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REALIZATIONS OF INVOLUTIVE AUTOMORPHISMS σ AND G^{σ} OF EXCEPTIONAL LINEAR LIE GROUPS G, PART I, $G=G_2$, F_4 AND E_6

By

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M. Berger [1] classified involutive automorphisms σ of simple Lie algebras $\mathfrak{g}$ and determined the type of the subalgebras $\mathfrak{g}^{\sigma}$ of fixed points. Now for connected exceptional universal linear Lie groups G, we shall find involutive automorphisms σ and realize the subgroups G^{σ} of fixed points explicitly. In this paper we consider the cases of type G_2, F_4 and E_6. Our results are as follows. (Results of E_7 will be soon appeared in this Journal).

G	G^{σ}	σ		
$G_2{}^C$	$(Sp(1, C)\times Sp(1, C))/\boldsymbol{Z}_2$	γ		
$G_2{}^C$	G_2	τ		
G_2	$(Sp(1)\times Sp(1))/\boldsymbol{Z}_2$	γ		
$G_2{}^C$	$G_{2(2)}$	$\tau\gamma$	$\tau\gamma c$	
$G_{2(2)}$	$(Sp(1)\times Sp(1))/\boldsymbol{Z}_2$	γ		
	$(Sp(1, \boldsymbol{R})\times Sp(1, \boldsymbol{R}))/\boldsymbol{Z}_2\times 2$		γ	
$F_4{}^C$	$(Sp(1, C)\times Sp(3, C))/\boldsymbol{Z}_2$	γ		
	$Spin(9, C)$	σ		
$F_4{}^C$	F_4	τ		
F_4	$(Sp(1)\times Sp(3))/\boldsymbol{Z}_2$	γ		
	$Spin(9)$	σ		
$F_4{}^C$	$F_{4(4)}$	$\tau\gamma$	$\tau\gamma c$	$\tau\gamma\sigma$
$F_{4(4)}$	$(Sp(1)\times Sp(3))/\boldsymbol{Z}_2$	γ		
	$(Sp(1, \boldsymbol{R})\times Sp(3, \boldsymbol{R}))/\boldsymbol{Z}_2\times 2$		γ	
	$(Sp(1)\times Sp(1, 2))/\boldsymbol{Z}_2$			γ
	$spin(4, 5)$	σ		
$F_4{}^C$	$F_{4(-20)}$	$\tau\sigma$	$\tau\sigma'$	
$F_{4(-20)}$	$(Sp(1)\times Sp(1, 2))/\boldsymbol{Z}_2$	γ		
	$Spin(9)$	σ		

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	$Spin(8, 1)$		σ			
$E_6{}^C$	$(Sp(1, C)\times SL(6, C))/\boldsymbol{Z}_2$	γ				
	$(C^*\times Spin(10, C))/\boldsymbol{Z}_4$	σ				
	$F_4{}^C$	λ				
	$Sp(4, C)/\boldsymbol{Z}_2$	$\lambda\gamma$				
$E_6{}^C$	E_6	$\tau\lambda$				
E_6	$(Sp(1)\times SU(6))/\boldsymbol{Z}_2$	γ				
	$(U(1)\times Spin(10))/\boldsymbol{Z}_4$	σ				
	F_4	λ				
	$Sp(4)/\boldsymbol{Z}_2$	$\lambda\gamma$				
$E_6{}^C$	$E_{6(6)}$	$\tau\gamma$	$\tau\gamma_C$	$\tau\gamma\gamma_C$	$\tau\gamma\sigma$	
$E_{6(6)}$	$(Sp(1)\times SU^*(6))/\boldsymbol{Z}_2$	γ				
	$(Sp(1, \boldsymbol{R})\times SL(6, \boldsymbol{R}))/\boldsymbol{Z}_2\times 2$		γ			
	$(\boldsymbol{R}^+\times spin(5, 5))\times 2$	σ				
	$F_{4(4)}$	λ				
	$Sp(4)/\boldsymbol{Z}_2$	$\lambda\gamma$				
	$Sp(4, \boldsymbol{R})/\boldsymbol{Z}_2\times 2$			$\lambda\gamma$		
	$Sp(2, 2)/\boldsymbol{Z}_2\times 2$				$\lambda\gamma$	
$E_6{}^C$	$E_{6(2)}$	$\tau\lambda\gamma$	$\tau\lambda\gamma_C$	$\tau\lambda\gamma_H$	$\tau\lambda\gamma\sigma$	$\tau\lambda\rho$
$E_{6(2)}$	$(Sp(1)\times SU(6))/\boldsymbol{Z}_2$	γ				
	$(Sp(1, \boldsymbol{R})\times SU(3, 3))/\boldsymbol{Z}_2\times 2$			γ		
	$(Sp(1)\times SU(2, 4))/\boldsymbol{Z}_2$				γ	
	$(U(1)\times spin(6, 4))/\boldsymbol{Z}_4$	σ				
	$U(1)\times spin^*(10))/\boldsymbol{Z}_4$					σ
	$F_{4(4)}$	λ				
	$Sp(1, 3)/\boldsymbol{Z}_2$	$\lambda\gamma$				
	$Sp(4, \boldsymbol{R})/\boldsymbol{Z}_2\times 2$		$\lambda\gamma$			
$E_6{}^C$	$E_{6(-14)}$	$\tau\lambda\sigma$	$\tau\lambda\sigma'$	$\tau\lambda\gamma\rho$	$\tau\lambda\gamma_H\rho$	
$E_{6(-14)}$	$(Sp(1)\times SU(2, 4))/\boldsymbol{Z}_2$	γ				
	$(Sp(1, \boldsymbol{R})\times SU(5, 1))/\boldsymbol{Z}_2$				γ	
	$(U(1)\times spin(10))/\boldsymbol{Z}_4$	σ				
	$(U(1)\times spin(8, 2))/\boldsymbol{Z}_4$		σ			
	$(U(1)\times spin^*(10))/\boldsymbol{Z}_4$			σ		
	$F_{4(-20)}$	λ				
	$Sp(2, 2)/\boldsymbol{Z}_2\times 2$	$\lambda\gamma$				
$E_6{}^C$	$E_{6(-26)}$	τ				
$E_{6(-26)}$	$(Sp(1)\times SU^*(6))/\boldsymbol{Z}_2$	γ				
	$\boldsymbol{R}^+\times Spin(9, 1)$	σ				

