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ON ALMOST KÄHLER MANIFOLDS OF CONSTANT CURVATURE

By

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§1. Introduction

An almost Hermitian manifold $M=(M, J, g)$ is called an almost Kähler manifold if the corresponding Kähler form is closed (or equivalently $\mathfrak{S}_{X,Y,Z}g((\nabla_X J)Y, Z)=0$ for $X, Y, Z \in \mathfrak{X}(M)$, where $\mathfrak{S}$ and $\mathfrak{X}(M)$ denotes the cyclic sum and the Lie algebra of all differentiable vector fields on M respectively). A Kähler manifold, which is defined by $\nabla J=0$, is necessarily an almost Kähler manifold. It is well-known that an almost Kähler manifold with integrable almost complex structure is a Kähler manifold. A non-Kähler almost Kähler manifold is called a strictly almost Kähler manifold. Concerning the integrability of almost Kähler manifolds, the following conjecture by S. I. Goldberg is known ([2]):

CONJECTURE. *A compact almost Kähler Einstein manifold is a Kähler manifold.*

K. Sekigawa proved the above conjecture is true for the case where the scalar curvature is nonnegative ([7]). However, the above conjecture is still open in the case where the scalar curvature is negative.

Concerning the above conjecture, Z. Olszak proved that, in dimensions ≥ 8, an almost Kähler manifold of constant curvature is a flat Kähler manifold ([6]). In dimension 4, D. E. Blair claimed that the same assertion is valid by making use of quaternionic analysis. However, there is a gap in the final step of his proof. The statement "each $a_i=0$" is not correct ([1], p. 1038). Recently, K. Sekigawa and the author proved that a $2n(\geq 4)$-dimensional complete almost Kähler manifold of constant sectional curvature is a flat Kähler manifold ([5]). The proof in [5] is essentially dependent on the completeness.

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The aim of the present paper is to prove that the hypothesis of completeness in the above result is needless, namely we prove the following.

THEOREM. *In dimensions* ≥ 4, *there are no almost Kähler manifolds of constant curvature unless the constant is* 0, *in which case the manifold is Kählerian.*

In dimensions ≥ 8, above Theorem is nothing but the result of Z. Olszak, but we shall give a proof which does not depend on the dimension.

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§ 2. Preliminaries

Let $M = (M, J, g)$ be a $2n$-dimensional almost Kähler manifold. We denote by ∇ and R the Riemannian connection and the curvature tensor of M with respect to the Riemannian metric g. Here, we assume that the curvature tensor R is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for $X, Y \in \mathfrak{X}(M)$. Further, we assume that M is oriented by the volume form $dM = (-1)^n \Omega^n / n!$, where Ω is the Kähler form defined by $\Omega(X, Y) = g(X, JX)$ for $X, Y \in \mathfrak{X}(M)$. We recall a curvature identity for almost Kähler manifold due to A. Gray ([3]):

$$\begin{aligned}(2.1)\quad & R(w,x,y,z) - R(w,x,Jy,Jz) - R(Jw,Jx,y,z) + R(Jw,Jx,Jy,Jz) \\ & \quad + R(Jw,x,Jy,z) - R(Jw,x,Jz,y) - R(Jx,w,Jy,z) + R(Jx,w,Jz,y) \\ & = 2g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)_z - (\nabla_z J)y)\end{aligned}$$

for $w, x, y, z \in T_pM, p \in M$. If M is also a space of constant curvature c, then the equality (2.1) becomes

$$\begin{aligned} & 2c\{g(x,y)g(w,z) - g(x,z)g(w,y) - g(x,Jy)g(w,Jz) + g(x,Jz)g(w,Jy)\} \\ & \quad = g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)z - (\nabla_z J)y)\end{aligned}$$

and hence, we have

$$(2.2)\qquad \|\nabla J\|^2 = -8cn(n-1).$$

Since we may assume that $n \geq 2$, this implies that $c \leq 0$ and that $c = 0$ if and only if M is a flat Kähler manifold.

In the present paper, unless otherwise specified, we assume that all manifolds are connected and of class C^∞ and that all tensor fields are of class C^∞.

