this condition on the initial data.

1. Introduction

We consider the Cauchy problem for the semilinear massless Dirac equation in Minkovski space-time R^{1+3}:

$$\begin{aligned} \mathscr{D}\psi &= F(\psi), \\ \psi(0,x) &= \eta(x), \end{aligned} \tag{1}$$

where $\mathscr{D} \equiv i\gamma^\mu \partial_\mu$ (with usual summation convention) is the Dirac operator, $\partial_0 = \partial_t$, $\partial_j = \partial_{x_j}$, $1 \leq j \leq 3$, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^t$ is the spinor field and γ^μ are the Dirac matrices. Dirac matrices satisfy the following commutator relations:

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$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g_{\mu\nu}I_4,$$
$$0 \le \mu, \quad \nu \le 3,$$

where $(g_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$ is the Minkovski metric and I_4 is the unit 4×4 matrice.

For the nonlinearity $F = (F_1, F_2, F_3, F_4)^t$ we assume $F(\psi) = O(|\psi|^p)$.

Various problems in quantum mechanics can be reduced to (1) (see [2]). We are interested for which p the Cauchy problem (1) has global solution. As an extension of the preveous work [16] we prove that (1) has a global solution for $p > 2$ providing the initial data are sufficiently small with respect to suitable Sobolev norms. With the method we prove global existence for $p > 2$ we can obtain local solution in the case $p = 2$. In a view of [7] one may conjecture that this local solution blows-up in finite time for some nonlinearities $F(\psi)$. If we follow ideas of T. Sideris (see [14]), we have to require some symmetry of the initial data η (spherical symmetry is not appropriate for the case of Dirac equation). One also may conjecture that global existence of (1) when $p = 2$ could be obtained if the nonlinearity F satisfy some structural condition similar to the well-known null condition of S. Klainerman (see Klainerman [10], Bachelot [1]). In the case $p = 2$ we show that for every $\varepsilon > 0$, there exists η such that $|\eta|_{L^2} = \varepsilon$ and $|\psi(t,\cdot)|_{L^2}$ tends to ∞ in finite time (see Theorem 2).

We construct global solutions of (1) for $p > 2$ in generalized Sobolev spaces following ideas of Klainerman (see [9], [10], [11]).

We introduce the following vector fields:

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j}, \quad 1 \le j \le 3,$$

$$\Omega_{0j} = t\frac{\partial}{\partial x_j} + x_j\frac{\partial}{\partial t}, \quad 1 \le j \le 3,$$

$$\Omega_{ij} = x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}, \quad 1 \le i, j \le 3,$$

$$L = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}.$$

The above vector fields are a part of the full conformal group in Minkovski space time.

Since $\Omega_{\mu\nu}$ does not commute with the Dirac operator we introduce Fermi vector fields (see [1]):

$$\hat{\Omega}_{\mu\nu} = \Omega_{\mu\nu} + \frac{1}{2}\gamma_\mu\gamma_\nu,$$

$$0 \leq \mu, \quad \nu \leq 3$$

Let us denote:

$$\Gamma = (\Gamma_1, \dots, \Gamma_{11}),$$

where $\Gamma_1, \dots \Gamma_{11}$ are ∂_μ, $\Omega_{\mu\nu}$ and L.

$$\hat{\Gamma} = (\hat{\Gamma}_1, \dots, \hat{\Gamma}_{11}),$$

where $\hat{\Gamma}_1, \dots, \hat{\Gamma}_{11}$ are ∂_μ, $\hat{\Omega}_{\mu\nu}$ and L.

We have:

$$[\Gamma_i, \Gamma_j] = c_{ij}^k \Gamma_k,$$

$$[\hat{\Gamma}_i, \hat{\Gamma}_j] = \hat{c}_{ij}^k \hat{\Gamma}_k,$$

$$1 \leq i, \quad j \leq 11,$$

where $c_{ij}^k, \hat{c}_{ij}^k$ are constants (see [9] and [1]) and $[\cdot, \cdot]$ is the usual commutator of vector fields. Therefore the linear span of vector fields from Γ forms a Lie algebra. The same is true for $\hat{\Gamma}$.

If $\alpha = (\alpha_1, \dots, \alpha_{11})$ is a multiindex, then we set:

$$\Gamma^\alpha = \Gamma_1^{\alpha_1} \dots \Gamma_{11}^{\alpha_{11}},$$

$$\hat{\Gamma}^\alpha = \hat{\Gamma}_1^{\alpha_1} \dots \hat{\Gamma}_{11}^{\alpha_{11}}.$$

We define the following norms:

$$|u(t)|_{\Gamma,s,p} = \sum_{|\alpha| \leq s} |\Gamma^\alpha u(t, \cdot)|_{L^p},$$

$$|\psi(t)|_{\hat{\Gamma},s,p} = \sum_{|\alpha| \leq s} |\hat{\Gamma}^\alpha \psi(t, \cdot)|_{L^p}.$$

Further we set:

$$\|u\|_{\Gamma,s,p} = \sup_{t \in R_+} |u(t)|_{\Gamma,s,p},$$

$$\|\psi\|_{\hat{\Gamma},s,p} = \sup_{t \in R_+} |\psi(t)|_{\hat{\Gamma},s,p},$$

$$W_{\Gamma}^{s,p} = \{u : \|u\|_{\Gamma,s,p} < \infty\},$$

$$W_{\hat{\Gamma}}^{s,p} = \{\psi : \|\psi\|_{\hat{\Gamma},s,p} < \infty\}$$

REMARK. For simplicity we shall omit in most of the cases the index Γ (respectively $\hat{\Gamma}$) from norms $\|\cdot\|_{\Gamma,s,p}$ (respectively $\|\cdot\|_{\hat{\Gamma},s,p}$).