F_4	λ
$F_{4(-20)}$	$\lambda\sigma$
$Sp(1, 3)/\boldsymbol{Z}_2$	$\lambda\gamma$

The proofs of some theorems about the complex Lie groups are somewhere obtained by the modifications of the preceding papers [4]$\sim$[7], but we give their proofs again. Notation $\sim$ in Theorems, for example, $(G_{2(2)})^{\gamma}\sim(\tau\gamma_C)^{\gamma}$ in Theorem 1.3.5 means $(G_{2(2)})^{\delta^{-1}\gamma\delta}=((G_2{}^C)^{\tau\gamma}c)^{\gamma}$ for some $\delta\in G_2$. Finally the author would like to thank Takeshi Miyasaka, Toshikazu Miyashita and Osamu Shukuzawa for their advices and encouragements.

0.1. Notations and preliminaries.

Let $\boldsymbol{R}$, $\boldsymbol{C}=\boldsymbol{R}\oplus\boldsymbol{R}\boldsymbol{i}$ ($\boldsymbol{i}^2=-1$) and $\boldsymbol{H}=\boldsymbol{C}\oplus\boldsymbol{C}\boldsymbol{j}$ ($\boldsymbol{j}^2=-1$) be the fields of real, complex and quaternion numbers, respectively. We define $\boldsymbol{R}$-algebras

$$\boldsymbol{C}'=\boldsymbol{R}\oplus\boldsymbol{R}\boldsymbol{i}, \quad \boldsymbol{i}'^2=1,$$

$$\boldsymbol{H}'=\boldsymbol{C}'\oplus\boldsymbol{C}'\boldsymbol{j}, \quad \boldsymbol{j}^2=-1, \qquad '\boldsymbol{H}=\boldsymbol{C}\oplus\boldsymbol{C}\boldsymbol{j}', \quad \boldsymbol{j}'^2=1,$$

called the algebras of split complex numbers and split quaternion numbers, respectively. $\boldsymbol{H}'$ and $'\boldsymbol{H}$ are isomorphic as algebras.

For a vector space V over $\boldsymbol{R}$, its complexification $\{u+iv\,|\,u, v\in V\}$ is denoted by V^C. For an $\boldsymbol{R}$-linear transformation $f: V\to V$, its complexification $f^C: V^C\to V^C$ is written by the same notation f. The complex conjugation in V^C is denoted by τ:

$$\tau(u+iv)=u-iv, \quad u, v\in V.$$

The complexification of $\boldsymbol{R}$ is briefly denoted by C: $C=\boldsymbol{R}^C$. The complexifications $\boldsymbol{C}^C$, $\boldsymbol{H}^C$ of $\boldsymbol{C}$, $\boldsymbol{H}$ have algebraic structures over C. Note that these algebras have the natural conjugations $\overline{}$, for example, $\overline{a+b\boldsymbol{i}}=a-b\boldsymbol{i}$, $a+b\boldsymbol{i}\in C\oplus C\boldsymbol{i}=\boldsymbol{C}^C$.

We use the following notations.

$M(n, K)$ (resp. $M(n, m, K)$): all of $n\times n$ (resp. $n\times m$) matrices with entries in K, $K=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{C}', \boldsymbol{H}, \boldsymbol{H}', '\boldsymbol{H}, C, \boldsymbol{C}^C, \boldsymbol{H}^C$ etc..

E: the $n\times n$ unit matrix (n is arbitrary).

$J_n=\mathrm{diag}(J, \cdots, J)\in M(2n, \boldsymbol{R})$ where $J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$, $I_n=\mathrm{diag}(I, \cdots, I)\in M(2n, \boldsymbol{R})$ where $I=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$ and $J'=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$. Hereafter the suffices n of J_n, I_n will be omitted (so I_n will have ho confusions with the following I_n).

$I_1=\mathrm{diag}(-1, 1, 1, \cdots)$, $I_2=\mathrm{diag}(-1, -1, 1, 1, \cdots)$, $\cdots\in M(n, \boldsymbol{R})$.

