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LIGHTNESS OF INDUCED MAPPINGS

By

Janusz J. CHARATONIK and Włodzimierz J. CHARATONIK

Abstract. Relations are studied concerning lightness of a mapping between continua and lightness of the two induced mappings: between hyperspaces of closed subsets and between hyperspaces of subcontinua.

1. Introduction

For a metric continuum X we denote by 2^X and $C(X)$ the hyperspaces of all nonempty closed and of all nonempty closed connected subsets of X, respectively. Given a mapping $f : X \to Y$ between continua X and Y, we let $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ to denote the corresponding induced mappings. Let $\mathfrak{M}$ be a class of mappings between continua. A general problem which is related to a given mapping and to the two induced mappings is to find all interrelations between the following three statements:

$$f \in \mathfrak{M}; \tag{1.1}$$

$$C(f) \in \mathfrak{M}; \tag{1.2}$$

$$2^f \in \mathfrak{M}. \tag{1.3}$$

There are some papers in which particular results concerning this problem are shown for various classes $\mathfrak{M}$ of mappings like open, monotone, confluent and some others, see [2], [3], [4], [5], [6], [9], [12]. In the present paper we discuss the problem in full for the class of light mappings, and we get some corollaries for local homeomorphisms.

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valuable discussions on the topic of this paper; especially for his idea of Example 3.9.

2. Preliminaries

All spaces considered in this paper are assumed to be metric. A *mapping* means a continuous function. We denote by $\boldsymbol{N}$ the set of all positive integers, and by $\boldsymbol{R}$ the space of real numbers.

A *continuum* means a compact connected space. Given a continuum X with a metric d, we let 2^X to denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A,B) = \max\{\sup\{d(a,B) : a \in A\}, \sup\{d(b,A) : b \in B\}\}$$

(equivalently: with the Vietoris topology: see e.g. [12, (0.1), p. 1 and (0.12), p. 10]. Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X. For $\Lambda \subset 2^X$ we let Λ^* to denote the union of all elements of Λ. It is well-known that if Λ is closed in 2^X, then Λ^* is a closed subset of X (see e.g. [8, §42, III, Theorem 5, p. 52]). The reader is referred to Kuratowski's monograph [8] and (mainly) to Nadler's book [12] for needed information on the structure of hyperspaces. In particular, the following fact is well-known (see [12, Theorem (1.13), p. 65]).

2.1. FACT. *For each continuum X the hyperspace $C(X)$ is a subcontinuum of the hyperspace 2^X.*

For each $n \in \boldsymbol{N}$ we put $F_n(X) = \{A \in 2^X : \operatorname{card} A \leq n\}$. Observe that

$$F_1(X) \subset F_2(X) \subset \cdots \subset F_n(X) \subset \cdots \subset 2^X,$$

and that each $F_n(X)$ is a closed subset of 2^X. Further, the following proposition is a consequence of definitions.

2.2. PROPOSITION. *For each continuum X the space $F_1(X)$ of singletons is homeomorphic (even isometric) to X, and thus it is a subcontinuum of the hyperspace $C(X)$. Consequently,*

$$X \simeq F_1(X) \subset C(X) \subset 2^X. \tag{2.3}$$

By an *order arc* in 2^X we mean an arc Φ in 2^X such that if $A, B \in \Phi$, then either $A \subset B$ or $B \subset A$. The following facts are known (see [12, Theorem (1.8), p. 59 and Lemma (1.11), p. 64]).

2.4. FACT. *Let A, $B \in 2^X$ with $A \neq B$. Then there exists an order arc in 2^X from A to B if and only if $A \subset B$ and each component of B intersects A.*

2.5. FACT. *If an order arc Φ in 2^X begins with $A \in C(X)$, then $\Phi \subset C(X)$.*

Given a mapping $f : X \to Y$ between continua X and Y, we consider mappings (called the *induced* ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = f(A) \quad \text{for every } A \in 2^X \quad \text{and} \quad C(f)(A) = f(A) \quad \text{for every } A \in C(X).$$

Thus, by Fact 2.1, the following is obvious.