We have the following relation between elements of $\hat{\Gamma}$ and Dirac operator:

$$[\mathscr{D}, \hat{\Omega}_{\mu,\nu}] = 0,$$

$$[\mathscr{D}, \partial_\mu] = 0,$$

$$[\mathscr{D}, L] = \mathscr{D},$$

$$0 \le \mu, \quad \nu \le 3.$$

Taking into account the above relation one can obtain the following form of the conservation law of charge for Dirac equation:

LEMMA 1. *If ψ is such that*:

$$\mathscr{D}\psi = F,$$

then for any $m \ge 0$ one has:

$$|\psi(t)|_{m,2} \le c\left(|\psi(0)|_{m,2} + \int_0^t |F(s)|_{m,2}\, ds\right).$$

Now we can formulate our first result:

THEOREM 1. (a) *Suppose that $p > 2$ and the initial data of* (1) *satisfy*:

$$\sum_{|\alpha| \le 2} |(1 + |\cdot|)^{|\alpha|} \partial^\alpha \eta(\cdot)|_{L^2} \le \varepsilon. \tag{2}$$

If:

$$X_\delta = \{\psi : \|\psi\|_{2,2} \le \delta\},$$

then there exists $\varepsilon_0 > 0$ such that for $p > 2$ the initial value problem (1) *has a unique global solution in X_δ for $0 < \varepsilon \le \varepsilon_0$, providing $\delta > 0$ sufficiently small.*

(b) *If $p = 2$ and the initial data satisfy* (2) *then we have*:

$$T_\varepsilon \ge \exp(c/\varepsilon) - 1,$$

where T_ε is the life span of the solution.

(c) *Suppose the initial data* (1) *satisfy*:

$$\sum_{|\alpha|\leq 2} |(1+|\cdot|)^{|\alpha|}\partial^\alpha \eta(\cdot)|_{L^2} + |(1+|\cdot|)^{p-1}\eta(\cdot)|_{L^\infty} + |(1+|\cdot|)^p \nabla \eta(\cdot)|_{L^\infty} \leq \varepsilon.$$

Then (1) *has global solution with the following decay property*:

$$|\psi(t,x)| \leq \frac{c}{(1+t+|x|)(1+|t-|x||)^\kappa}$$

where $\kappa = max(1/2, p-3)$.

REMARK. It seems to be true that the local solution obtained in part (b) blows-up in finite time for some special nonlinearities.

We can prove the following.

THEOREM 2. *Suppose the nonlinearity F in* (1) *has the form*:

$$F(\psi) = G(\psi)\gamma^0\psi,$$

where $G(\psi) = O(|\psi|)$ *and* $G(\psi) \geq c|\psi|$.

Then for every $\varepsilon > 0$ *there exists* η, *such that*:

$$|\eta|_{L^2} = \varepsilon$$

and the solution of (1) *with initial data* η *is such that* $|\psi(t,\cdot)|_{L^2}$ *tends to* ∞ *in finite time*.

REMARK 1. We can take for example:

$$F(\psi) = (|\psi_1| + |\psi_2| + |\psi_3| + |\psi_4|)\gamma^0\psi,$$

$$F(\psi) = |\psi|\gamma^0\psi.$$

REMARK 2. If $F(\psi) = \langle\gamma^0\psi,\psi\rangle e$ or $F(\psi) = \langle\gamma^0\gamma^5\psi,\psi\rangle e$, then due to Klainerman and Bachelot (see [10] and [1]), one may conjecture that there exists global solution of (1) (here e is a constant vector and $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$).

In this paper we shall also consider the Cauchy problem for the nonlinear wave equation:

$$
\begin{aligned}
u_{tt} - \Delta u &= |u_t|^{\nu}, \\
u(0,x) &= f(x), \qquad (3)\\
u_t(0,x) &= g(x),
\end{aligned}
$$

where $(t,x) \in [0,\infty) \times R^n$, f,g are the initial data and Δ is the Laplace operator on R^n. In [14] T. Sideris obtained global solution of (3) in the case $n=3$ for $\nu > 2$ requiring the initial data are spherically symmetric functions and they have compact support. H. Takamura [15] removed the assumption on the support of data for integer ν. The case of spherically symmetrical data is in fact one dimensional problem and classical Sobolev inequalities are well adapted to the Cauchy problem (3).

Here in the case $n=3$ we obtain solution of (3) for $\nu > 2$ in generalised Sobolev spaces with the same idea of the proof of Theorem 1. We have the following.

