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JACOBI OPERATORS ON REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

By

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Abstract. In this paper, we investigate a real hypersurface of a complex projective space CP^n in terms of the Jacobi operators. We give a local structure theorem of a real hypersurface of CP^n satisfying $R_\xi = k(I - \eta \otimes \xi)$, where $R_\xi = R(\cdot, \xi)\xi$ is the Jacobi operator with respect to ξ and k is a function. Further, we classify real hypersurfaces of CP^n satisfying $\phi R_\xi = R_\xi \phi$ under the condition that $A\xi$ is a principal curvature vector. Also, we show that a complex projective space does not admit a locally symmetric real hypersurface.

0. Introduction

Let $CP^n = (CP^n, J, \tilde{g})$ be an n-dimensional complex projective space with Fubini-Study metric $\tilde{g}$ of constant holomorphic sectional curvature 4, and let M be an orientable real hypersurface of CP^n and N be a unit normal vector on M. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kählerian structure $(J, \tilde{g})$ of CP^n (see section 1). One of the typical examples of M is a geodesic hypersphere. R. Takagi ([10]) classified homogeneous real hypersurfaces of CP^n by means of six model spaces of type A_1, A_2, B, C, D, and E, further he explicitly write down their principal curvatures and multiplicities in the table in [11]. T. E. Cecil and P. J. Ryan ([2]) extensively investigated a real hypersurface which is realized as a tube of constant radius r over a complex submanifold of CP^n on which ξ is a principal curvature vector with principal

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curvature $\alpha = 2\cot 2r$ and the corresponding focal map $\varphi_r : M \to CP^n$ (defined by $\varphi_r(p) = exp_p(rN)$) has constant rank.

We remark that, in particular, a homogeneous real hypersurface of type A_1, A_2 has a lot of nice geometric properties. For example, M. Okumura ([9]) showed that a real hypersurface of CP^n is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 if and only if the structure vector field ξ is Killing or if and only if the structure tensor ϕ commutes with the shape operator A ($\phi A = A\phi$).

We denote by ∇ the Levi-Civita connection with respect to g. The curvature tensor field R on M is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, where X and Y are vector fields on M. We define the Jacobi operator field $R_X = R(\cdot, X)X$ with respect to a unit vector field X. Then we see that R_X is a self-adjoint endomorphism of the tangent space. It is related with the Jacobi vector fields, which are solutions of the second order differential equation (the Jacobi equation) $\nabla_{\dot\gamma}(\nabla_{\dot\gamma} Y) + R(Y, \dot\gamma)\dot\gamma = 0$ along a geodesic γ. It is well-known that the notion of Jacobi vector fields involve many important geometric properties. In section 2, particularly we show that the Jacobi operator R_ξ with respect to the structure vector field ξ of a geodesic hypersphere is represented by $R_\xi = k(I - \eta \otimes \xi)$ where I denotes the identity transformation and k is a constant. Further, we give a local structure theorem of a real hypersurface of CP^n satisfying $R_\xi = k(I - \eta \otimes \xi)$ where k is a function.

In section 3, we prove that a real hypersurface of CP^n is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 if and only if the structure tensor ϕ commutes with the Jacobi operator R_ξ ($\phi R_\xi = R_\xi \phi$) and $A\xi$ is a principal curvature vector. In section 4, we give another characterization of homogeneous real hypersurfaces of type A_1, A_2 by the property that $(*)$ the structure vector field ξ is a geodesic vector field and further the Jacobi operator R_ξ is diagonalizable by a parallel orthonormal frame field along each trajectory of ξ and at the same time their eigenvalues are constant along each trajectory of ξ. We easily see that the property $(*)$ is equivalent to the condition $R'_\xi = 0$ where we denote $R'_X = (\nabla_X R)(\cdot, X)X$ for any unit vector field X. Also, in section 4 we show that CP^n does not admit a locally symmetric ($\nabla R = 0$) real hypersurface.