$\varGamma_1=\mathrm{diag}(i, 1, 1, \cdots)$, $\varGamma_2=\mathrm{diag}(i, i, 1, 1, \cdots)$, $\cdots\in M(n, C)$.

For a vector space V over $K=\boldsymbol{R}$, C, $\mathrm{Iso}_K(V)$ denotes all of K-linear isomorphisms of V. For a K-linear transformation f of V, V_f denotes $\{v\in V \mid f(v)=v\}$. When V has the non-degenerate inner produdt (u, v), for a K-linear transformation of f of V, ${}^tf$ denotes the transpose of $f:({}^tf(u), v)=(u, f(v))$.

$\boldsymbol{Z}_r$ (resp. Z_r): the cyclic group of order r.

Let G be a group. For $a, b\in G$, $a\sim b$ means that a and b are conjugate in $G: da=bd$ for some $d\in G$.

For a topological group G, G_0 denotes the identity connected component and $G=G_0\times 2$ means that G has two connected components. When G is a transformation group of a space X, G_x denotes the isotropy subgroup of G at $x\in X$: $G_x=\{g\in G \mid gx=x\}$.

If two groups G, G' (resp. algebras A, A') are isomorphic: $G\cong G'$ (resp. $A\simeq A'$), then G, G' (resp. A, A') are often identified: $G=G'$ (resp. $A=A'$).

We arrange here some of classical Lie groups used in this paper.

$$SL(n, K)=\{A\in M(n, K) \mid \det A=1\},\ K=\boldsymbol{R}, \boldsymbol{C}, C, \boldsymbol{C}^C,$$

$$SO(n, K)=\{A\in M(n, K) \mid {}^tAA=E, \det A=1\},\ K=\boldsymbol{R}, C,$$

$$O(m, n-m)=\{A\in M(n, \boldsymbol{R}) \mid {}^tAI_mA=I_m\},$$

$$SO^*(2n)=\{A\in M(2n, C) \mid {}^tAA=E, JA=(\tau A)J, \det A=1\},$$

$$SU(n, K)=\{A\in M(n, K) \mid A^*A=E, \det A=1\},\ K=\boldsymbol{C}, \boldsymbol{C}', \boldsymbol{C}^C,$$

$$SU(m, n-m, K)=\{A\in M(n, K) \mid A^*I_mA=I_m, \det A=1\},\ K=\boldsymbol{C}, \boldsymbol{C}', \boldsymbol{C}^C,$$

$$SU^*(2n, K)=\{A\in M(2n, K) \mid JA=\bar{A}J, \det A=1\},\ K=\boldsymbol{C}, \boldsymbol{C}', \boldsymbol{C}^C,$$

$$Sp(n, K)=\{A\in M(n, K) \mid A^*A=E\},\ K=\boldsymbol{H}, \boldsymbol{H}', '\boldsymbol{H}, \boldsymbol{H}^C,$$

$$Sp(m, n-m, K)=\{A\in M(n, K) \mid A^*I_mA=I_m\},\ K=\boldsymbol{H}, \boldsymbol{H}', \boldsymbol{H}^C,$$

$$Sp(n, K)=\{A\in M(2n, K) \mid {}^tAJA=J\},\ K=\boldsymbol{R}, C$$

where ${}^tA$ is the transposed matrix of A and $A^*={}^t\bar{A}$. Usually the following notations are used.

$$SO(n)=SO(n, \boldsymbol{R}), \quad SU(n)=SU(n, \boldsymbol{C}), \quad SU(m, n-m)=SU(m, n-m, \boldsymbol{C}),$$

$$SU^*(2n)=SU^*(2n, \boldsymbol{C}), \quad Sp(n)=Sp(n, \boldsymbol{H}), \quad Sp(m, n-m)=Sp(m, n-m, \boldsymbol{H}).$$

The Lie algebra of a Lie group G is denoted by the corresponding German small letter $\mathfrak{g}$. For example, $\mathfrak{su}(n)$ denotes the Lie algebra of $SU(n)$.

LEMMA 0.1. $U(n, \boldsymbol{C}')\cong U(m, n-m, \boldsymbol{C}')\cong GL(n, \boldsymbol{R})$.

PROOF. $f: GL(n, \boldsymbol{R}) = \{A \in M(n, \boldsymbol{R}) \mid \det A \neq 0\} \to U(n, \boldsymbol{C}') = \{B \in M(n, \boldsymbol{C}') \mid B^*B = E\}$,

$$f(A) = \varepsilon A + \bar{\varepsilon}\, {}^tA^{-1}, \qquad \varepsilon = \frac{1}{2}(1+\boldsymbol{i}')$$

is an isomorphism (note $\varepsilon^2 = \varepsilon$, $\bar{\varepsilon}^2 = \bar{\varepsilon}$, $\varepsilon\bar{\varepsilon} = 0$, $\varepsilon + \bar{\varepsilon} = 1$). The inverse mapping $f^{-1}: U(n, \boldsymbol{C}') \to GL(n, \boldsymbol{R})$ of f is given by $f^{-1}(P + Q\boldsymbol{i}') = P + Q$, $P, Q \in M(n, \boldsymbol{R})$. Similarly, $f: GL(n, \boldsymbol{R}) \to U(m, n-m, \boldsymbol{C}') = \{B \in M(n, \boldsymbol{C}') \mid B^* I_m B = I_m\}$, $f(A) = \varepsilon A + \bar{\varepsilon} I_m {}^tA^{-1} I_m$, is an isomorphism.