THEOREM 3. (a) *Suppose $n=3$ and the initial data of* (3) *satisfy*:

$$\sum_{|\alpha|\le 3} |(1+|\cdot|)^{|\alpha|-1}\partial^{\alpha} f(\cdot)|_{L^2} + \sum_{|\alpha|\le 2} |(1+|\cdot|)^{|\alpha|}\partial^{\alpha} g(\cdot)|_{L^2} \le \varepsilon.$$

Let H_D be the closure of C_0^{∞} with respect to the seminorm:

$$\|u\|_{H_D} = \|\nabla u\|_{2,2}$$

If:

$$Y_{\delta} = \{u \in H_D : \|u\|_{H_D} \le \delta\}.$$

then there exists $\varepsilon_0 > 0$ such that for $\nu > 2$ the initial value problem (3) *has a unique global solution in Y_{δ} for $0 < \varepsilon \le \varepsilon_0$, providing $\delta > 0$ sufficiently small.*

(b) *Suppose the initial data of* (3) *satisfy*:

$$\sum_{|\alpha|\le 3} |(1+|\cdot|)^{|\alpha|-1}\partial^{\alpha} f(\cdot)|_{L^2} + \sum_{|\alpha|\le 2} |(1+|\cdot|)^{|\alpha|}\partial^{\alpha} g(\cdot)|_{L^2}$$

$$+ |(1+|\cdot|)^{\nu-1} f(\cdot)|_{L^{\infty}} + |(1+|\cdot|)^{\nu}(|g(\cdot)| + |\nabla f(\cdot)|)|_{L^{\infty}} \le \varepsilon.$$

Then (3) *has global solution with the following decay property*:

$$|u(t,x)| \le \frac{c}{(1+t+|x|)(1+|t-|x||)^{\kappa}}$$

where $\kappa = max(1/2, \nu - 2)$.

REMARK 1. The same result holds when the nonlinearity has the form $|\nabla u|^{\nu}$, where $\nabla = (\partial_t, \nabla_x)$.

REMARK 2. This result is sharp taking into account [7], where it is shown that for $\nu = 2$ the solution of (3) blows-up in finite time.

REMARK 3. If $n = 2$ one can obtain solution of (3) for $\nu > 3$ with the same arguments of the proof of Theorem 3.

After this work was completed, the author learned that earlier K. Hidano and K. Tsutaya obtained independently the same global existence result for the Cauchy problem (3) (see [4]), by using similar methods.

For the general case for n we can define in the same way the spaces $W_{\Gamma}^{s,p}$. For the general case for n we can obtain result just for the integer case for ν.

THEOREM 4. *If $n \geq 4$ and the initial data $(\nabla f, g) \in W_{\Gamma}^{s,2} \times W_{\Gamma}^{s-1,2} (s > n)$ is sufficiently small then* (3) *has global solution for $\nu = 2, 3, 4, \ldots$*

2. Proof of Theorem 1 and Theorem 2

We need the following Sobolev inequality due to Hörmander (see [5]) and Klainerman (see [11]).

LEMMA 2. *If $u(t,x) \in C_0^{\infty}(R_+ \times R^n)$ then the following inequalities hold:*

$$\text{(a)} \quad |u(t,x)| \leq c(1+t+|x|)^{-(n-1)/p}(1+|t-|x||)^{-1/p}|u(t)|_{s,p},$$

where $s > n/p$.

$$\text{(b)} \quad |u(t,\cdot)|_{L^q} \leq c(1+t)^{-(n-1)(1/p-1/q)}|u(t)|_{s,p},$$

where $p \leq q < \infty$ and $s \geq n(1/p - 1/q)$.

PROOF OF THEOREM 1. We define a map:

$$M : \psi \mapsto \phi$$

For given ψ we define ϕ as a solution of the following linear Dirac equation:

$$\mathscr{D}\phi = F(\psi),$$

$$\phi(0,x) = \eta(x).$$

Using lemma 1, lemma 2 and Hölder inequality, one can obtain:

$$
\begin{aligned}
|\phi(t)|_{2,2} &\le c\left(\varepsilon + \int_0^t |F(\psi(\tau,\cdot))|_{2,2}\,d\tau\right) \\
&\le c\left(\varepsilon + \int_0^t \left(\sum_{|\alpha|\le 2} |\hat{\Gamma}^\alpha \psi(\tau,\cdot)|_{L^2} |\psi(\tau,\cdot)|_{L^\infty}^{p-1} \right.\right. \\
&\quad \left.\left. + \sum_{|\alpha|\le 1, |\beta|\le 1} |\hat{\Gamma}^\alpha \psi(\tau,\cdot)|_{L^4} |\hat{\Gamma}^\beta \psi(\tau,\cdot)|_{L^4} |\psi(\tau,\cdot)|_{L^\infty}^{p-2}\right) d\tau\right) \\
&\le c\left(\varepsilon + \int_0^t ((1+\tau)^{-p+1} |\psi(\tau)|_{2,2}^p + (1+\tau)^{-1/2-1/2-(p-2)} |\psi(\tau)|_{2,2})^p\,d\tau\right) \\
&\le c(\varepsilon + \|\psi\|_{2,2}^p)
\end{aligned}
$$

Hence:

$$
\|M\psi\|_{2,2} \le c(\varepsilon + \|\psi\|_{2,2}^p). \tag{4}
$$

Therefore, if we take ε and δ such that $c(\varepsilon + \delta^p) < \delta$, then we obtain $M\psi \in X_\delta$, when $\psi \in X_\delta$.