In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

1. Preliminaries

At first, we review the fundamental facts on a real hypersurface of CP^n. Let M be a real hypersurface of CP^n and N be a unit normal vector on M. By $\tilde\nabla$ we denote the Levi-Civita connection with respect to the Fubini-Study metric of CP^n. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$. An eigenvector (resp. eigenvalue) of the shape operator A is called a principal curvature vector (resp. principal curvature). For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi. \tag{1.1}$$

Then we may see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y). \end{aligned} \tag{1.2}$$

From (1.2), we get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{1.3}$$

From the fact $\tilde{\nabla} J = 0$ and (1.1), making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{1.4}$$

$$\nabla_X \xi = \phi AX. \tag{1.5}$$

Since the ambient space is of constant holomorphic sectional curvature 4, we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi. \tag{1.7}$$

We recall the following:

PROPOSITION 1 ([8]). *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

PROPOSITION 2 ([8]). *Assume that ξ is a principal curvature vector and corresponding principal curvature α. If $AX = \lambda X$ for X orthogonal to ξ, then we have $A\phi X = (\alpha\lambda + 2)/(2\lambda - \alpha)\phi X$.*

THEOREM 1 ([9]). *Let M be a real hypersurface of CP^n. Then the followings are equivalent*:

(i) *M is locally congruent to one of homogeneous real hypersurfaces of type A_1 and A_2.*

(ii) $\phi A = A\phi$.

A ruled real hypersurface of CP^n is defined by a foliated one by complex hyperplanes CP^{n-1} and its shape operator is written down in [5]. Namely,

$$
\begin{aligned}
A\xi &= \alpha\xi + \mu W \qquad (\mu \neq 0),\\
AW &= \mu\xi, \\
AZ &= 0
\end{aligned}
\tag{1.8}
$$

for any $Z \perp \xi$, W, where W is unit vector orthogonal to ξ, α and μ are functions on M. For more details about a ruled real hypersurface of CP^n, we refer to [6]. The ϕ-holomorphic sectional curvature $H(X)$ is defined by a sectional curvature of $span\{X, \phi X\}$. Further in [5], it was proved that

THEOREM 2. *Let M be a real hypersurface of CP^n $(n \geq 3)$ with constant ϕ-holomorphic sectional curvature. Then M is locally congruent to the following spaces*:

(1) *a geodesic hypersphere (that is, a homogeneous real hypersurface which lies on a tube of radius r over a hyperplane CP^{n-1}, where $0 < r < (\pi/2)$)*;

(2) *a ruled real hypersurface*;

(3) *a real hypersurface on which there is a foliation of codimension two such that each leaf of the foliation is contained in some complex hyperplane CP^{n-1} as a ruled hypersurface.*

We denote $\alpha = g(A\xi, \xi)$ and $\beta = g(A^2\xi, \xi)$. We define a vector field U on M by $U = \nabla_\xi \xi$. Then from (1.2) and (1.5) we easily observe that U is orthogonal to ξ and also to $A\xi$. Since $\|U\|^2 = g(U, U) = \beta - \alpha^2$, from (1.2), (1.5) and (1.9) we have at once

LEMMA 1. *The followings are equivalent*:

(i) *ξ is a geodesic vector field.*

(ii) *ξ is a principal curvature vector field.*

(iii) $\beta - \alpha^2 = 0$.

2. Real hypersurfaces of CP^n satisfying $R_\xi = k(I - \eta \otimes \xi)$

For each point $p \in M$ and each unit tangent vector $X \in T_pM$, we define a self-adjoint operator R_X of T_pM by $R_X = R(\cdot, X)X$. We call R_X Jacobi operator with respect to X. It is well-known that a geodesic hypersphere M of CP^n is η-umbilical, i.e., M satisfies $AX = aX + b\eta(X)\xi$ for any tangent vector field X on M, where a and b are constants on M (cf. [11]). Thus from (1.6) we have

PROPOSITION 3. *Let M be a geodesic hypersphere of CP^n. Then M satisfies $R_\xi = k(I - \eta \otimes \xi)$ where I denotes the identity transformation and k is a constant on M.*

Furthermore, we prove

THEOREM 3. *Let M be a real hypersurface of CP^n $(n \geq 3)$. Suppose that M satisfies $R_\xi = k(I - \eta \otimes \xi)$, where k is a function on M.*

In case that ξ is a principal curvature vector field with the associated principal curvature $\alpha = 2\cot 2r$ and the rank of corresponding focal map φ_r is constant, then M is locally congruent to one of the following spaces:

(1) *a geodesic hypersphere;*

(2) *a homogeneous tube of radius $\pi/4$ over a totally geodesic CP^l $(1 \leq l \leq n-2)$;*