2.6. FACT. *For every continua X and Y and for each mapping $f : X \to Y$ we have $2^f | C(X) = C(f)$.*

A proof of the following fact is straightforward.

2.7. FACT. *Let a mapping $f : X \to Y$ between continua X and Y be given. Then $C(f)(F_1(X)) \subset F_1(Y)$.*

Recall that two mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are said to be *equivalent* provided that there exist homeomorphisms $h_X : X_1 \to X_2$ and $h_Y : Y_1 \to Y_2$ such that $f_2(h_X(x)) = h_Y(f_1(x))$ for each point $x \in X$. Observe that this relation is an equivalence in the class of mappings between topological spaces (see [13, p. 127]).

2.8. PROPOSITION. *The mappings*

$$f : X \to Y, \quad 2^f | F_1(X) : F_1(X) \to F_1(Y) \quad \textit{and} \quad C(f) | F_1(X) : F_1(X) \to F_1(Y)$$

are mutually equivalent.

PROOF. In fact, by Proposition 2.2 there are homeomorphisms $h : X \to F_1(X)$ and $g : Y \to F_1(Y)$. Now, for each $x \in X$, the equalities

$$2^f | F_1(X)(h(x)) = g(f(x)) \quad \text{and} \quad C(f) | F_1(X)(h(x)) = g(f(x))$$

hold by the definitions.

We start our discussion concerning the problem mentioned in the introduction from a result of a more general nature, which is related to all hereditary classes of mappings.

Recall that a class $\mathfrak{M}$ of mappings between continua is said to have the *composition property* provided that compositions of two mappings belonging to $\mathfrak{M}$ also belongs to $\mathfrak{M}$, and it is said to be *hereditary* (compare [11]) provided that for each subcontinuum S of the domain the partial mapping restricted to S also is in $\mathfrak{M}$.

2.9. THEOREM. *If a class $\mathfrak{M}$ of mappings between continua contains homeomorphisms, has the composition property and is hereditary, then* (1.3) *implies* (1.2), *and* (1.2) *implies* (1.1).

PROOF. Assume (1.3). Since $C(X)$ is a subcontinuum of 2^X by Fact 2.1, and since $\mathfrak{M}$ is hereditary, the partial mapping $2^f|C(X)$ is in $\mathfrak{M}$. Applying Fact 2.6 we get (1.2). Similarly, assuming (1.2) and using Proposition 2.2 we infer that $C(f)|F_1(X)$ is in $\mathfrak{M}$. Now (1.1) follows by Proposition 2.8.

3. Light mappings

A mapping $f : X \to Y$ between spaces X and Y is said to be *light* provided that for each point $y \in Y$ the set $f^{-1}(y)$ has one-point components (equivalently, if $f^{-1}(f(x))$ is totally disconnected for each $x \in X$; note that if the inverse images of points are compact, this condition is equivalent to the property that they are zero-dimensional).

3.1. THEOREM. *Let a mapping $f : X \to Y$ between continua X and Y be given. Then the following conditions are equivalent*:

(3.2) $$f \text{ is light;}$$

(3.3) $$(C(f))^{-1}(F_1(Y)) \subset F_1(X);$$

(3.4) $$F_1(X) = (C(f))^{-1}(F_1(Y));$$

(3.5) $$F_1(X) \text{ is a component of } (C(f))^{-1}(F_1(Y)).$$

PROOF. Assume (3.2). If a continuum K is an element of $(C(f))^{-1}(F_1(Y))$, then $C(f)(K) \in F_1(Y)$, i.e., $f(K)$ is a singleton, say $\{y\}$, which implies that $K \subset f^{-1}(y)$, and therefore K is degenerate by (3.2). Hence $K \in F_1(X)$ and so (3.3) follows. (3.3) obviously implies (3.4). Since $F_1(X)$ is connected (see Proposition 2.2), (3.4) implies (3.5). To show that (3.5) implies (3.2) suppose f is not light. Then there exists a nondegenerate subcontinuum Q of X such that $f(Q)$ is a singleton, say $\{q\}$, in Y. So, Q is an element of $C(X)\backslash F_1(X)$ and $C(f)(Q) \in F_1(Y)$. Obviously, for each subcontinuum Q' of Q we have $f(Q') = f(Q) = \{q\}$, whence it follows that $C(f)(C(Q)) = \{q\} \in F_1(Y)$. Thus $C(Q) \subset (C(f))^{-1}(F_1(Y))$, contrary to (3.5). The proof is complete.

3.6. THEOREM. *For each mapping $f : X \to Y$ between continua X and Y the induced mapping $2^f : 2^X \to 2^Y$ is light if and only if for every A, $B \in 2^X$ the conditions $A \subsetneq B$ and each component of B intersects A imply the condition $f(A) \subsetneq f(B)$.*

PROOF. "Only if". Suppose there are A, $B \in 2^X$ such that $A \subsetneq B$, each component of B intersects A, and $f(A) = f(B)$. Put $D = f(A)$. Then A, $B \in (2^f)^{-1}(D)$ and, by Fact 2.3, there exists an order arc Φ from A to B in 2^X. Thus for each $K \in \Phi$ we have $A \subset K \subset B$, whence $f(K) = D$, and consequently $\Phi \subset (2^f)^{-1}(D)$, so 2^f is not light.

"If". Suppose there is $D \in 2^Y$ such that $(2^f)^{-1}(D)$ contains a nondegenerate continuum Δ. Then $\Delta^* \in 2^X$ and $f(\Delta^*) = D$. Since Δ is nondegenerate, there exists $A \in \Delta$ such that $A \neq \Delta^*$. Put $B = \Delta^*$. Then $A \subsetneq B$ and $f(A) = f(B) = D$. We shall show that each component of B intersects A. Suppose on the contrary that there is a component C of B such that $C \cap A = \emptyset$. Thus there exists a set U open in X and such that

$$A \subset U, \quad C \cap \operatorname{cl} U = \emptyset \quad \text{and} \quad B \cap \operatorname{bd} U = \emptyset.$$

Define $\Gamma = 2^U$. Then Γ is open in 2^X, and $A \in \Gamma$. Since C is a component of B, there is an element E of Δ such that $E \cap C \neq \emptyset$. Thus $E \in \Delta\backslash\Gamma$. By connectedness of Δ it follows that there is $F \in \Delta \cap \operatorname{bd}\Gamma$. Thus $F \cap \operatorname{bd} U \neq \emptyset$, whence $\Delta^* \cap \operatorname{bd} U \neq \emptyset$, a contradiction. The proof is complete.

As a consequence of Theorem 3.6 and of Facts 2.4 and 2.5 we get the following known result due to J. B. Fugate and S. B. Nadler, Jr., which is formulated as an exercise in [12, (1.212.3), p. 204].

3.7. THEOREM. *For each mapping $f : X \to Y$ between continua X and Y the induced mapping $C(f) : C(X) \to C(Y)$ is light if and only if for every A, $B \in C(X)$ the condition $A \subsetneq B$ implies the condition $f(A) \subsetneq f(B)$.*

Now the intend to study all the possible implications between lightness of a mapping between continua and lightness of the two induced mappings between the hyperspaces. We shall show that lightness of f implies lightness of neither 2^f nor $C(f)$, and that lightness of $C(f)$ does not imply that of 2^f. The other implications are true. We start with the needed examples.

Denote by $\boldsymbol{C}$ the plane of complex numbers equipped with the Euclidean metric, and let S^1 stand for the unit circle, i.e., $S^1 = \{z \in \boldsymbol{C} : |z| = 1\}$.