If $\psi, \tilde{\psi} \in X_\delta$, then using lemma 1, we obtain:

$$
|(M\psi - M\tilde{\psi})(t)|_{1,2} \le c \int_0^t \sum_{|\alpha|\le 1} |\hat{\Gamma}^\alpha (F(\psi) - F(\tilde{\psi}))(\tau,\cdot)|_{L^2}\,d\tau
$$

Further, for any α such that $|\alpha| \le 1$ via Hölder inequality, one has:

$$
\begin{aligned}
|\hat{\Gamma}^\alpha (F(\psi) - F(\tilde{\psi}))(\tau,\cdot)|_{L^2} &\le |\psi(\tau,\cdot)|_{L^\infty}^{p-1} |\hat{\Gamma}^\alpha(\psi - \tilde{\psi})(\tau,\cdot)|_{L^2} \\
&\quad + |\hat{\Gamma}^\alpha \tilde{\psi}(\tau,\cdot)|_{L^4} |(\psi - \tilde{\psi})(\tau,\cdot)|_{L^4} (|\psi(\tau,\cdot)|_{L^\infty}^{p-2} + |\tilde{\psi}(\tau,\cdot)|_{L^\infty}^{p-2}).
\end{aligned}
$$

Using lemma 2 one obtains:

$$
|(M\psi - M\tilde{\psi})(t)|_{1,2} \le c \int_0^t (1+\tau)^{-p+1} |(\psi - \tilde{\psi})(\tau)|_{1,2} |\psi(\tau)|_{2,2}^{p-1}\,d\tau \le c\delta^{p-1} \|\psi - \tilde{\psi}\|_{1,2}.
$$

Hence, if we take δ such that $c\delta^{p-1} < c_1 < 1$, then for any $\psi, \tilde{\psi} \in X_\delta$ we have:

$$
\|M\psi - M\tilde{\psi}\|_{1,2} \le c_1 \|\psi - \tilde{\psi}\|_{1,2}, \tag{5}
$$

Further for arbitrary $\psi, \tilde{\psi} \in X_\delta$ using lemma 1 one has:

$$|(M\psi - M\tilde{\psi})(t)|_{2,2} \le c\int_0^t |(F(\psi) - F(\tilde{\psi}))(\tau)|_{2,2}\, d\tau$$

$$\le c\int_0^t \sum_{|\alpha|\le 2} |(F'(\psi)\hat{\Gamma}^\alpha\psi - F'(\tilde{\psi})\hat{\Gamma}^\alpha\tilde{\psi})(\tau,\cdot)|_{L^2}\, d\tau$$

$$+ c\int_0^t \sum_{|\alpha|\le 1, |\beta|\le 1} \sum_{j=1}^4 |(F_j''(\psi)\hat{\Gamma}^\alpha\psi\hat{\Gamma}^\beta\psi - F_j''(\tilde{\psi})\hat{\Gamma}^\alpha\tilde{\psi}\hat{\Gamma}^\beta\tilde{\psi})(\tau,\cdot)|_{L^2}\, d\tau,$$

where F' is the matrice with entries $\partial_k F_j$, $1 \le k,\ j \le 4$, and F_j'' is the Hesse matrice of F_j.

Using Hölder inequality we obtain:

$$\begin{aligned}(6)\qquad & |(F'(\psi)\hat{\Gamma}^\alpha\psi - F'(\tilde{\psi})\hat{\Gamma}^\alpha\tilde{\psi})(\tau,\cdot)|_{L^2} \\ &= |(F'(\psi)(\hat{\Gamma}^\alpha\psi - \hat{\Gamma}^\alpha\tilde{\psi}) + (F'(\psi) - F'(\tilde{\psi}))\hat{\Gamma}^\alpha\tilde{\psi})(\tau,\cdot)|_{L^2} \\ &\le |\psi(\tau,\cdot)|_{L^\infty}^{p-1} |\hat{\Gamma}^\alpha(\psi - \tilde{\psi})(\tau,\cdot)|_{L^2} \\ &\quad + |\hat{\Gamma}^\alpha\tilde{\psi}(\tau,\cdot)|_{L^2} |(\psi - \tilde{\psi})(\tau,\cdot)|_{L^\infty} (|\psi(\tau,\cdot)|_{L^\infty}^{p-2} + |\tilde{\psi}(\tau,\cdot)|_{L^\infty}^{p-2}).\end{aligned}$$

Further we have:

$$\begin{aligned}(7)\qquad & |(F_j''(\psi)\hat{\Gamma}^\alpha\psi\hat{\Gamma}^\beta\psi - F_j''(\tilde{\psi})\hat{\Gamma}^\alpha\tilde{\psi}\hat{\Gamma}^\beta\tilde{\psi})(\tau,\cdot)|_{L^2} \\ &\le |((F_j''(\psi) - F_j''(\tilde{\psi}))\hat{\Gamma}^\alpha\psi\hat{\Gamma}^\beta\psi + F_j''(\tilde{\psi})\hat{\Gamma}^\alpha\psi(\hat{\Gamma}^\beta\psi - \hat{\Gamma}^\beta\tilde{\psi}) \\ &\quad + F_j''(\tilde{\psi})(\hat{\Gamma}^\alpha\psi - \hat{\Gamma}^\alpha\tilde{\psi})\hat{\Gamma}^\beta\tilde{\psi})(\tau,\cdot)|_{L^2} \\ &\le |((F_j''(\psi) - F_j''(\tilde{\psi}))\hat{\Gamma}^\alpha\psi\hat{\Gamma}^\beta\psi)(\tau,\cdot)|_{L^2} \\ &\quad + |\tilde{\psi}(\tau,\cdot)|_{L^\infty}^{p-2} |\hat{\Gamma}^\alpha\psi(\tau,\cdot)|_{L^4} |\hat{\Gamma}^\beta(\psi - \tilde{\psi})(\tau,\cdot)|_{L^4} \\ &\quad + |\tilde{\psi}(\tau,\cdot)|_{L^\infty}^{p-2} |\hat{\Gamma}^\beta\tilde{\psi}(\tau,\cdot)|_{L^4} |\hat{\Gamma}^\alpha(\psi - \tilde{\psi})(\tau,\cdot)|_{L^4}.\end{aligned}$$