(3) *a non-homogeneous tube of radius $\pi/4$ over a $\varphi_{\pi/4}(M)$ with non-zero principal curvatures $\neq \pm 1$.*

Or in case that ξ is not a principal curvature vector field, then M is locally congruent to

(4) *a non-homogeneous real hypersurface whose shape operator A is written as*

$$A\xi = \alpha\xi + \mu W \qquad (\alpha \neq 0, \mu \neq 0),$$

$$AW = \mu\xi + \nu W,$$

$$AZ = (k-1)/\alpha Z, \qquad k = 1 + \alpha\nu - \mu^2$$

for any $Z \perp \xi$, W, where W is a unit vector field orthogonal to ξ, α, μ and ν are functions on M.

PROOF. From (1.6) it follows that

$$R_\xi X = R(X, \xi)\xi = X - \eta(X)\xi + \alpha AX - \eta(AX)A\xi \tag{2.1}$$

for any vector field X on M. Suppose that $R_\xi X = k(X - \eta(X)\xi)$, where k is a function. Then from (2.1) we get

$$\alpha AX = (k-1)(X - \eta(X)\xi) + \eta(AX)A\xi \tag{2.2}$$

for any vector field X on M.

First we consider the case that ξ is a principal curvature vector field, that is, $A\xi = \alpha\xi$. Then from (2.2) we get

$$\alpha AX = (k-1)X \tag{2.3}$$

for any vector field X orthogonal to ξ. Since α is constant (Proposition 1), we divide our arguments into two cases: (i) $\alpha = 0$ or (ii) $\alpha \neq 0$.

(i) $\alpha = 0$. First, from (2.1) we see that $k = 1$ and M satisfies $R_\xi X = X - \eta(X)\xi$. Since the rank of the corresponding focal map $\varphi_{\pi/4}$ is constant, by virtue of [2] we see that M is locally congruent to a homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic $\boldsymbol{C}P^k$ $(1 \leq k \leq n-1)$ or locally congruent to a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kähler submanifold $\tilde{N}$ with non-zero principal curvatures $\neq \pm 1$. (See also [7]).

(ii) $\alpha \neq 0$. From (2.3) we see that M has at most two distinct principal curvatures. Hence Theorem 3 in [2] implies that M is locally congruent to geodesic hypersphere.

Next, we consider the case that ξ is not principal. We may assume that

$$A\xi = \alpha\xi + \mu W, \tag{2.4}$$

where W is unit and orthogonal to ξ, $\mu \neq 0$. Then from (2.2) and (2.4) we get

$$\alpha AW = \alpha\mu\xi + (\mu^2 + k - 1)W. \tag{2.5}$$

Also, from (2.2) we get

$$\alpha AZ = (k-1)Z, \tag{2.6}$$

where Z is unit and orthogonal to ξ and W. Now we prove that $\alpha \neq 0$. If $\alpha = 0$ on M, then from (2.2) we get

$$(k-1)(X - \eta(X)\xi) + \eta(AX)A\xi = 0 \tag{2.7}$$

for any vector field X on M. Putting $X = U$ in (2.7), then we obtain

$$(k-1)U = 0. \tag{2.8}$$

Set $\Omega_1 = \{p \in M : k(p) = 1\}$ and $\Omega_2 = \{p \in M : k(p) \neq 1\}$. Then $M = \Omega_1 \cup \Omega_2$. If $M = \Omega_1$ (Ω_2 is empty), then from (2.7) we find $\beta = 0$ on M, and hence by applying Lemma 1 we have $A\xi = 0$ on M. If $M = \Omega_2$ (Ω_1 is empty), then from (2.8) and Lemma 1 we see that $A\xi = \alpha\xi$ on M. Or, in case that both Ω_1 and Ω_2 are non-empty, by Proposition 1 and the continuity of α yield that ξ is a principal curvature vector field on M. Hence all the cases yield a contradiction. Thus $\alpha \neq 0$ on M.