§ 3. Proof of the theorem

If there exists a strictly almost Kähler structure on a space of constant curvature, then we have, in the view of the argument in section 2, that locally hyperbolic space must carry such a structure. We denote by $\boldsymbol{H}^{2n}$ the $2n$-dimensional hyperbolic space of constant curvature -1. As a model of $\boldsymbol{H}^{2n}$, we take the upper half space $\boldsymbol{R}_+^{2n} = \{(x_1, \dots, x_{2n}) \in \boldsymbol{R}^{2n} | x_1 > 0\}$ of $\boldsymbol{R}^{2n}$ and the metric g given by

$$g = \frac{1}{x_1^2} \sum_{i=1}^{2n} dx_i \otimes dx_i.$$

Let $\{X_i = x_1(\partial/\partial x_i)\}_{i=1,\dots,2n}$ be a global orthonormal frame field. Then

$$[X_1, X_i] = -[X_i, X_1] = X_i \qquad \text{for} \quad i = 2, \dots, 2n,$$

and are otherwise zero. If we put $\Gamma_{ijk} = g(\nabla_{X_i} X_j, X_k)$, then

$$\Gamma_{ii1} = -\Gamma_{i1i} = 1 \qquad \text{for} \quad i = 2, \dots, 2n, \tag{3.1}$$

and are otherwise zero.

Now, we assume that there exists a compatible almost Kähler structure (J, g) on a connected open neighborhood U of a point $p \in \boldsymbol{H}^{2n}$. If we put $J_{ij} = g(JX_i, X_j)$, then

$$J_{ij} = -J_{ji}, \qquad \sum_{u=1}^{2n} J_{iu} J_{ju} = \delta_{ij}.$$

We can choose isometries $\phi_{(1)}, \dots, \phi_{(2n)}$ of a neighborhood of p in U such that

(1) $\phi_{(a)}(p) = p$ for $a = 1, \dots, 2n$;

(2) $(\phi_{(1)})_{*p}$ is the identity mapping of the tangent space at p;

(3) $(\phi_{(a)})_{*p}(X_1) = (X_a)_p, (\phi_{(a)})_{*p}(X_a) = (X_1)_p$ and $(\phi_{(a)})_{*p}(X_i) = (X_i)_p (i \neq 1, a)$ for $a = 2, \dots, 2n$.

We note that $\phi_{(1)}$ is the identity mapping. For brevity, we shall write $\phi_{(a)}$ instead of $(\phi_{(a)})_*$. We put $\phi_{(a)}(X_i) = \sum_{j=1}^{2n} B_{ij}^{(a)} X_j$. Then, from (2) and (3), we have

$$(B_{ij}^{(1)}(p)) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

and

$$(B^{(a)}_{ij}(p)) = \begin{array}{cc} & \begin{array}{cccccccc} & & & & (a) & & & \end{array} \\ \begin{array}{c} \\ \\ \\ \\ (a) \\ \\ \\ \\ \end{array} & \begin{pmatrix} 0 & & & & 1 & & & \\ & 1 & & & 0 & & & \\ & & \ddots & & \vdots & & & \\ & & & 1 & 0 & & & \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ & & & & 0 & 1 & & \\ & & & & \vdots & & \ddots & \\ & & & & 0 & & & 1 \end{pmatrix} \end{array} \quad \text{for } a = 2, \dots, 2n.$$

Thus, it is easy to verify that

$$B^{(a)}_{11}(p) B^{(a)}_{1k}(p) = B^{(a)}_{11}(p) B^{(a)}_{k1}(p) = \delta^a_1 \delta_{1k}. \tag{3.2}$$

Since $\phi_{(a)}(\nabla_{X_i} X_j) = \nabla_{\phi_{(a)}(X_i)} \phi_{(a)}(X_j)$, we have

$$\sum_{u=1}^{2n} \Gamma_{iju} B^{(a)}_{uk} = \sum_{u=1}^{2n} B^{(a)}_{iu} (X_u B^{(a)}_{jk}) + \sum_{u,v=1}^{2n} B^{(a)}_{iu} B^{(a)}_{jv} \Gamma_{uvk},$$

and hence

$$X_i B^{(a)}_{jk} = \sum_{u,v=1}^{2n} \Gamma_{ujv} B^{(a)}_{ui} B^{(a)}_{vk} - \sum_{u=1}^{2n} \Gamma_{iuk} B^{(a)}_{ju}. \tag{3.3}$$

Thus, from (3.1) $\sim$ (3.3), we have

$$\begin{aligned} &(X_1 B^{(a)}_{jk})(p) = B^{(a)}_{j1}(p) B^{(a)}_{1k}(p) \qquad \text{for} \quad j, k \geq 2, \\ &(X_1 B^{(a)}_{1k})(p) = (X_1 B^{(a)}_{k1})(p) = 0 \qquad \text{for} \quad k \geq 2, \\ &(X_1 B^{(a)}_{11})(p) = (B^{(a)}_{11})^2(p) - 1 = \delta^a_1 - 1. \end{aligned} \tag{3.4}$$