Using (6), (7) and lemma 2 one has:

$$\begin{aligned}(8)\qquad & |(M\psi - M\tilde{\psi})(t)|_{2,2} \\ &\le c\int_0^t (1+\tau)^{-p+1} \|\psi - \tilde{\psi}\|_{2,2} (\|\psi\|_{2,2}^{p-1} + \|\tilde{\psi}\|_{2,2}^{p-1})\, d\tau \\ &\quad + c\int_0^t \sum_{|\alpha|\le 1, |\beta|\le 1} \sum_{j=1}^4 |((F_j''(\psi) - F_j''(\tilde{\psi}))\hat{\Gamma}^\alpha\psi\hat{\Gamma}^\beta\psi)(\tau,\cdot)|_{L^2}\, d\tau.\end{aligned}$$

We shall consider two cases for p.

Case 1. Let $2 < p \le 3$. We shall use the following inequality which is valid for $0 < k \le 1$

$$|x^k - y^k| \le c|x - y|^k. \tag{9}$$

Using (9) and Hölder inequality we obtain:

$$\begin{aligned} |((F_j''(\psi) - F_j''(\tilde{\psi}))\hat{\Gamma}^\alpha\psi\hat{\Gamma}^\beta\psi)(\tau,\cdot)|_{L^2} &\le c|(|\psi - \tilde{\psi}|^{p-2}\hat{\Gamma}^\alpha\psi\hat{\Gamma}^\beta\psi)(\tau,\cdot)|_{L^2} \\ &\le c|\hat{\Gamma}^\alpha\psi(\tau,\cdot)|_{L^q}|\hat{\Gamma}^\beta\psi(\tau,\cdot)|_{L^q}|(\psi - \tilde{\psi})(\tau,\cdot)|_{L^6}^{p-2}, \end{aligned}$$

where:

$$1/q + 1/q + (p-2)/6 = 1/2, \quad \text{i.e. } q = 12/(5-p).$$

Hence:

$$6 \ge q > 4$$

Now we are in a situation to apply lemma 2:

$$|((F_j''(\psi) - F_j''(\tilde{\psi}))\hat{\Gamma}^\alpha\psi\hat{\Gamma}^\beta\psi)(\tau,\cdot)|_{L^2} \le c(1+\tau)^{-p+1}|\psi(\tau)|_{2,2}^2|(\psi - \tilde{\psi})(\tau)|_{1,2}^{p-2}.$$

Hence for any $\psi, \tilde{\psi} \in X_\delta$ one has:

$$\begin{aligned} \|M\psi - M\tilde{\psi}\|_{2,2} &\le \|\psi - \tilde{\psi}\|_{2,2}(\|\psi\|_{2,2}^{p-1} + \|\tilde{\psi}\|_{2,2}^{p-1}) + c\|\psi - \tilde{\psi}\|_{1,2}^{p-2}\|\psi\|_{2,2}^2 \\ &\le c(\delta^{p-1}\|\psi - \tilde{\psi}\|_{2,2} + \delta^2\|\psi - \tilde{\psi}\|_{1,2}^{p-2}). \end{aligned}$$

Therefore if $c(\delta^{p-2} + \delta^2) < c_2/2$, then:

$$\|M\psi - M\tilde{\psi}\|_{2,2} \le c_2(\|\psi - \tilde{\psi}\|_{2,2} + \|\psi - \tilde{\psi}\|_{1,2}^{p-2}), \tag{10}$$

where $0 < c_2 < 1$.

We take ψ_0 to be the solution of the linear Dirac equation:

$$\mathscr{D}\psi = 0,$$
$$\psi(0,x) = \eta(x).$$

We consider the usual iteration:

$$\psi_{m+1} = M\psi_m.$$

For ε sufficiently small $\psi_0 \in X_\delta$. Hence $\psi_m \in X_\delta$ for every m. Therefore, taking into account (5) and (10) we obtain:

$$(11) \qquad \|\psi_{m+1} - \psi_m\|_{2,2} + \|\psi_{m+1} - \psi_m\|_{1,2}^{p-2}$$
$$\leq 1/2(\|\psi_m - \psi_{m-1}\|_{2,2} + \|\psi_m - \psi_{m-1}\|_{1,2}^{p-2}),$$

providing c_1 and c_2 sufficiently small (i.e. δ).

Inductively from (11) we get:

$$\|\psi_{m+1} - \psi_m\|_{2,2} + \|\psi_{m+1} - \psi_m\|_{1,2}^{p-2} \leq \frac{c}{2^m}.$$

Hence:

$$\|\psi_{m+1} - \psi_m\|_{2,2} \leq \frac{c}{2^m},$$

and we conclude that ψ_m is a Cauchy sequence in $W_{\hat{\Gamma}}^{2,2}$. Hence ψ_m converges in $W_{\hat{\Gamma}}^{2,2}$ to the solution of (1) providing ε sufficiently small (i.e. the initial data).