PROPOSITION 0.2. (1) $SU(n, \boldsymbol{C}') \cong SU(m, n-m, \boldsymbol{C}') \cong SL(n, \boldsymbol{R})$, $SU^*(2n, \boldsymbol{C}') \cong SL(2n, \boldsymbol{R})$.

(2) $SU(n, \boldsymbol{C}^C) \cong SU(m, n-m, \boldsymbol{C}^C) \cong SL(n, C)$, $SU^*(2n, \boldsymbol{C}^C) \cong SL(2n, C)$.

PROOF. (1) The restriction $f: SL(n, \boldsymbol{R}) \to SU(n, \boldsymbol{C}')$ of f in Lemma 0.1 is an isomorphism. In fact, the calculations of $\det(f(A)) = 1$, $A \in SL(n, \boldsymbol{R})$ and $\det(f^{-1}(B)) = 1$, $B \in SU(n, \boldsymbol{C}')$ follow from

LEMMA 0.3. (1) *For $A, B \in M(n, \boldsymbol{R})$, we have*

$$\det(\varepsilon A + \bar{\varepsilon} B) = \varepsilon \det A + \bar{\varepsilon} \det B, \qquad \varepsilon = \frac{1}{2}(1+\boldsymbol{i}').$$

(The above is also valid for $A, B \in M(n, C)$ and $\varepsilon = \frac{1}{2}(1+i\boldsymbol{i})$).

(2) *Let $P(x_1, \cdots, x_m)$ be a polynomial with integral coefficients. If $P(p_1 + q_1\boldsymbol{i}', \cdots, p_m + q_m\boldsymbol{i}') = 1$ for $p_i + q_i\boldsymbol{i}' \in \boldsymbol{R} \oplus \boldsymbol{R}\boldsymbol{i}' = \boldsymbol{C}'$ (resp. $p_i + q_i i\boldsymbol{i} \in C \oplus Ci\boldsymbol{i} = \boldsymbol{C}^C$), then $P(p_1 + q_1, \cdots, p_m + q_m) = 1$.*

Similarly, $f: SL(n, \boldsymbol{R}) \to SU(m, n-m, \boldsymbol{C}')$, $f(A) = \varepsilon A + \bar{\varepsilon} I_m {}^tA^{-1} I_m$ and $f: SL(2n, \boldsymbol{R}) \to SU^*(2n, \boldsymbol{C}')$, $f(A) = \varepsilon A - \bar{\varepsilon} JAJ$ where $\varepsilon = \frac{1}{2}(1+\boldsymbol{i}')$, are isomorphisms, respectively.

(2) These are corollaries of (1). In fact, for example, $f: SL(n, \boldsymbol{C}) \to SU(n, \boldsymbol{C}^C)$, $f(A) = \varepsilon A + \bar{\varepsilon}\, {}^tA^{-1}$ where $\varepsilon = \frac{1}{2}(1+i\boldsymbol{i})$, is an isomorphism.

PROPOSITION 0.4. (1) $Sp(n, \boldsymbol{H}') \cong Sp(m, n-m, \boldsymbol{H}') \cong Sp(n, \boldsymbol{R})$.

(2) $Sp(n, \boldsymbol{H}^C) \cong Sp(m, n-m, \boldsymbol{H}^C) \cong Sp(n, C)$. *In particular,* $Sp(1, \boldsymbol{H}') \cong Sp(1, \boldsymbol{R}) = SL(2, \boldsymbol{R})$, $Sp(1, \boldsymbol{H}^C) \cong Sp(1, C) = SL(2, C)$.

PROOF. (1) Let $k': M(n, \boldsymbol{H}') \to \{B \in M(2n, \boldsymbol{C}') \mid JB = \bar{B}J\}$ be the algebraic $\boldsymbol{R}$-isomorphism defined by

$$k'((a+bj))=\left(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\right) \qquad a,\ b\in \boldsymbol{C}'$$

Then $f^{-1}k' : Sp(n, \boldsymbol{H}')\to Sp(n, \boldsymbol{R})$ is an isomorphism. In fact,

$$\begin{aligned} Sp(n, \boldsymbol{H}') &= \{ D\in M(n, \boldsymbol{H}') \mid D^*D=E \} \\ &\overset{k'}{\longrightarrow} \{ B\in M(2n, \boldsymbol{C}') \mid B^*B=E,\ JB=\bar{B}J \} \\ &= \{ B\in U(2n, \boldsymbol{C}') \mid {}^tBJB=J \} \\ &\overset{f^{-1}}{\longrightarrow} \{ A\in M(2n, \boldsymbol{R}) \mid {}^tAJA=J \} \ \text{((Lemma 0.1)} = Sp(n, \boldsymbol{R}). \end{aligned}$$