3.8. EXAMPLE. *The mapping $f : S^1 \to S^1$ defined by $f(z) = z^2$ is light, while $C(f)$ and 2^f are not.*

PROOF. Lightness of f is evident. To see that $C(f)$ and 2^f are not light, we apply Theorems 3.7 and 3.6, respectively. To this aim consider the right semi-circle $A = \{z \in S^1 : |\arg z| \le \pi/2\}$ of the domain S^1, and put $B = S^1$. Then $A \subsetneq B$, while $f(A) = f(B) = S^1$. The proof is complete.

3.9. EXAMPLE (Illanes). *There is a light mapping $f : X \to Y$ between continua X and Y such that $C(f)$ is light, while 2^f is not.*

PROOF. Let

$$L_1 = \{(x, \sin(1/x)) \in \boldsymbol{R}^2 : x \in (0,1]\} \quad \text{and} \quad I_1 = \{(0,y) \in \boldsymbol{R}^2 : y \in [-1,1]\},$$

and let L_2 and I_2 be the images of L_1 and I_1, respectively, under the symmetry of the plane $\boldsymbol{R}^2$ with respect to the straight line $x = 1$. Let $L = L_1 \cup L_2$, and put $X = L \cup I_1 \cup I_2$. Then X is a continuum having L, I_1 and I_2 as its arc-components. Let a relation ρ on X identify points $(0,y)$ of I_1 and $(2,y)$ of I_2 only. Define $Y = X/\rho$ as the quotient space, and let $f : X \to Y$ be the quotient mapping. Then the continuum Y has two arc-components: $f(L)$ and $f(I_1) = f(I_2)$. Obviously for each $y \in Y$ we have $\operatorname{card} f^{-1}(y) \le 2$, so f is light.

To see $C(f)$ is light take $K \in C(Y)$ and consider two cases. If $K \cap f(L) \ne \emptyset$, then the set $(C(f))^{-1}(K)$ is either one-element or empty. If $K \cap f(L) = \emptyset$, then $K \subset f(I_1) = f(I_2)$, and therefore $(C(f))^{-1}(K)$ has two elements.

To verify that 2^f is not light take a point $p \in I_2$ and put $A = I_1 \cup \{p\}$ and $B = I_1 \cup I_2$. Then $A \subsetneq B$ and each component of B intersects A, while $f(A) = f(B) = f(I_1)$, so 2^f is not light by Theorem 3.6. The proof is then complete.

3.10. THEOREM. *Let a mapping $f : X \to Y$ between continua X and Y be given. Consider the following conditions:*

(3.2) $$f \text{ is light};$$

(3.11) $$C(f) : C(X) \to C(Y) \text{ is light};$$

(3.12) $$2^f : 2^X \to 2^Y \text{ is light}.$$

Then (3.12) *implies* (3.11), *and* (3.11) *implies* (3.2). *Consequently*, (3.12) *implies* (3.2). *The other implications do not hold.*

Proof. In fact, it is enough to observe that the class of light mappings contains homeomorphisms, is closed with respect to compositions, and is hereditary, and next to apply Theorem 2.9. The implications listed in the formulation of the theorem are the only true, as it can be seen from Examples 3.8 and 3.9. The proof is finished.

In connection with Example 3.8 we have the following observation.

3.13. Observation. *For every continuum Y, for every point $p \in Y$ and for every number $n \in N$ with $n > 1$ there exists a continuum X and a surjective light mapping $f : X \to Y$ such that*

(3.14) $$\operatorname{card} f^{-1}(p) = 1 \quad \textit{and} \quad \operatorname{card} f^{-1}(q) = n \quad \textit{for each point } q \in Y \setminus \{p\};$$

(3.15) $$\textit{neither } C(f) \textit{ nor } 2^f \textit{ is light.}$$

Proof. Take the disjoint union U of n copies (Y_i, p_i) of the pointed continuum (Y, p) for $i \in \{1, \ldots, n\}$. Define a relation ρ on U by

$$x\rho y \qquad \text{if } x = y \quad \text{or} \quad x, y \in \{p_i : i \in \{1, \ldots, n\}\}.$$

Let $X = U/\rho$ denote the quotient space, and let $f : X \to Y$ be the natural projection. Then (3.14) holds by the definition, whence lightness of f follows. Applying Theorem 3.7 we see that $C(f)$ is not light, and thus 2^f also is not, by Theorem 3.10. This ends the proof.