Case 2. Let $p > 3$. When $p > 3$ one has:

$$|((F_j''(\psi) - F_j''(\tilde{\psi}))\hat{\Gamma}^\alpha\psi\hat{\Gamma}^\beta\psi)(\tau,\cdot)|_{L^2}$$
$$\leq c|\hat{\Gamma}^\alpha\psi(\tau,\cdot)|_{L^4}|\hat{\Gamma}^\beta\psi(\tau,\cdot)|_{L^4}|(\psi-\tilde{\psi})(\tau,\cdot)|_{L^\infty}(|\psi(\tau,\cdot)|_{L^\infty}^{p-3} + |\tilde{\psi}(\tau,\cdot)|_{L^\infty}^{p-3}).$$

Using the above estimate in (8) we obtain for any $\psi, \tilde{\psi} \in X_\delta$:

$$\|M\psi - M\tilde{\psi}\|_{2,2} \leq c\delta^{p-1}\|\psi - \tilde{\psi}\|_{2,2}.$$

It remains to use similar arguments like in case 1. This completes the proof of part (a).

Proof of part (b). We denote:

$$X_{z,T} = \left\{\psi : \sup_{0\leq\tau\leq T} |\psi(\tau)|_{2,2} \leq z\right\}.$$

Application of lemma 1 and lemma 2, like in the proof of part (a) leads to:

$$|\psi_{m+1}(\tau)|_{2,2} \leq c\left(\varepsilon + \ln(1+t)\sup_{0\leq\tau\leq t}|\psi_m(\tau)|_{2,2}^2\right).$$

In order the iteration to be such that $\psi_{m+1} \in X_{z,T}$, when $\psi_m \in X_{z,T}$ it is necessary that the following quadratic equation has real roots:

$$(12) \qquad \ln(1+t)\cdot z^2 - z + c\varepsilon = 0$$

It is easy to be seen that (12) has real roots, iff:

$$t \leq \exp(c/\varepsilon) - 1.$$

If we denote by z_0 the smaller positive root of (12) then the iteration provides local solution of (1) in $X_{z,T}$, where:

$$0 \leq z \leq z_0,$$
$$T \leq \exp(c/\varepsilon) - 1.$$

This completes the proof of part (b).

PROOF OF PART (c). The proof is direct consequence of lemma 2 and Theorem 4 of [16].

This completes the proof of Theorem 1.

PROOF OF THEOREM 2. We shall choose $\eta \in L^2$ such that:

$$\operatorname{supp} \eta \subset \{x : |x| < R\}$$

Hence for solutions of (1) we have:

$$\operatorname{supp} \psi(t, \cdot) \subset \{x : |x| < t + R\}$$

Multiplying the equation (1) with $-\gamma^0 \psi$ and integrating over x we obtain:

$$\frac{d}{dt}\int_{|x| \leq t+R} |\psi(t,x)|^2\, dx = \int_{|x| \leq t+R} G(\psi(t,x))|\psi(t,x)|^2\, dx$$

Taking into account the requirements for G and via Hölder inequality, one obtains:

$$\frac{d}{dt}\int_{R^3} |\psi(t,x)|^2\, dx \geq \int_{|x| \leq t+R} |\psi(t,x)|^3\, dx \geq c(t+R)^{-3/2}\left(\int_{R^3} |\psi(t,x)|^2\, dx\right)^{3/2}$$

Further we set:

$$y(t) = \int_{R^3} |\psi(t,x)|^2\, dx$$

Hence we have that:

$$y' \geq \frac{cy^{3/2}}{(t+R)^{3/2}}, \tag{13}$$
$$0 \leq y(0) = y_0.$$

Integrating the inequality (13) we obtain:

$$y(t) \geq \left(\frac{c}{(t+R)^{1/2}} + \frac{1}{y_0^{1/2}} - \frac{c}{R^{1/2}}\right)^{-2}$$

Hence if $R < c^2 y_0$ then $y(t)$ goes to infinity in finite time. We can choose $\eta \in L^2$ such that $|\eta|_{L^2} = \varepsilon$ and supp $\eta \subset \{x : |x| < c^2 y_0\}$. A direct computation shows that we can set:

$$\eta(x) = \frac{1}{|x|^k} \Phi\left(\frac{x}{R}\right),$$

where $2 < k < 3$, and:

$$\Phi \in C_0^\infty, \quad \operatorname{supp} \Phi \subset \{x : |x| < 1\}.$$

This completes the proof of Theorem 2.

3. Proof of Theorem 3 and Theorem 4

PROOF OF THEOREM 3.

