TSUKUBA J. MATH.
Vol. 22 No. 1 (1998), 157–163

WALL INVARIANT FOR THE SPACE SATISFYING CONDITION (T^{**})

By

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Abstract. In this paper, we define the locally nilpotent space and condition (T^{**}), and study their properties.

Furthermore, we find the vanishing condition of the Wall invariant of the space satisfying the condition (T^{**}) and locally nilpotent space.

1. Introduction

Since C. T. C. Wall defined the Wall obstruction, V. J. Lal, G. Mislin, E. K. Pedersen, L. R. Taylor and R. Oliver have studied the finiteness condition of the nilpotent space and homologically nilpotent space [4, 13, 15, 18]. And there are many results on the nilpotent space [5, 6, 7, 8].

In this paper, we define the condition (T^{**}) and locally nilpotent space as the extensive concept of the nilpotent space and study their properties.

Furthermore, we study the Wall invariant of the space satisfying the condition (T^{**}).

All spaces are arcwise connected CW complexes unless otherwise stated and we denote the category T.

We assert the following:

THEOREM 3.2. *For X satisfying condition (T^{**}) with $\pi_1(X)$ finite, and the action $\pi_1(X)\times H_n(\tilde{X})\to H_n(\tilde{X})$ is nilpotent for all $n\geq 0$, then $X\in T_N$.*

THEOREM 3.3. *Let $F\to E\xrightarrow{p} B$ be a fibration with F a finitely dominated space. If B is a finite space satisfying condition (T^{**}), the action $\pi_1(B)\times H_n(\tilde{B})\to H_n(\tilde{B})$ is nilpotent for all $n\geq 0$ and $\pi_1(B)(\neq 0)$ is finite then $\omega(E)=0$.*

This work was partially supported by BSRI program, Ministry of education under the project number BSRI-96-1425.
Received August 19, 1996.
Revised February 24, 1997.

THEOREM 3.4. *Let $F \to E \xrightarrow{p} B$ be a fibration with F a finitely dominated space. For finite $B(\in T_{LN})$ if $\pi_1(B)$ is finite and $\pi_1(B) \neq 0$ or $\pi_1(B)$ is infinite with the maximal condition on a normal subgroup of $\pi_1(B)$ then $\omega(E) = 0$.*

THEOREM 3.6. *Let $F \to E \xrightarrow{p} B$ be a fibration under the following conditions that B is a finitely dominated space and $\pi_1(B)$ acts nilpotently on the homology of the fiber F. If F is a finite complex, such that $\pi_1(F)$ is nontrivial, E is a space satisfying condition (T^{**}), and the action $\pi_1(E) \times H_n(\tilde{E}) \to H_n(\tilde{E})$ is nilpotent where $n \geq 0$, with $\pi_1(E)(\neq 0)$ finite, then $\omega(E) \in Ker\, p_*$, where $p_* : K_0(Z\pi_1(E)) \to K_0(Z\pi_1(B))$.*

2. Preliminaries

For a space X, we consider the group ring $Z\pi_1(X)$. Let $K_0(Z\pi_1(X))$ denote the Grothendieck group of the group ring $Z\pi_1(X)$.

DEFINITION 2.1. *A space is called of type FP, if the singular chain complex $C_i\tilde{X}$ of the universal covering $\tilde{X}$ of X is chain homotopy equivalent (as $Z\pi_1(X)$-complex) to a finite projective complex, i.e., a complex $\bar{C}_i$ with $\bar{C}_i = 0$ for i big enough, and with each $\bar{C}_i$ a finitely generated projective $Z\pi_1(X)$ module.*

If X is of type FP, the Wall obstruction $\omega(X)$ is defined by

$$\omega(X) = \Sigma(-1)^i[\bar{C}_i] \in K_0(Z\pi_1(X))$$

where $\bar{C}_i$ is a finite projective complex equivalent to $C_i\tilde{X}$, and $[\bar{C}_i]$ denotes the class of $\bar{C}_i$ in the projective class group $K_0(Z\pi_1(X))$. It is evident that $w(X)$ is independent of the choice of $\bar{C}_i$.

Furthermore, a space X of type FP is dominated by a finite complex if and only if $\pi_1(X)$ is finitely presented [12].

DEFINITION 2.2. *A π-module M is called nilpotent if $I^kM = 0$ for some $k > 0$ where I denotes the augmentation ideal of $Z\pi$* [9, 10]. *Furthermore, a space X is called nilpotent if $\pi_1(X)$ is nilpotent and the $\pi_1(X)$-module $\pi_i(X)$ $(i > 1)$ are all nilpotent.*

We know the fact that: if $\pi_1(X)$ is nilpotent then X is of type FP if and only if X is finitely dominated [14].