4. Mappings of a constant degree. Local homeomorphisms

A mapping $f : X \to Y$ is said to be:
— *of a constant degree* if there is an $n \in N$ such that $\operatorname{card} f^{-1}(y) = n$ for each $y \in Y$ (in some papers these mappings are called n-to-1 ones);
— a *local homeomorphism* provided that every point $x \in X$ has an open

neighborhood U such that $f(U)$ is an open subset of Y and the partial mapping $f|U : U \to f(U)$ is a homeomorphism;
— *open*, if f maps each open set in X onto an open set in Y.

It is known that a mapping $f : X \to Y$ of a compact space X onto a connected space Y is a local homeomorphism if and only if it is open and of a constant degree ([10, Proposition 2, p. 855 and Theorem 4, p. 856]). Since each mapping of a constant degree is obviously light, we see that any local homeomorphism between continua is light.

For homeomorphisms we have the following known result (see [12, (0.52), p. 29].

4.1. THEOREM. *If $\mathfrak{M}$ means the class of homeomorphisms, then conditions* (1.1), (1.2) *and* (1.3) *are equivalent.*

For local homeomorphisms however, such an equivalence does not hold. Indeed, note that the mapping $f : S^1 \to S^1$ of Example 3.8 is a local homeomorphism, while $C(f)$ and 2^f are not light even. Moreover, the following result concerns mappings of a constant degree.

4.2. THEOREM. *Let $f : X \to Y$ be a mapping between continua X and Y. If, for some $n \in N$, the induced mapping either 2^f or $C(f)$ is of the constant degree n, then $n = 1$ and f, 2^f and $C(f)$ are homeomorphisms.*

PROOF. Suppose 2^f is of the constant degree $n > 1$ (for $C(f)$ the argument is the same). Thus for the element Y of 2^Y there are at least two nonempty closed subsets $B = X$ and $A \subsetneq X$ of X such that $f(A) = f(B) = Y$. Thus 2^f is not light by Theorem 3.6. Hence $n = 1$, and the rest of the conclusion follows from Theorem 4.1.

4.3. COROLLARY. *Let $f : X \to Y$ be a mapping between continua X and Y. If the induced mapping either 2^f or $C(f)$ is a local homeomorphism, then f, 2^f and $C(f)$ are homeomorphisms.*

4.4. REMARK. In the proof of the above theorem we use the "top" Y of the hyperspace $C(Y)$ to show that $C(f)$ is never of a constant degree $n > 1$. The following example shows that if we delete the "top", then the partial mapping restricted to $C(X)\setminus\{X\}$ can be of degree 2.

4.5. Example. *There are a continuum X and a mapping $f : X \to X$ such that f and the partial mapping $C(f)|(C(X)\setminus\{X\})$ are of the constant degree 2, $(C(f))^{-1}(X) = \{X\}$ (thus f and $C(f)$ are light), while 2^f is not light.*

Proof. Let $\{X_i, \phi_i : i \in \mathbf{N}\}$ be an inverse sequence with $X_i = S^1$ and $\phi_i : X_{i+1} \to X_i$ defined by $\phi_i(z) = z^3$ for each $i \in \mathbf{N}$. Then the inverse limit $X = \lim_{\leftarrow}\{X_i, \phi_i\}$ is the triadic solenoid. Define $f_i : X_i \to X_i$ by $f_i(z) = z^2$ and note that the diagram

$$\begin{array}{ccc} X_i & \xleftarrow{\phi_i} & X_{i+1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i+1}} \\ X_i & \xleftarrow{\phi_i} & X_{i+1} \end{array}$$

commutes. Denote by $f : X \to X$ the limit mapping $f = \lim f_i$. Further, take $h : X \to X$ defined by $h((x_1, x_2, \ldots)) = (-x_1, -x_2, \ldots)$ and note that $f^{-1}(f(x)) = \{x, h(x)\}$ for $x \in X$. Since $h(x) \neq x$, the mapping f is of the constant degree 2. We claim that

(4.6) for each $x \in X$ there is no proper subcontinuum of X containing both x and $h(x)$.