PROOF OF PART (a). As in the proof of Theorem 1 we define a map:

$$L : v \mapsto w$$

For any v we define w as a solution of the following linear wave equation:

$$(\partial_t^2 - \Delta)w = |u_t|^\nu$$
$$w(0, x) = f(x)$$
$$w_t(0, x) = g(x)$$

We have the following relations between the elements of Γ and $\partial_t^2 - \Delta$:

$$[\partial_t^2 - \Delta, \Omega_{\mu\nu}] = [\partial_t^2 - \Delta, \partial_\mu] = 0,$$
$$[\partial_t^2 - \Delta, L] = \partial_t^2 - \Delta,$$
$$0 \leq \mu, \quad \nu \leq 3.$$

Hence we have the following form of the classical energy estimate:

LEMMA 3. *If $u(t, x)$ is such that*:

$$(\partial_t^2 - \Delta)u = F(t, x),$$

then for any $m \geq 0$ *one has*:

$$|\nabla u(t)|_{\Gamma,m,2} \leq c\left(|\nabla u(0)|_{\Gamma,m,2} + \int_0^t |F(s)|_{\Gamma,m,2}\, ds\right).$$

Using lemma 2 and lemma 3 like in the proof of Theorem 1 one can obtain:

$$\|\nabla Lv\|_{2,2} \leq c(\varepsilon + \|\nabla v\|_{2,2}^{\nu}). \tag{14}$$

Moreover for any $v, \tilde{v} \in Y_\delta$ we have:

$$\|\nabla(Lv - L\tilde{v})\|_{1,2} \leq c_1 \|\nabla(v - \tilde{v})\|_{1,2} \tag{15}$$

$$\|\nabla(Lv - L\tilde{v})\|_{2,2} \leq c_2(\|\nabla(v - \tilde{v})\|_{2,2} + \|\nabla(v - \tilde{v})\|_{1,2}^{\nu-2}), \tag{16}$$

where $0 < c_1 < 1$, $0 < c_2 < 1$, providing δ sufficiently small.

Further we take u_0 to be the solution of the homogeneous problem. We consider the iteration:

$$u_{m+1} = Lu_m$$

Taking into account (14) we see that $u_{m+1} \in Y_\delta$, when $u_m \in Y_\delta$ for ε, δ sufficiently small. Considering (15) and (16) one obtains:

$$\begin{aligned}&\|\nabla(u_{m+1} - u_m)\|_{2,2} + \|\nabla(u_{m+1} - u_m)\|_{1,2}^{\nu-2}\\ &\leq \frac{1}{2}(\|\nabla(u_m - u_{m-1})\|_{2,2} + \|\nabla(u_m - u_{m-1})\|_{1,2}^{\nu-2}),\end{aligned}$$

providing c_1 and c_2 sufficiently small (i.e. δ).

Hence:

$$\|\nabla(u_{m+p} - u_m)\|_{2,2} \to 0, \tag{17}$$

when $m, p \to \infty$.

In fact u_m converges to the solution of (3) because via Newton formula for every $T > 0$ and $t \in [0, T)$ one has:

$$|(u_{m+1} - u_m)(t)|_{2,2} \leq CT \sup_{0 \leq \tau \leq T} |\nabla(u_{m+1} - u_m)(\tau)|_{2,2}.$$

Therefore, from (17), for any $T > 0$, we see that u_m converges with respect to norm:

$$\sup_{0 \leq \tau \leq T} |\cdot(\tau)|_{2,2}$$

to the solution of (3).

This completes the proof of part (a).

PROOF OF PART (b). To prove part (b) we need the following two lemmas:

LEMMA 4 (see [12]). *If f is a continuous function and $r := |x|$, then*:

$$\int_{|y-x|=t} f(|y|)\, dS_y = 2\pi \frac{t}{r} \int_{|r-t|}^{r+t} \lambda f(\lambda)\, d\lambda.$$

LEMMA 5. *If $n \geq 2$, then*:

$$\int_a^b \frac{ds}{(c+s)^n} \leq \frac{b-a}{(c+a)^{n-1}(c+b)},$$

where $b \geq a \geq 0$, and $c \geq 0$.

PROOF. We have that:

$$\int_a^b \frac{ds}{(c+s)^n} \leq \frac{1}{(c+a)^{n-2}} \int_a^b \frac{ds}{(c+s)^2} = \frac{b-a}{(c+a)^{n-1}(c+b)}.$$

This completes that proof of the lemma 5.

Using the representation formula for the solution of the wave equation we see that for every solution $u(t,x)$ of (3) we have:

$$u(t,x) = u_0(t,x) + \frac{1}{4\pi} \int_0^t \frac{1}{t-s} \left(\int_{|y-x|=t-s} |u_t(s,y)|^\nu\, dS_y \right) ds,$$

where $u_0(t,x)$ is the solution of the homogeneous problem with initial data (f,g).

We have the following estimate for the linear equation due to Pecher (see [12]):

$$|u_0(t,x)| \leq \frac{c}{(1+t+|x|)(1+|t-|x||)^{\nu-2}}.$$

Using lemma 2 and lemma 4 one obtains:

$$\left| \int_0^t \frac{1}{t-s} \left(\int_{|y-x|=t-s} |u_t(s,y)|^\nu\, dS_y \right) ds \right|$$

$$\leq c\|\nabla u\|_{2,2}^\nu \int_0^t \left(\int_{|y-x|=t-s} (1+s+|y|)^{-\nu}(1+|s-|y||)^{-\nu/2}\, dS_y \right) ds$$

$$= c\|\nabla u\|_{2,2}^\nu \cdot r^{-1} \cdot \int_0^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda d\lambda}{(1+s+\lambda)^\nu (1+|s-|\lambda||)^{\nu/2}}\, ds.$$