Therefore from (2.4), (2.5) and (2.6) we have our real hypersurface M of the case (4), where we have put $\mu^2 + k - 1 = \alpha\nu$. From (1.6) we can easily see that a real hypersurface CP^n whose shape operator written as (2.4), (2.5) and (2.6) satisfies $R_\xi X = k(X - \eta(X)\xi)$ for any vector field X on M. Also, since the structure vector field ξ on a homogeneous real hypersurface of CP^n is a principal curvature vector field ([10]), we see that a real hypersurface of this case is non-homogeneous. (Q.E.D.)

REMARK 1. From Theorem 3 and the table in [11], we see that the normal Jacobi vector field along each geodesic trajectory of ξ on a geodesic hypersphere, a homogeneous tube of radius $\pi/4$ over a totally geodesic CP^l ($1 \leq l \leq n-2$), or a non-homogeneous tube of radius $\pi/4$ of the case (3) in Theorem 3 satisfies the spherical space form type Jacobi equation, i.e., $Y'' + kY = 0$ where k is a positive constant and $'$ denotes covariant derivative along a geodesic trajectory of ξ.

The rank of A at a point p in M is called a type number and is denoted by $t(p)$. Let M be a real hypersurface of CP^n which satisfies $R_\xi X = X - \eta(X)\xi$, i.e., $k = 1$. Then from (2.2) it follows that

$$\alpha AX = \eta(AX)A\xi \tag{2.9}$$

for any vector field X on M. If there exist a point p in M such that $\alpha(p) \neq 0$, then (2.9) implies that the type number $t(p)$ at p is at most 1. It is however seen (cf. [12]) that the point p is geodesic. So it is contradiction to the assumption that $\alpha(p) \neq 0$. Thus $\alpha = 0$ on M, and hence from (2.9) $\beta = 0$ on M. Therefore by Lemma 1, we see that $A\xi = 0$ on M.

REMARK 2. The above arguements together with (1.8) in section 1 and (20) in [7] imply that a non-homogeneous real hypersurface of the case (4) is neither a ruled real hypersurface nor of the case (3) in Theorem 2. But, we do not yet know the construction of the case (4) in Theorem 3.

Due to Theorems 2, 3 and Remark 2, we characterized a geodesic hypersphere of CP^n by following

COROLLARY 1. *Let M be a real hypersurface of CP^n ($n \geq 3$) with constant ϕ-holomorphic sectional curvature. In addition that M satisfies $R_\xi = k(I - \eta \otimes \xi)$, where k is a function, then M is locally congruent to a geodesic hypersphere.*

3. Real hypersurfaces of CP^n satisfying $\phi R_\xi = R_x \phi$

We see that all the cases appeared in Theorem 3 satisfies $\phi R_\xi = R_\xi \phi$. In this section, we prove

THEOREM 4. *Let M be a real hypersurface of CP^n. The structure tensor ϕ commutes with the Jacobi operator R_ξ* and *$A\xi$ is principal curvature vector field on M. Then ξ is principal curvature field on M. Further assume that $\alpha = 2\cot 2r$ and the rank of the focal map φ_r is constant, then M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 or a non-homogeneous tube of radius $\pi/4$ of the case (3) in Theorem 3.*

PROOF. Assume that $\phi R_\xi = R_\xi \phi$ and $A^2\xi = \lambda A\xi$. From (1.6) we get

$$R_\xi(\phi X) = \phi X + \alpha A\phi X + g(X, U)A\xi, \tag{3.1}$$
$$\phi(R_\xi X) = \phi X + \alpha\phi AX - g(AX, \xi)U.$$

From (3.1) and the assumption $\phi R_\xi = R_\xi \phi$, we find

$$\alpha(\phi A - A\phi)X = g(X, U)A\xi + g(AX, \xi)U. \tag{3.2}$$

First, we prove that ξ is principal curvature vector on M. We put $X = A\xi$ in (3.2) and using the another assumption $A^2\xi = \lambda A\xi$, then we get $\alpha AU = (\alpha\lambda - \beta)U$, and hence we have

$$\alpha AU = 0, \tag{3.3}$$

because $\beta = \alpha\lambda$. If there exists a point $p \in M$ such that $\alpha(p) = 0$, then we see that $\beta(p) = 0$, and hence by using Lemma 1, we conclude that $A\xi = 0$ at p. So, from now we discuss where α has not zero. Then from (3.3), it follows that

$$AU = 0. \tag{3.4}$$

With (3.4) we easily see that

$$g((\nabla_X A)\xi, \xi) = d\alpha(X),$$

where d denotes the exterior differential. Since $U = \phi A\xi$, from (1.4), (1.7) and (3.4) we have

$$\nabla_{\xi} U = \alpha A\xi - \beta\xi + \phi\nabla\alpha, \tag{3.5}$$

where $\nabla\alpha$ denotes the gradient vector of α. Differentiating (3.4) covariantly, then by using (1.7) and (3.5) we have

$$(\nabla_U A)\xi = -\phi U - \alpha A^2\xi + \beta A\xi + A\phi\nabla\alpha. \tag{3.6}$$