Similarly, $Sp(m, n-m, \boldsymbol{H}') = \{ D\in M(n, \boldsymbol{H}') \mid D^*I_mD=I_m \} \overset{k'}{\to} \{ B\in M(2n, \boldsymbol{C}') \mid B^*I_{2m}B=I_{2m},\ JB=\bar{B}J \} = \{ B\in M(2n, \boldsymbol{C}') \mid B^*I_{2m}B = I_{2m},\ {}^tBJI_{2m}B = JI_{2m} \}$ (since J and JI_{2m} are conjugate in $O(2n)$: $J_m'J=JI_{2m}J_m'$ where $J_m'=\mathrm{diag}(J', \cdots, J, 1, \cdots, 1)$, $J'=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, by the correspondence $B\to J_m'BJ_m'$) $\cong \{ B\in M(2n, \boldsymbol{C}') \mid B^*I_{2m}B=I_{2m},\ {}^tBJB=J \} = \{ B\in U(2m, 2n-2m, \boldsymbol{C}') \mid {}^tBJB=J \} \overset{f^{-1}}{\to} \{ A\in M(2n, \boldsymbol{R}) \mid {}^tAJA=J \} = Sp(n, \boldsymbol{R})$.

(2) These are corollaries of (1).

0.2. Automorphisms of a gronp.

Let G be a group and σ an automorphism of G. G^σ denotes $\{ g\in G \mid \sigma g= g \}$. For $s\in G$, $\tilde{s}$ denotes the inner automorphism induced by s: $\tilde{s}(g)=sgs^{-1}$, $g\in G$, then $G^{\tilde{s}}=\{g\in G \mid sg=gs\}$. Hereafter $G^{\tilde{s}}$ will be written by G^s. Moreover when G is indicated, G^σ, G^s will be written by σ, s, respectively.

LEMMA 0.5. *Let σ_1, σ_2, σ_3 are involutive automorphisms of a group G satisfying $\sigma_i\sigma_j=\sigma_j\sigma_i$, then*

$$(G^{\sigma_1})^{\sigma_2}=(G^{\sigma_2})^{\sigma_1}, \qquad (G^{\sigma_1\sigma_2})^{\sigma_1}=(G^{\sigma_2})^{\sigma_1}, \qquad (G^{\sigma_1\sigma_3})^{\sigma_2\sigma_3}=(G^{\sigma_1\sigma_2})^{\sigma_2\sigma_3}.$$

By the simple representation, these are written by $(\sigma_1)^{\sigma_2}=(\sigma_2)^{\sigma_1}$, $(\sigma_1\sigma_2)^{\sigma_1}=(\sigma_2)^{\sigma_1}$, $(\sigma_1\sigma_3)^{\sigma_2\sigma_3}=(\sigma_1\sigma_2)^{\sigma_2\sigma_3}$, respectively.

For a given group G and an involutive automorphism σ of G, our aim is to determine the group structure of G^σ. After this, for a homomorphism $\phi: G'\to G^\sigma$ of groups, it needs often to prove that ϕ is well-defined and onto. When G', G^σ are Lie groups, these properties can reduce to their Lie algebras, that is,

LEMMA 0.6. *Let $\phi: G'\to G^\sigma$ be a homomorphism of Lie groups.*

(1) *When G' is connected, if $d\phi: \mathfrak{g}'\to\mathfrak{g}^\sigma$ is well-defined, then ϕ is so.*

(2) *When G^σ is connected, if $d\phi: \mathfrak{g}'\to\mathfrak{g}^\sigma$ is onto, then ϕ is so.*

To use Lemma 0.6. (2), the following Lemma is useful.

LEMMA 0.7 (E. Cartan-P. K. Raševskii [3]). *Let G be a simply connected Lie group and σ an involutive automorphism of G, then G^σ is connected.*

In the following we will somewhere try to give elementary proof not using Lemmas 0.6, 0.7. The author thinks that the elementary proof finds out occasionally essential properties of the group G^σ.

Group G_2

1.1. Cayley algebras and Lie groups of type G_2.

Let $\mathfrak{C}=\boldsymbol{H}\oplus\boldsymbol{H}e$ be the division Cayley algebra with the multiplication

$$(m+ae)(n+be)=(mn-\bar{b}a)+(a\bar{n}+bm)e,$$

the conjugation $\overline{m+ae}=\bar{m}-ae$ and the inner product $(m+ae, n+be)=(m, n)+(a, b)\ \left(=\frac{1}{2}((m\bar{n}+n\bar{m})+(a\bar{b}+b\bar{a}))\right)$. Another Cayley algebra $\mathfrak{C}'=\boldsymbol{H}\oplus\boldsymbol{H}e'$, called the split Cayley algebra, is defined as the algebra with the multiplication

$$(m+ae')(n+be')=(mn+\bar{b}a)+(a\bar{n}+bm)e',$$

the conjugation $\overline{m+ae'}=\bar{m}-ae'$ and the inner product $(m+ae', n+be')=(m, n)-(a, b)$.