Indeed, suppose on the contrary that there is $A \in C(X)\setminus\{X\}$ such that x, $h(x) \in A$. Then there is an index $m \in \mathbf{N}$ such that if $\pi_m : X \to X_m$ denotes the m-th projection, then $\pi_m(A) \subsetneq X_m$. Then $\pi_{m+1}(A)$ contains two antipodal points $\pi_{m+1}(x)$ and $\pi_{m+1}(h(x))$. Note that $\pi_m(A) = \phi_m(\pi_{m+1}(A))$, and since ϕ_m maps every continuum containing any two antipodal points onto X_m, we conclude that $\pi_m(A) = X_m$, a contradiction. Thus (4.6) is established.

To show that the partial mapping $C(f)|(C(X)\setminus\{X\})$ is of the constant degree 2, take $A \in C(X)\setminus\{X\}$ and note that A, $h(A) \in (C(f))^{-1}(C(f)(A))$. By (4.6) we have $A \neq h(A)$. Assume that $f(A) = f(B)$ for some $B \in C(X)$. We will show that either $B = A$ or $B = h(A)$. Recall that $f^{-1}(f(x)) = \{x, h(x)\}$ and thus

(4.7) for each $x \in A$ either $x \in B$ or $h(x) \in B$.

Thus $B = (A \cap B) \cup (h(A) \cap B)$. By (4.6) this is a decomposition of the set B into two disjoint closed subsets; so one of them, say $A \cap B$, is empty. Therefore $B \subset h(A)$, and by (4.7) it follows that $h(A) \subset B$. If $h(A) \cap B = \emptyset$, the argument is the same.

To see that 2^f is not light take an arbitrary proper subcontinuum P of X and a point $p \in P$. Then for $A = P \cup \{h(p)\}$ and $B = P \cup h(P)$ we have $A \subsetneq B$, the sets P and $h(P)$ are components of B by (4.6), thereby each component of B intersects A, while $f(A) = f(B)$. Thus 2^f is not light by Theorem 3.6. The proof is then complete.

4.8. Remark. Observe that the mapping f of Example 4.5 is open, thus weakly confluent, and therefore the induced mapping $C(f)$ is a surjection (see [12, (0.49.1), p. 24]), unlike one of Example 3.9.

5. An intermediate condition

The continua of Examples 3.9 and 4.5 (showing that the conditions (1.1), (1.2) and (1.3) are not equivalent if $\mathfrak{M}$ stands for the class of light mappings) are not locally connected. They are not arcwise connected even. Thus the following question is natural.

5.1. Question. Are lightness of the induced mappings 2^f and $C(f)$ equivalent conditions for a mapping f between arcwise connected (in particular, locally connected) continua?

Observe that in all the constructed examples, i.e., in Examples 3.8, 3.9 and 4.5, nonlightness of 2^f is implied by the existence of two nondegenerate disjoint subcontinua of X having the same images under f (for Example 3.8 one can take $\{z \in S^1 : \arg z \in [0, \pi/2]\}$ as one of them and $\{z \in S^1 : \arg z \in [\pi, 3\pi/2]\}$ as the other; for Example 3.9 we have I_1 and I_2; and in Example 4.5 we can take any $A \in C(X) \setminus (F_1(X) \cup \{X\})$ and $h(A)$ as the needed subcontinua). Generalizing this, we formulate a condition (viz. condition (5.3) below) which is shown in the next theorem to be intermediate between (but not equivalent to) lightness of the two induced mappings. We start with the following example.