We put $J^{(a)} = \phi^{-1}_{(a)} \circ J \circ \phi_{(a)}$ and $J^{(a)}_{ij} = g(J^{(a)} X_i, X_j)$. We note that each $(J^{(a)}, g)$ is also a compatible almost Kähler structure. It is obvious that

$$J^{(a)}_{ij} = \sum_{u,v=1}^{2n} B^{(a)}_{iu} B^{(a)}_{jv} J_{uv}.$$

Thus, we have

$$X_1 J^{(a)}_{ij} = \sum_{u,v=1}^{2n} (X_1 B^{(a)}_{iu}) B^{(a)}_{jv} J_{uv} + \sum_{u,v=1}^{2n} B^{(a)}_{iu} (X_1 B^{(a)}_{jv}) J_{uv} + \sum_{u,v=1}^{2n} B^{(a)}_{iu} B^{(a)}_{jv} X_1 J_{uv}.$$

From this equality, by direct calculation, we have

$$
\begin{aligned}
(3.5)\qquad \sum_{i,j=1}^{2n}(X_1J_{ij}^{(a)})^2 &= \sum_{i,j=1}^{2n}(X_1J_{ij})^2 + 2\sum_{i,u=1}^{2n}(X_1B_{iu}^{(a)})^2 \\
&\quad + 2\sum_{i,j,u,u',v,v'=1}^{2n}(X_1B_{iu}^{(a)})(X_1B_{jv'}^{(a)})B_{jv}^{(a)}B_{iu'}^{(a)}J_{uv}J_{u'v'} \\
&\quad + 4\sum_{j,u,v,v'=1}^{2n}(X_1B_{jv}^{(a)})(X_1J_{uv'})B_{jv'}^{(a)}J_{uv}.
\end{aligned}
$$

From (3.4), we have

$$\sum_{i,u=1}^{2n}(X_1B_{iu}^{(a)})^2(p) = (X_1B_{11}^{(a)})^2(p) + \sum_{i,u\geq 2}(X_1B_{iu}^{(a)})^2(p) = 2(\delta_1^a - 1)^2.$$

Now, we set

$$\xi^{(a)} = \sum_{u,v=1}^{2n} B_{1u}^{(a)}J_{1u}B_{1v}^{(a)}J_{1v}, \qquad \eta^{(a)} = \sum_{u,v=1}^{2n}(X_1J_{1u})B_{1v}^{(a)}J_{uv}.$$

Then, from (3.2) and (3.4), by direct calculation, we can derive

$$
\begin{aligned}
&\sum_{i,j,u,u',v,v'=1}^{2n}(X_1B_{iu}^{(a)})(p)(X_1B_{jv'}^{(a)})(p)B_{jv}^{(a)}(p)B_{iu'}^{(a)}(p)J_{uv}(p)J_{u'v'}(p) \\
&= \sum_{u',v=1}^{2n}(X_1B_{11}^{(a)})(p)(X_1B_{11}^{(a)})(p)B_{1v}^{(a)}(p)B_{1u'}^{(a)}(p)J_{1v}(p)J_{u'1}(p) \\
&\quad + \sum_{u',v=1}^{2n}\sum_{j,v'\geq 2}(X_1B_{11}^{(a)})(p)(X_1B_{jv'}^{(a)})(p)B_{jv}^{(a)}(p)B_{1u'}^{(a)}(p)J_{1v}(p)J_{u'v'}(p) \\
&\quad + \sum_{u',v=1}^{2n}\sum_{i,u\geq 2}(X_1B_{iu}^{(a)})(p)(X_1B_{11}^{(a)})(p)B_{1v}^{(a)}(p)B_{iu'}^{(a)}(p)J_{uv}(p)J_{u'1}(p) \\
&\quad + \sum_{u',v=1}^{2n}\sum_{i,j,u,v'\geq 2}(X_1B_{iu}^{(a)})(p)(X_1B_{jv'}^{(a)})(p)B_{jv}^{(a)}(p)B_{iu'}^{(a)}(p)J_{uv}(p)J_{u'v'}(p) \\
&= -(\delta_1^a - 1)^2\xi^{(a)}(p) + (1-\delta_1^a)^2\sum_{u,v'\geq 2}B_{1u}^{(a)}(p)B_{1v'}^{(a)}(p)J_{u1}(p)J_{1v'}(p) \\
&= -2(\delta_1^a - 1)^2\xi^{(a)}(p),
\end{aligned}
$$