The connected linear Lie groups of type G_2 are obtained as the automorphism groups of the Cayley algebras, respectively.

$$G_2{}^C=G_2(\mathfrak{C}^C)=\{\alpha\in\mathrm{Iso}_C(\mathfrak{C}^C) \mid \alpha(xy)=(\alpha x)(\alpha y)\},$$
$$G_2=G_2(\mathfrak{C})=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}) \mid \alpha(xy)=(\alpha x)(\alpha y)\},$$
$$G_{2(2)}=G_2(\mathfrak{C}')=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}') \mid \alpha(xy)=(\alpha x)(\alpha y)\}.$$

(Similarly the group $G_2(\boldsymbol{H}^C)$ is defined). $G_2{}^C$, G_2 are simply connected (see Appendix).

1.2. Involutions of Lie groups of type G_2.

We define $\boldsymbol{R}$-linear transformations γ, γ_C, γ_H of $\mathfrak{C}$ by

$$\gamma(m+ae)=m-ae, \qquad m+ae\in\boldsymbol{H}\oplus\boldsymbol{H}e=\mathfrak{C},$$

$$\gamma_C(m+ae)=\gamma_C m+(\gamma_C a)e, \qquad \gamma_H(m+ae)=\gamma_H m+(\gamma_H a)e$$

where $\gamma_C, \gamma_H : \boldsymbol{H}\to\boldsymbol{H}$ are defined as $\gamma_C(x+yj)=\bar{x}+\bar{y}j$, $\gamma_H(x+yj)=x-yj$, $x+yj\in \boldsymbol{C}\oplus\boldsymbol{C}j=\boldsymbol{H}$, respectively. Then $\gamma, \gamma_C, \gamma_H\in G_2\subset G_2{}^C$ and $\gamma^2=\gamma_C{}^2=\gamma_H{}^2=1$.

LEMMA 1.2.1. (1) $(\boldsymbol{H}^C)_\tau=\boldsymbol{H}$, $(\boldsymbol{H}^C)_{\tau\gamma_C}\simeq\boldsymbol{H}'$, $(\boldsymbol{H}^C)_{\tau\gamma_H}\simeq{}'\boldsymbol{H}$.

(2) $(\mathfrak{C}^C)_\tau=\mathfrak{C}$, $(\mathfrak{C}^C)_{\tau\gamma}\simeq\mathfrak{C}'$.

PROOF. For example, the correspondence

$$(\mathfrak{C}^C)_{\tau\gamma}\ni m+iae \longrightarrow m+ae'\in\mathfrak{C}' \qquad (m, a\in\boldsymbol{H})$$

gives an isomorphism as algebras.

The semi-linear transformations τ, $\tau\gamma$ of $\mathfrak{C}^C$ induce involutive automorphisms $\tilde{\tau}$, $\widetilde{\tau\gamma}$ of $G_2{}^C$:

$$\tilde{\tau}(\alpha)=\tau\alpha\tau, \qquad \widetilde{\tau\gamma}(\alpha)=\tau\gamma\alpha\gamma\tau, \qquad \alpha\in G_2{}^C.$$

THEOREM 1.2.2. $(G_2{}^C)^\tau=G_2$, $(G_2{}^C)^{\tau\gamma}\cong G_{2(2)}$.

PROOF. ($(G_2{}^C)^\tau$, $(G_2{}^C)^{\tau\gamma}$ mean $(G_2{}^C)^{\tilde{\tau}}$, $(G_2{}^C)^{\widetilde{\tau\gamma}}$, respectively). These are direct results of Lemma 1.2.1. (2).

PROPOSITION 1.2.3. *γ, γ_C, γ_H, $\gamma\gamma_C$, $\gamma\gamma_H$ are conjugate in G_2 with one another (moreover γ is conjugate to the others under $\delta=\delta^{-1}\in G_2$).*

PROOF. Define four $\boldsymbol{R}$-linear isomorphisms $\delta:\mathfrak{C}\to\mathfrak{C}$ satisfying $\delta(1)=1$ and

$$\begin{array}{llll}
i\longrightarrow e & i\longrightarrow i & i\longrightarrow ie & i\longrightarrow i\\
j\longrightarrow j & j\longrightarrow e & j\longrightarrow j & j\longrightarrow je\\
k\longrightarrow -je & k\longrightarrow ie & k\longrightarrow -ke & k\longrightarrow -ke\\
e\longrightarrow i, & e\longrightarrow j, & e\longrightarrow -e, & ie\longrightarrow -e\\
ie\longrightarrow -ie & ie\longrightarrow k & ie\longrightarrow i & ie\longrightarrow -ie\\
je\longrightarrow -k & je\longrightarrow -je & je\longrightarrow -je & je\longrightarrow j\\
ke\longrightarrow -ke & ke\longrightarrow -ke & ke\longrightarrow -k & ke\longrightarrow -k
\end{array}$$

where $\boldsymbol{k}=\boldsymbol{ij}$, respectively. Then $\delta=\delta^{-1}\in G_2$ and $\delta\gamma=\gamma_C\delta$, $\delta\gamma=\gamma_H\delta$, $\delta\gamma=\gamma\gamma_C\delta$, $\delta\gamma=\gamma\gamma_H\delta$, respectively.

1.3. Subgroups of type $C_1\oplus C_1$ of Lie groups of type G_2.

PROPOSITION 1.3.1. $G_2(\boldsymbol{H}^C)\cong Sp(1, C)/\boldsymbol{Z}_2$.