5.2. Example. *There are continua X and Y and a surjective mapping $f : X \to Y$ such that*

(5.3) *for every two continua $P, Q \in C(X) \setminus F_1(X)$ with $P \cap Q = \emptyset$ the inequality $f(P) \setminus f(Q) \neq \emptyset$ holds,*

and that 2^f is not light.

Proof. Let C stand for the Cantor ternary set. Denote by F_C the Cantor fan, i.e., the cone $(C \times [0, 1])/(C \times \{1\})$ over C, and let $v = C \times \{1\}$ be the top of the cone. Then $F_C \setminus \{v\}$ is a locally compact, noncompact metric space. It is known (see [1, Theorem, p. 35]) that if S a locally compact, noncompact, metric space, then each continuum is a remainder of S in some compactification of S. So, take an arc L as the remainder of $S = F_C \setminus \{v\}$ in a compactification γ of S, and let X be the obtained compact space, i.e.,

$$\gamma : S \to \gamma(S) \subset \operatorname{cl} \gamma(S) = X = \gamma(S) \cup L,$$

with $\gamma(S) \cap L = \emptyset$. Thus X is a continuum. To define Y and f we need two auxiliary mappings. The first is an arbitrary homeomorphism $h : L \to [0, 1]$ of the arc L onto the closed unit interval. The second one, $g : C \to [0, 1]$, is defined as the well-known Cantor-Lebesgue step-function from the Cantor set C onto $[0, 1]$ (compare e.g. [7, §16, II, (8) and footnote 1, p. 150] or [13, p. 35]). Consider now a decomposition $\mathscr{D}$ of X having the sets $h^{-1}(t) \cup \gamma((g^{-1}(t)) \times \{0\})$, for each $t \in [0, 1]$, as the only nondegenerate elements; let $Y = X/\mathscr{D}$ be the decomposition space, and take $f : X \to Y$ as the quotient mapping. In other words, for the set $E = \{e = \gamma((c, 0)) \in X : c \in C\}$ of the end points of X we consider a mapping $\psi : E \to L$ defined by $\psi(e) = h^{-1}(g(c))$ (which is equivalent to g); then f identifies each end point $e \in E$ of X with its image $\psi(e)$, and is a homeomorphism on $X \setminus E$.

To see that 2^f is not light choose a point $p \in L$ and put $A = E \cup \{p\}$ and $B = E \cup L$. Then we have $f(A) = f(B)$ and the conclusion follows from Theorem 3.6.

To verify that (5.3) holds suppose on the contrary that there are two nondegenerate disjoint subcontinua P and Q of X such that $f(P) \subset f(Q)$. Since the partial mapping $f|(X \setminus (E \cup L))$ is a homeomorphism, we conclude that P, $Q \subset E \cup L$. Therefore, since P and Q are nondegenerate, we have P, $Q \subset L$. But the partial mapping $f|L$ is a homeomorphism, too, thus $f(P)$ and $f(Q)$ have to be disjoint, which is a contradiction finishing the proof.

The above mentioned theorem, which supplies Theorem 3.10, runs as follows.

5.4. Theorem. *Let continua X and Y and a mapping $f : X \to Y$ be given. Consider the following conditions*:

(3.11) $\quad C(f) : C(X) \to C(Y)$ *is light*;

(5.3) *for every two continua P, $Q \in C(X) \setminus F_1(X)$ with $P \cap Q = \emptyset$ the inequality $f(P) \setminus f(Q) \neq \emptyset$ holds*;

(3.12) $\quad 2^f : 2^X \to 2^Y$ *is light*.