and

$$\sum_{j,u,v,v'=1}^{2n}(X_1B_{jv}^{(a)})(p)(X_1J_{uv'})(p)B_{jv'}^{(a)}(p)J_{uv}(p)$$
$$=\sum_{u,v'=1}^{2n}(X_1B_{11}^{(a)})(p)(X_1J_{uv'})(p)B_{1v'}^{(a)}(p)J_{u1}(p)$$
$$+\sum_{u,v'=1}^{2n}\sum_{j,v\geq 2}(X_1B_{jv}^{(a)})(p)(X_1J_{uv'})(p)B_{jv'}^{(a)}(p)J_{uv}(p)$$
$$=(\delta_1^a-1)\eta^{(a)}(p)+(1-\delta_1^a)\sum_{u=1}^{2n}\sum_{v\geq 2}(X_1J_{u1})(p)B_{1v}^{(a)}(p)J_{uv}(p)$$
$$=2(\delta_1^a-1)\eta^{(a)}(p).$$

Therefore, from (3.5), we obtain

$$\sum_{i,j=1}^{2n}(\nabla_1J_{ij}^{(a)})^2(p)=\sum_{i,j=1}^{2n}(\nabla_1J_{ij})^2(p)+4(\delta_1^a-1)^2 \tag{3.6}$$
$$-4(\delta_1^a-1)^2\xi^{(a)}(p)+8(\delta_1^a-1)\eta^{(a)}(p).$$

Thus, taking account of

$$\xi^{(a)}(p)=(J_{1a})^2(p), \qquad \eta^{(a)}(p)=-\sum_u(\nabla_1J_{1u})(p)J_{au}(p),$$

we have

$$\sum_{a,i,j=1}^{2n}(\nabla_1J_{ij}^{(a)})^2(p)=2n\sum_{i,j=1}^{2n}(\nabla_1J_{ij})^2(p)+8(n-1) \tag{3.7}$$
$$+8\sum_{a,u=1}^{2n}(\nabla_1J_{1u})(p)J_{au}(p).$$

Since

$$g((\nabla_XJ^{(a)})Y,Z)=g((\nabla_{\phi_{(a)}(X)}J)\phi_{(a)}(Y),\phi_{(a)}(Z)), \tag{3.8}$$

we have

$$\sum_{i,j}(\nabla_1J_{ij}^{(a)})^2(p)=\sum_{i,j}(\nabla_aJ_{ij})^2(p).$$

Thus, (3.7) becomes

$$\|\nabla J\|^2(p) = 2n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij})^2(p) + 8(n-1) + 8 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u})(p) J_{au}(p).$$

Therefore, from (2.2), we have

$$n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij})^2(p) = 4(n-1)^2 - 4 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u})(p) J_{au}(p). \tag{3.9}$$

Since the above argument does not depend on the choice of the almost complex structure J, corresponding to (3.9), we can obtain

$$n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij}^{(c)})^2(p) = 4(n-1)^2 - 4 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u}^{(c)})(p) J_{au}^{(c)}(p). \tag{3.10}$$

for $c = 2, \ldots, 2n$. Therefore, from (3.9) and (3.10), we have

$$n \sum_{c,i,j=1}^{2n} (\nabla_1 J_{ij}^{(c)})^2(p) = 8(n-1)^2 - 4 \sum_{c,a,u=1}^{2n} (\nabla_1 J_{1u}^{(c)})(p) J_{au}^{(c)}(p).$$

Again from (3.8), the above equality becomes

$$n \sum_{c,i,j=1}^{2n} (\nabla_c J_{ij})^2(p) = 8n(n-1)^2 - 4 \sum_{c,a,u=1}^{2n} (\nabla_c J_{cu})(p) J_{au}(p),$$

namely,

$$n\|\nabla J\|^2(p) = 8n(n-1)^2 - 4 \sum_{c,a,u} (\nabla_c J_{cu})(p) J_{au}(p). \tag{3.11}$$

Since an almost Kähler manifold is necessarily a semi-Kähler manifold, the second term in the right-hand-side of (3.11) must vanish. Therefore, (3.11) yields

$$\|\nabla J\|^2(p) = 8(n-1)^2.$$

Hence, from (2.2), we have

$$8n(n-1) = 8(n-1)^2.$$

This implies $n = 1$.

Therefore, we have finally our Theorem.

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