And we denote the category of nilpotent spaces and continuous maps as T_N.

DEFINITION 2.3. *A space $X(\in T)$ is said to be a locally nilpotent space if*
(1) *$\pi_1(X)$ is a locally nilpotent group,*

(2) *the action* $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ *is nilpotent for all* $n \geq 2$.

And we denote the category of locally nilpotent spaces and continuous maps as T_{LN}.

We know that the category T_N *is a full subcategory of* T_{LN}.

Generally, for a group G *and a fixed* $g \in G$, *we denote by* $[g, G]$ *the subgroup of* G *generated by all commutators in* G.

Since $[g, a]^b = [g, b]^{-1}[g, ab]$ *for each* $a, b \in G$ (*where* $a^b = b^{-1}ab$), $[g, G]$ *is a normal subgroup of* G [17].

Definition 2.4. *We say that a space* $X(\in T)$ *satisfies condition* (T^*) *if for all* $g, t \in \pi_1(X)$

$$\text{either } g[g, \pi_1(X)] = t[t, \pi_1(X)]$$

$$\text{or } g[g, \pi_1(X)] \cap t[t, \pi_1(X)] = \phi.$$

Lemma 2.5 [2]. *Let* G *be an arbitrary group. If* $b \in a[a, G]$ $(a, b \in G)$ *then* $b[b, G] \subset a[a, G]$.

Lemma 2.6 [2]. *For* $X \in T_{LN}$, *then* X *satisfies the condition* (T^*).

Proof. Since $\pi_1(X)$ is a locally nilpotent group, suppose $c \in a[a, \pi_1(X)] \cap b[b, \pi_1(X)]$ for some $a, b, c \in \pi_1(X)$. We only show that $a[a, \pi_1(X)] = b[b, \pi_1(X)]$. By Lemma 2.5,

$$c[c, \pi_1(X)] \subset a[a, \pi_1(X)] \cap b[b, \pi_1(X)] \cdots (*)$$

Clearly, $c = h^{-1}a$ for some $h = \prod_{i=1}^{m}[a, g_i]^{\varepsilon_i} \in [a, \pi_1(X)]$ $(g_i \in \pi_1(X), \varepsilon_i = \pm 1)$. Let $G_1 = \langle a, g_1, \dots, g_m \rangle$. Since $a = hc$, $h \equiv \prod_{i=1}^{m}[h, g_i]^{\varepsilon_i}$ modulo $[c, G_1]$, that is, $h = \prod_{i=1}^{m}[h, g_i]^{\varepsilon_i}$ in $G_1/[c, G_1]$. However, since the latter group is nilpotent it follows that $h = 1$ in $G_1/[c, G_1]$ and $h \in [c, G_1]$. Therefore, $a = hc \in c[c, \pi_1(X)]$ and by Lemma 2.5, $a[a, \pi_1(X)] \subset c[c, \pi_1(X)]$. It follows from $(*)$ that $a[a, \pi_1(X)] = c[c, \pi_1(X)]$. Similarly, $b[b, \pi_1(X)] = c[c, \pi_1(X)]$ and consequently, $a[a, \pi_1(X)] = b[b, \pi_1(X)]$.

Definition 2.7. *For* $X \in T$, *we say that* X *satisfies the condition* (T^{**}) *if for all* $g(\neq 1) \in \pi_1(X)$, *then* $g \notin [g, \pi_1(X)]$.

Since the $[g, \pi_1(X)]$ is a normal subgroup of $\pi_1(X)$, the condition (T^{**}) is homotopy invariant property. And the condition (T^{**}) is a very useful tool in the study of the locally nilpotent space.

THEOREM 2.8. *For $X \in T_{LN}$, X satisfies the condition* (T^{**}).

PROOF. Assume that $g \in [g, \pi_1(X)]$ for some $g(\neq 1) \in \pi_1(X)$. Then $g^{-}1 \in [g, \pi_1(X)]$ and $1 \in g[g, \pi_1(X)]$. Thus $g[g, \pi_1(X)] \cap 1[1, \pi_1(X)] \neq \phi$. Since X satisfies the condition (T^*) by Lemma 2.6, $g[g, \pi_1(X)] = 1$.

But $g(\neq 1) \in g[g, \pi_1(X)]$. Thus we have a contradiction.