It is easy to be seen that:

$$|r-t| \leq s+\lambda \leq r+t,$$

when:

$$s \in [0,t],$$
$$\lambda \in [|r-t+s|, r+t-s].$$

Therefore by changing variables:

$$\alpha = s+\lambda, \quad \beta = s-\lambda,$$

one can obtain via lemma 5:

$$\left|\int_0^t \frac{1}{t-s}\left(\int_{|y-x|=t-s} |u_t(s,y)|^\nu \, dS_y\right) ds\right|$$
$$\leq c\|\nabla u\|_{2,2}^\nu \cdot r^{-1} \cdot \int_{r-t}^{r+t}\int_{-\infty}^{\infty} \frac{d\beta}{(1+\alpha)^{\nu-1}(1+\beta)^{\nu/2}} d\alpha$$
$$\leq c\|\nabla u\|_{2,2}^\nu \cdot r^{-1} \cdot \int_{r-t}^{r+t} \frac{d\alpha}{(1+\alpha)^{\nu-1}}$$
$$\leq c\|\nabla u\|_{2,2}^\nu \cdot r^{-1} \cdot \frac{r+t-|r-t|}{(1+r+t)(1+|r-t|)^{\nu-2}}$$
$$\leq \frac{c\|\nabla u\|_{2,2}^\nu}{(1+t+x)(1+|t-|x||)^{\nu-2}},$$

since $r+t-|r-t| \leq 2r$.

Hence:

$$|u(t,x)| \leq \frac{c}{(1+t+|x|)(1+|t-|x||)^{\nu-2}}.$$

Combining the last estimate with lemma 2 completes the prove of part (b). This completes the proof of Theorem 3.

REMARK. One has difficulty to use the idea of proving theorem 3 for the general space dimensions, since Sobolev inequality costs losses of derivatives. One may try to overcome the difficulty by estimates similar to these obtained in Georgiev [3]. One also may try to use some generalised form of Strihartz inequality (see [8]). Another approach is this developed by Ruis and Vega (see [13]). The idea is to estimate more refined mean values of fractional derivatives of the solution of the equation (3).

PROOF OF THEOREM 4.

To prove Theorem 4 we need the following lemma:

LEMMA 6. (a) *If* $f, g \in C_0^\infty$ *then the following inequality holds for* $m \geq 0$ *integer*:

$$|fg(t)|_{2m,2} \leq c(|f(t)|_{m,\infty}|g(t)|_{2m,2} + |f(t)|_{m,2}|g(t)|_{2m,\infty}).$$

(b) *If* $f \in C_0^\infty$ *and* $m \geq 0$ *is an integer then the following inequality holds*:

$$|f^\nu(t)|_{2m,2} \leq c|f(t)|_{m,\infty}^{\nu-1}|f(t)|_{2m,2}.$$

PROOF. Obviously (b) is a direct consequence of (a). To prove (a) we should notice that for any multiindex α such that $|\alpha| \leq 2m$ we have:

$$|\Gamma^\alpha fg(t)| \leq c \sum_{\alpha_i+\beta_i=\alpha} |\Gamma^{\alpha_i} f(t)|\,|\Gamma^{\beta_i} g(t)|,$$

where the sum is taken over all pairs (α_i, β_i) such that $\alpha_i + \beta_i = \alpha$. Obviously $|\alpha_i| \leq m$ or $|\beta_i| \leq m$. Now it remains to use Hölder inequality:

$$|ab|_{L^2} \leq |a|_{L^\infty}|b|_{L^2},$$

as many times as it is necessary in order to derive the desired estimate. This completes the proof of lemma 6.

As in the proof of Theorem 3 we consider the map:

$$M : v \mapsto w = Mv,$$

where w is the solution of the linear equation:

$$(\partial_t^2 - \Delta)w = |v_t|^\nu,$$

$$w(0,x) = f(x),$$

$$w_t(0,x) = g(x).$$

For $m > n/2$ one has via lemma 2, lemma 3 and lemma 6:

$$
\begin{aligned}
(18)\qquad |\nabla w(t)|_{2m,2} &\leq c\left(|\nabla w(0)|_{2m,2} + \int_0^t |v_t^{\nu}(\tau)|_{2m,2}\,d\tau\right)\\
&\leq c\left(|\nabla w(0)|_{2m,2} + \int_0^t |\nabla v(\tau)|_{2m,2}|\nabla v(\tau)|_{m,\infty}^{\nu-1}\,d\tau\right)\\
&\leq c\left(|\nabla w(0)|_{2m,2} + \|\nabla v\|_{2m,2}^{\nu}\int_0^t (1+\tau)^{-(n-1)(\nu-1)/2}\,d\tau\right).
\end{aligned}
$$

For the converges of the last integral we need:

$$
\frac{n-1}{2}(\nu-1) > 1 \Rightarrow \nu > \frac{n+1}{n-1}
$$

For $n \geq 4$, $\nu = 2, 3, 4, \ldots$ fulfiled the inequality $\nu > (n+1)/(n-1)$.

Hence we obtained:

$$
\|\nabla Mv\|_{2m,2} \leq c(\varepsilon + \|\nabla v\|_{2m,2}^{\nu}),
$$

providing f and g sufficiently small. Now it remains to use the same arguments like in the proof of Theorem 3 to complete the proof of Theorem 4.

REMARK. In a view of (18) one may conjecture that for $\nu > (n+1)/(n-1)$ (3) has global solution.

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Institute of Mathematics
Section of Mathematical Physics
Bulgarian Academy of Sciences
Sofia 1113, Acad. G. Bonchev bl. 8
Bulgaria
e-mail: Mathph@Bgearn.bitnet