PROOF. We define $\phi: Sp(1, \boldsymbol{H}^C)\to G_2(\boldsymbol{H}^C)$ by

$$\phi(q)m=qm\bar{q}\,, \qquad m\in \boldsymbol{H}^C.$$

It is clear that ϕ is well-defined and a homomorphism. We shall show ϕ is onto. Let $\alpha\in G_2(\boldsymbol{H}^C)$. Since $\boldsymbol{H}^C$ is a central simple C-algebra, by Noether-Skolem's theorem, there exists an invertible element $q\in\boldsymbol{H}^C$ such that $\alpha m=qmq^{-1}$, $m\in\boldsymbol{H}^C$. We may assume $q\bar{q}=1$, that is, $q\in Sp(1, \boldsymbol{H}^C)$. Hence ϕ is onto. $\operatorname{Ker}\phi=\{1, -1\}=\boldsymbol{Z}_2$. Thus we have $G_2(\boldsymbol{H}^C)\cong Sp(1, \boldsymbol{H}^C)/\boldsymbol{Z}_2\cong Sp(1, C)/\boldsymbol{Z}_2$ (Proposition 0.4).

THEOREM 1.3.2. $(G_2{}^C)^\gamma\cong(Sp(1, C)\times Sp(1, C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(1, 1), (-1, -1)\}$.

PROOF ([5]). We define $\phi: Sp(1, \boldsymbol{H}^C)\times Sp(1, \boldsymbol{H}^C)\to(G_2{}^C)^\gamma$ by

$$\phi(p, q)(m+ae)=qm\bar{q}+(pa\bar{q})e, \qquad m+ae\in\boldsymbol{H}^C\oplus\boldsymbol{H}^Ce=\mathfrak{C}^C.$$

It is easy to verify that ϕ is well-defined and a homomorphism. We shall show ϕ is onto. Let $\alpha\in(G_2{}^C)^\gamma$. Since $(\mathfrak{C}^C)_\gamma=\boldsymbol{H}^C$ is invariant under α, α induces an automorphism of $\boldsymbol{H}^C$. Hence there exists $q\in Sp(1, \boldsymbol{H}^C)$ such that

$$\alpha m=qm\bar{q}\,, \qquad m\in\boldsymbol{H}^C \quad \text{(Proposition 1.3.1)}.$$

Put $\beta=\phi(1, q)^{-1}\alpha$, then $\beta\in(G_2{}^C)^\gamma$ and $\beta|\boldsymbol{H}^C=1$. Since $(\mathfrak{C}^C)_{-\gamma}=\boldsymbol{H}^Ce$ is also invariant under β, we can put

$$\beta e=pe\,, \qquad p\in\boldsymbol{H}^C.$$

$p\in Sp(1, \boldsymbol{H}^C)$ because $-1=\beta(ee)=(\beta e)(\beta e)=(pe)(pe)=-p\bar{p}$, and $\beta(m+ae)=m+a(\beta e)=m+a(pe)=m+(pa)e=\phi(p, 1)(m+ae)$, that is, $\beta=\phi(p, 1)$. Hence $\alpha=\phi(1, q)\beta=\phi(1, q)\phi(p, 1)=\phi(p, q)$. Therefore ϕ is onto. $\operatorname{Ker}\phi=\{(1, 1), (-1, -1)\}=\boldsymbol{Z}_2$ Thus we have the required isomorphism. (Remark. $(Sp(1, C)\times Sp(1, C))/\boldsymbol{Z}_2\cong SO(4, C)$).

LEMMA 1.3.3. *$\phi: Sp(1, \boldsymbol{H}^C)\times Sp(1, \boldsymbol{H}^C)\to G_2{}^C$ of Theorem 1.3.2 satisfies*

(1) $\gamma=\phi(-1, 1)$, $\gamma_C=\phi(\boldsymbol{j}, \boldsymbol{j})$, $\gamma_H=\phi(\boldsymbol{i}, \boldsymbol{i})$.

(2) $\tau\phi(p, q)\tau=\phi(\tau p, \tau q)$, $\gamma_C\phi(p, q)\gamma_C=\phi(\gamma_C p, \gamma_C q)$.

THEOREM 1.3.4. $(G_2)^\gamma\cong(Sp(1)\times Sp(1))/\boldsymbol{Z}_2\cong(G_{2(2)})^\gamma$.

PROOF. $(G_2)^\gamma=((G_2{}^C)^\tau)^\gamma$ (Theorem 1.2.2)$=((G_2{}^C)^\gamma)^\tau$ (Lemma 0.5)$=(\phi(Sp(1, \boldsymbol{H}^C)\times Sp(1, \boldsymbol{H}^C)))^\tau$ (Theorem 1.3.2). Hence for $\alpha\in(G_2)^\gamma$ there exist $p, q\in Sp(1, \boldsymbol{H}^C)$ such that $\alpha=\phi(p, q)$. From the condition $\tau\alpha=\alpha\tau$, we have $\phi(p, q)=\alpha=\tau\alpha\tau=\tau\phi(p, q)\tau=\phi(\tau p, \tau q)$ (Lemma 1.3.3). Hence

$$\tau p=p\,, \quad \tau q=q \quad \text{or} \quad \tau p=-p\,, \quad \tau q=-q\,.$$