LEMMA 2.9 [14, THEOREM 2.1]. *Let $F \xrightarrow{j} E \to B$ be a fibration with F a finitely dominated complex and B a finite complex. Then E is a finitely dominated complex and $w(E) = j_* w(F)\chi(B)$, where $j_* : K_0(Z\pi_1(F)) \to K_0(Z\pi_1(E))$ is a group homomorphisim and χ means the Euler characteristic.*

LEMMA 2.10 [16, 10, THEOREM 3]. *Let $F \to E \xrightarrow{p} B$ be a fibration under the condition that $\pi_1(B)$ acts nilpotently on the homology of the fiber F. B and F are dominated by a finite CW-complex then $p_* w(E) = w(B)\chi(F)$ where $p_* : K_0(Z\pi_1(E)) \to K_0(Z\pi_1(B))$ is the group homomorphisim.*

If a space X is nilpotent, then $\pi_1(X)$ is also nilpotent and for all $i \geq 0$ the $\pi_1(X)$-modules $H_i(\tilde{X}, Z)$ are nilpotent. Next, suppose that $\pi_1(X)$ is nilpotent and operates nilpotently on $H_i(\tilde{X})$ for all i, then we have the followings [14]: there exists

a Cartan-Whitehead decomposition of X:

$$\cdots \to X(m) \to X(m-1) \to \cdots \to X(2) = \tilde{X} \to X, \text{ such that}$$

(1) the fibrations

$$K(\pi_m X, m-1) \to X(m+1) \to X(m),$$

where $X(m)$ is $(m-1)$ connected, K means the Eilenberg-Maclane space

(2) $\pi_m X \cong H_m(X(m))$ for $m \geq 2$.

Assume inductively that $\pi_1(X)$ operates nilpotently on $H_i(X(m))$ for all i and all m with $2 \leq m \leq M$. Then $\pi_1(X)$ operates nilpotently on $H_j(K(\pi_M(X), M-1))$. The Serre spectral sequence associated to the fibration

$$K(\pi_M(X), M-1) \to X(M+1) \to X(M)$$

has an E^2-term

$$E_{ij}^2 = H_i(X(M); H_j(K(\pi_M(X), M-1))$$

which is a nilpotent $\pi_1(X)$-module for every pair (i, j). Hence $\pi_1(X)$ operates nilpotently on $H_k(X(M+1))$ for all k and $\pi_1(X)$ operates nilpotently on $\pi_{M+1}(X) \cong H_{M+1}(X(M+1))$. With the facts above, the below lemma is followed.

LEMMA 2.11 [11, PROPOSITION 2.1]. *A space X is nilpotent if and only if $\pi_1(X)$ is nilpotent and for all $i \geq 0$ the $\pi_1(X)$-modules $H_i(\tilde{X} : Z)$ are nilpotent.*

3. Main Theorems

In this section, we make several results on the Wall invariant of the space satisfying condition (T^{**}).

LEMMA 3.1 [3, THEOREM]. *If $\pi_1(X)$ contains a torsion free nontrivial normal abelian subgroup which acts nilpotently on $H_*(\tilde{X})$ then Euler characteristic $\chi(X) = 0$, where X is a finite complex.*

We know the following; if $\pi_1(X)$ is a nilpotent group then there exist finite upper central series of $\pi_1(X)$ by virtue of the center of $\pi_1(X)$.

THEOREM 3.2. *For X satisfying condition (T^{**}) with $\pi_1(X)$ finite, and the action $\pi_1(X) \times H_n(\tilde{X}) \to H_n(\tilde{X})$ is nilpotent for all $n \geq 0$, then $X \in T_N$.*

PROOF. We only prove that $\pi_1(X)$ is a nilpotent group under the above hypothesis. So assume that $\pi_1(X)$ is not nilpotent, then we don't have finite upper central series of $\pi_1(X)$. If $Z_n(\pi_1(X))$ denote the n-th center of $\pi_1(X)$, we can find an integer n such that $Z_{n+1}(\pi_1(X)) = Z_n(\pi_1(X)) \subsetneq \pi_1(X)$. It follows that if $x \notin Z_n(\pi_1(X))$, then $[x, \pi_1(X)] \nsubseteq Z_n(\pi_1(X))$. Choose any $x_1 \notin Z_n(\pi_1(X))$, we know $[x_1, \pi_1(X)] \nsubseteq Z_n(\pi_1(X))$ by above. If $x_1 \in [x_1, \pi_1(X)]$, then we have shown that the condition (T^{**}) does not hold, as required, so assume $x_1 \notin [x_1, \pi_1(X)]$. Then choose $x_2 \in [x_1, \pi_1(X)]$, $x_2 \notin Z_n(\pi_1(X))$. Since $[x_1, \pi_1(X)]$ is a normal subgroup of $\pi_1(X)$, $[x_2, \pi_1(X)] \subseteq [x_1, \pi_1(X)]$. If $x_2 \in [x_2, \pi_1(X)]$, we are done.