Also, differentiating $A^2\xi = \lambda A\xi$ covariantly along M, then together with (1.5) we have

$$\begin{aligned} & g(A\xi, (\nabla_X A)Y) + g((\nabla_X A)\xi, AY) + g(\phi AX, A^2 Y) \\ & \quad = d\lambda(X)g(A\xi, Y) + \lambda g((\nabla_X A)\xi, Y) + \lambda g(\phi AX, AY). \end{aligned} \tag{3.7}$$

From (1.7) and (3.7) we have

$$\begin{aligned} & \eta(X)g(A\xi, \phi Y) - \eta(Y)g(A\xi, \phi X) - 2\alpha g(\phi X, Y) \\ & \quad + g((\nabla_X A)\xi, AY) - g((\nabla_Y A)\xi, AX) + g(\phi AX, A^2 Y) - g(\phi AY, A^2 X) \\ & = d\lambda(X)g(A\xi, Y) - d\lambda(Y)g(A\xi, X) + \lambda g((\nabla_X A)\xi, Y) - \lambda g((\nabla_Y A)\xi, X) \\ & \quad + 2\lambda g(\phi AX, AY) \end{aligned}$$

for any vector fields X and Y on M. We put $X = U$ and making use of (1.7), (3.4) and (3.6), then we have

$$g((\nabla_U A)\xi, AY) = 2(\alpha - \lambda)g(\phi U, Y) - \eta(Y)g(U, U) + d\lambda(U)g(A\xi, Y). \tag{3.8}$$

Thus, from (3.6) and (3.8) we obtain

$$\begin{aligned} & 2(\alpha - \lambda)g(\phi U, Y) - \eta(Y)g(U, U) + d\lambda(U)g(A\xi, Y) \\ & \quad = -g(\phi U, AY) - \alpha g(A^2\xi, AY) + \beta g(A\xi, AY) - d\alpha(\phi A^2 Y). \end{aligned} \tag{3.9}$$

Putting $Y = \xi$ in (3.9), then we get

$$d(\lambda\alpha)(U) = 2(\beta - \alpha^2).$$

Also, we put $Y = A\xi$ in (3.9), we get

$$\lambda d(\lambda\alpha)(U) = (\beta - \alpha^2)(3\alpha - \lambda).$$

Thus, we have $\beta - \alpha^2 = \alpha(\lambda - \alpha) = 0$, from which using Lemma 1 we see that $A\xi = \alpha\xi$ on M.

From (3.2) and Lemma 1, we see that

$$\alpha(\phi A - A\phi)X = 0.$$

Since α is constant, by a similar way as in the proof of Theorem 3 and using Theorem 1, we have our assertions. (Q.E.D.)

REMARK 3. If we omit the condition that $A\xi$ is a principal curvature vector, then Theorem 4 is not true. In fact, if a non-homogeneous real hypersurface of the case (4) in Theorem 3 satisfies $A^2\xi = \lambda A\xi$, then we can see that $\alpha\nu - \mu^2 = 0$ where α and μ have not zero, which yields a contradiction.

4. Real hypersurfaces of CP^n satisfying $R'_X = 0$

For each point $p \in M$ and each unit tangent vector $X \in T_pM$, we define R'_X by $R'_X = (\nabla_X R)(\cdot, X)X$. Then, in particular supposing that the structure vector field ξ of M is a geodesic vector field, it is easily seen that $R'_\xi = 0$ on M if and only if the Jacobi operator R_ξ is diagonalizable by a parallel orthonormal frame field along each trajectory of ξ and at the same time their eigenvalues are constant along each trajectory of ξ (cf. [1] or [3]).