Then (3.12) *implies* (5.3), *and* (5.3) *implies* (3.11). *Consequently*, (3.12) *implies* (3.11). *The other implications do not hold.*

PROOF. To show that (3.12) implies (5.3) suppose that there are two continua P, $Q \in C(X) \setminus F_1(X)$ with $P \cap Q = \emptyset$ and $f(P) \subset f(Q)$. Choose a point $p \in P$. Then for the sets $A = \{p\} \cup Q$ and $B = P \cup Q$ we see that $A \subsetneq B$, each component of B intersects A, while $f(A) = f(B)$. Thus 2^f is not light by Theorem 3.6.

To see that implication from (5.3) to (3.11) holds assume (5.3) and suppose that $C(f)$ is not light. By Theorem 3.7 there are A, $B \in C(X)$ such that $A \subsetneq B$ and $f(A) = f(B)$. Hence B is nondegenerate. Enlarging A in B if necessary, we can assume that A is nondegenerate, too. Let P be a nondegenerate subcontinuum contained in $B \setminus A$ and put $Q = A$. Then $f(P) \subset f(B) = f(A) = f(Q)$ contrary to (5.3).

Example 5.2 shows that implication from (3.12) to (5.3) is not reversible. Taking in Example 4.5 any $P \in C(X) \setminus (F_1(X) \cup \{X\})$ and $Q = h(P)$ we get the needed subcontinua showing that (3.11) does not imply (5.3). This finishes the proof.

Theorems 3.10 and 5.4 can be summarized in the following corollary.

5.5. COROLLARY. *Let continua X and Y and a mapping $f : X \to Y$ be given. Consider the following conditions.*

(3.2) f *is light*;

(3.11) $C(f) : C(X) \to C(Y)$ *is light*;

(5.3) *for every two continua P, $Q \in C(X) \setminus F_1(X)$ with $P \cap Q = \emptyset$ the inequality $f(P) \setminus f(Q) \neq \emptyset$ holds*;

(3.12) $2^f : 2^X \to 2^Y$ *is light.*

Then the implications

$$(3.12) \Rightarrow (5.3) \Rightarrow (3.11) \Rightarrow (3.2)$$

hold, and none of them can be reversed.

5.6. THEOREM. *Let an arcwise connected continuum X, a continuum Y and a mapping $f : X \to Y$ be given. Then* (3.11) *implies* (5.3).

PROOF. Suppose that (5.3) does not hold, i.e., that there are two continua P, $Q \in C(X) \setminus F_1(X)$ with $P \cap Q = \emptyset$ and $f(P) \subset f(Q)$. Denote by L an irreducible arc between P and Q. Putting $A = L \cup Q$ and $B = P \cup L \cup Q$ we have $A \subsetneq B$ and $f(A) = f(B)$. Thus by Theorem 3.7 we see that $C(f)$ is not light.

5.7. COROLLARY. *For mappings f with an arcwise connected (in particular with a locally connected) domain X conditions* (3.11) *and* (5.3) *are equivalent.*

5.8. REMARK. The implication from lightness of $C(f)$ to lightness of f cannot be reversed even for mappings f between locally connected continua (see Example 3.8). The authors do not know whether the implication from (3.12) to (5.3) of Corollary 5.5 can be replaced by the equivalence under this additional assumption. Thus we have the following question that is equivalent to Question 5.1.

5.9. QUESTION. Is the implication (5.3) $\Rightarrow$ (3.12) true if the domain space X is an arcwise connected (in particular locally connected) continuum?

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Mathematical Institute,
University of Wrocław, pl.
Grunwaldzki 2/4, 50-384
Wrocław, Poland

Instituto de Matemáticas,
UNAM, Circuito Exterior,
Ciudad Universitaria, 04510
México, D. F., México

Departamento de Matemáticas,
Facultad de Ciencias,
UNAM, Circuito Exterior,
Ciudad Universitaria, 04510
México, D. F., México
E-mail address: jjc@hera.math.uni.wroc.pl
jjc@gauss.matem.unam.mx
wjcharat@hera.math.uni.wroc.pl
wjcharat@lya.fciencias.unam.mx