Otherwise, we have $[x_2, \pi_1(X)] \subsetneq [x_1, \pi_1(X)]$ but also we noted $[x_2, \pi_1(X)] \nsubseteq Z_n(\pi_1(X))$. So pick $x_3 \in [x_2, \pi_1(X)]$, $x_3 \notin Z_n(\pi_1(X))$ and continue. Since $\pi_1(X)$ is finite, this process must stop. After all we have α for which $x_\alpha(\neq 1) \in [x_\alpha, \pi_1(X)]$. This is a contradiction to the fact that X satisfies the condition (T^{**}). Thus we know that $\pi_1(X)$ is a nilpotent group. Next, by Lemma 2.11, we get $X \in T_N$.

Recall that for a nilpotent space X if $\pi_i(X)$ is finitely generated for $i \geq 0$, we say that X is of finite type.

THEOREM 3.3. *Let $F \to E \xrightarrow{p} B$ be a fibration with F a finitely dominated space. If B is a finite space satisfying condition (T^{**}), the action $\pi_1(B) \times H_n(\tilde{B}) \to H_n(\tilde{B})$ is nilpotent for all $n \geq 0$ and $\pi_1(B)(\neq 0)$ is finite then $\omega(E) = 0$.*

PROOF. By Theorem 3.2, $B \in T_N$. Thus $\chi(B) = 0$, because $\pi_1(B) \neq 0$. By Lemma 2.9, our proof is completed.

We recall that a group G satisfies the maximal condition if it has no infinite strictly increasing chain of subgroups [16].

THEOREM 3.4. *Let $F \to E \xrightarrow{p} B$ be a fibration with F a finitely dominated space. For finite $B(\in T_{LN})$ if $\pi_1(B)$ is finite and $\pi_1(B) \neq 0$ or $\pi_1(B)$ is infinite with the maximal condition on a normal subgroup of $\pi_1(B)$ then $\omega(E) = 0$.*

PROOF. When $\pi_1(B)$ is infinite under the above hypothesis, $\pi_1(B)$ is finitely generated nilpotent group. Then $\pi_1(B)$ has the infinite normal abelian center group of $\pi_1(B)$ which acts nilpotently on $H_*\tilde{B}, * \geq 0$. Then by Lemma 3.1, our proof is completed.

Next, when $\pi_1(B)$ is finite, by the similar method of Theorem 3.3 and Theorem 2.8, our proof is completed.

COROLLARY 3.5. *Let $F \to E \xrightarrow{p} B$ be a fibration with F a finitely dominated space. If B is finite nilpotent space and $\pi_1(B) \neq 0$ then $\omega(E) = 0$.*

THEOREM 3.6. *Let $F \to E \xrightarrow{p} B$ be a fibration under the following conditions that B is a finitely dominated space and $\pi_1(B)$ acts nilpotently on the homology of the fiber F. If F is a finite complex, such that $\pi_1(F)$ is nontrivial, E is a space satisfying condition (T^{**}), and the action $\pi_1(E) \times H_n(\tilde{E}) \to H_n(\tilde{E})$ is nilpotent where $n \geq 0$, with $\pi_1(E)(\neq 0)$ finite, then $\omega(E) \in Ker\, p_*$, where $p_* : K_0(Z\pi_1(E)) \to K_0(Z\pi_1(B))$.*

PROOF. Since E is a nilpotent space by Lemma 2.11 and Theorem 3.2, the fiber F is also a nilpotent space [1]. From the fact that F is a finite complex, $\pi_1(F)$ is finitely generated. By Lemma 2.11 $\pi_1(F)$ acts nilpotently on $H_i(\tilde{F})$. Now we consider the Euler characteristic of F. By the similar method of proof Theorem 3.4, if $\pi_1(F)$ is infinite, we get the infinite center subgroup of $\pi_1(F)$ which acts nilpotently on $H_n(\tilde{F})$. By Lemma 3.1, $\chi(F) = 0$.

Next, if $\pi_1(F)$ is finite we know that $\chi(F) = \chi(\tilde{F})$ and $\chi(\tilde{F}) = |\pi_1(F)|\chi(F)$ where $|\ |$ means the order of $\pi_1(F)$ [11]. Since $\pi_1(F) \neq 0$, $\chi(F) = 0$. In two cases of $\pi_1(F)$ above, $\chi(F) = 0$. By Lemma 2.10, $\omega(E) \in Ker\, p_*$.

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