Now we prove

PROPOSITION 4. *Let M be a real hypersurface of CP^n. Suppose that ξ is a geodesic vector field on M and M satisfies $R'_\xi = 0$. Then ξ is principal curvature field on M. Further assume that $\alpha = 2\cot 2r$ and the rank of the focal map φ_r is constant, then M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 or a non-homogeneous tube of radius $\pi/4$ of the case* (3) *in Theorem* 3.

PROOF. Assume that ξ is a geodesic vector field on M. Then by Lemma 1, we immediately see that $A\xi = \alpha\xi$. Then from (1.6), taking account of (1.4), (1.7) and Proposition 1, we get

$$\begin{aligned} R'_\xi Y = (\nabla_\xi R)(Y, \xi)\xi = \alpha(\nabla_\xi A)Y \\ = \alpha(\alpha\phi AY - A\phi AY + \phi Y), \end{aligned}$$

for any vector field Y on M. Thus from the hypothesis we get

$$\alpha(\alpha\phi A - A\phi A + \phi)Y = 0.$$

Assume $AY = \lambda Y$ for Y orthogonal to ξ. Then from Proposition 2 we have

$$\alpha(\lambda^2 - \alpha\lambda - 1) = 0.$$

We see that $\lambda^2 - \alpha\lambda - 1 = 0$ implies $\lambda(2\lambda - \alpha) = \alpha\lambda + 2$, that is $\lambda = (\alpha\lambda + 2)/(2\lambda - \alpha)$. From this we also see that $\phi A = A\phi$, and hence from Theorem 1 and by similar arguments as in the proof of Theorem 3 in section 2, we have our assertions. (Q.E.D.)

PROPOSITION 5. *There does not exist a real hypersurface of CP^n whose structure vector field ξ is principal curvature vector field and satisfying $R'_V = 0$ for any vector field V orthogonal to ξ.*

PROOF. From (1.6), taking account of (1.4), we get

$$\begin{aligned}(4.2)\qquad (\nabla_V R)(Y, V)V = &-3\{\eta(Y)g(AV, V)\phi V - g(\phi Y, V)g(AV, V)\xi\}\\ &+ g((\nabla_V A)V, V)AY + g(AV, V)(\nabla_V A)Y\\ &- g((\nabla_V A)Y, V)AV - g(AY, V)(\nabla_V A)V\end{aligned}$$

for any vector field Y on M and any vector field V orthogonal to ξ. Assume that $A\xi = \alpha\xi$ and suppose that M satisfies $R'_V = (\nabla_V R)(\cdot, V)V = 0$ for any vector field V orthogonal to ξ. Then of course $R'_V = (\nabla_V R)(\xi, V)V = 0$ and from (4.2)

$$\begin{aligned}(4.3)\qquad &-3g(AV, V)\phi V + \alpha g((\nabla_V A)V, V)\xi + g(AV, V)(\alpha\phi AV - A\phi AV)\\ &-\alpha g(\phi AV, V)AV + g(A\phi AV, V)AV = 0\end{aligned}$$

for any vector field V orthogonal to ξ. From (4.3) we easily see that $\alpha g((\nabla_V A)V, V)\xi = 0$ and have

$$\begin{aligned}(4.4)\qquad &-3g(AV, V)\phi V + \alpha g((\nabla_V A)V, V)\xi + g(AV, V)(\alpha\phi AV - A\phi AV)\\ &+ g(A\phi AV, V)AV = 0\end{aligned}$$

Assume $AV = \lambda V$ and $g(V, V) = 1$. Then from (4.4) and Proposition 2 we have

$$(4.5)\qquad \lambda\{\alpha\lambda^2 - (8 + \alpha^2)\lambda + 3\alpha\} = 0.$$

From (4.5) and Proposition 1 we see that M has at most four distinct constant principal curvatures including α. But by the table in [11] we see that $\lambda \neq 0$, and (4.5) yield a contradiction. (Q.E.D.)

It is well-known that a locally symmetric space ($\nabla R = 0$) is locally homogeneous. Thus by virtue of R. Takagi's result ([10]) and Proposition 5, we have

COROLLARY 2. *There does not exist a locally symmetric real hypersurface of* CP^n.

REMARK 4. It was proved by the second author ([4]) that there does not exist a real hypersurface M with the parallel Ricci tensor in CP^n, $n \geq 3$.

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