The latter case is impossible. In fact, put $p=ip'$, $p'\in \boldsymbol{H}$, then $1=p\bar{p}=(ip')(\overline{ip'})=-p'\overline{p}'\leqq 0$, a contradiction. Therefore $p, q\in Sp(1)$. Thus $(G_2)^{\gamma}=(\phi(Sp(1, \boldsymbol{H}^C)\times Sp(1, \boldsymbol{H}^C)))^{\tau}=\phi(Sp(1)\times Sp(1))\cong(Sp(1)\times Sp(1))/\boldsymbol{Z}_2$. $(G_{2(2)})^{\gamma}=((G_2{}^C)^{\tau\gamma})^{\gamma}$ (Theorem 1.2.2)$=((G_2{}^C)^{\tau})^{\gamma}$ (Lemma 0.5) $\cong(Sp(1)\times Sp(1))/\boldsymbol{Z}_2$ (as above). (This fact is written as $(G_{2(2)})^{\gamma}=(\tau\gamma)^{\gamma}=(\tau)^{\gamma}$). (REMARK. $(Sp(1)\times Sp(1))/\boldsymbol{Z}_2\cong SO(4)$).

THEOREM 1.3.5. $(G_{2(2)})^{\gamma}\sim(\tau\gamma_C)^{\gamma}\cong(Sp(1, \boldsymbol{R})\times Sp(1, \boldsymbol{R}))/\boldsymbol{Z}_2\times 2$.

PROOF.
$$G_{2(2)}=(G_2{}^C)^{\tau\gamma}\cong(G_2{}^C)^{\tau\gamma_C}.$$

In fact, since γ and γ_C are conjugate in $G_2: \delta\gamma=\gamma_C\delta$, $\delta\tau=\tau\delta$ (Proposition 1.2.3), the correspondence $(G_2{}^C)^{\tau\gamma}\ni\alpha\to\delta\alpha\delta^{-1}\in(G_2{}^C)^{\tau\gamma_C}$ gives an isomorphism. Now let $\alpha\in((G_2{}^C)^{\tau\gamma_C})^{\gamma}$, $\alpha=\phi(p, q)$, $p, q\in Sp(1, \boldsymbol{H}^C)$ (Theorom 1.3.2). From the condition $\tau\gamma_C\alpha=\alpha\tau\gamma_C$, we have $\phi(\tau\gamma_C p, \tau\gamma_C q)=\phi(p, q)$ (Lemma 1.3.3). Hence

$$\tau\gamma_C p=p,\quad \tau\gamma_C q=q \quad\text{or}\quad \tau\gamma_C p=-p,\quad \tau\gamma_C q=-q.$$

Therefore $p, q\in Sp(1, \boldsymbol{H}')$ or $p, q\in \boldsymbol{i}Sp(1, \boldsymbol{H}')$ (Lemma 1.2.1). Thus $((G_2{}^C))^{\tau\gamma_C})^{\gamma}\cong(Sp(1, \boldsymbol{H}')\times Sp(1, \boldsymbol{H}')\cup \boldsymbol{i}Sp(1, \boldsymbol{H}')\times \boldsymbol{i}Sp(1, \boldsymbol{H}'))/\boldsymbol{Z}_2\cong(Sp(1, \boldsymbol{R})\times Sp(1, \boldsymbol{R}))/\boldsymbol{Z}_2\times 2$. $(\phi(\boldsymbol{i}, \boldsymbol{i})=\gamma_{\boldsymbol{H}})$. (REMARK. This group is isomorphic to the group $SO(2, 2)=\{A\in M(4, \boldsymbol{R})\mid {}^tAI_2A=I_2, \det A=1\}$).

Group F_4

2.1. Jordan algebras and Lie groups of type F_4.

Let K be $\boldsymbol{H}$, $\boldsymbol{H}^C$, $\mathfrak{C}$, $\mathfrak{C}'$ or $\mathfrak{C}^C$. $\mathfrak{J}(K)$ denotes one of the Jordan algebras

$$\mathfrak{J}(3, K)=\{X\in M(3, K)\mid X^*=X\},$$

$$\mathfrak{J}(1, 2, K)=\{X\in M(3, K)\mid I_1X^*I_1=X\}$$

with the Jordan multiplication $X\circ Y$, the inner product (X, Y) and the trilinear form $\operatorname{tr}(X, Y, Z)$:

$$X\circ Y=\frac{1}{2}(XY+YX),\quad (X, Y)=\operatorname{tr}(X\circ Y),\quad \operatorname{tr}(X, Y, Z)=(X, Y\circ Z).$$

In $\mathfrak{J}(K)$, we define another multiplication $X\times Y$, called the Freudenthal multiplication, by

$$X\times Y=\frac{1}{2}(2X\circ Y-\operatorname{tr}(X)Y-\operatorname{tr}(Y)X+(\operatorname{tr}(X)\operatorname{tr}(Y)-(X, Y))E)$$

and the trilinear form (X, Y, Z), the determinant $\det X$ by

$$(X, Y, Z)=(X, Y\times Z),\qquad \det X=\frac{1}{3}(X, X, X).$$

The algebra $\mathfrak{J}(K)$ with the Freudenthal multiplication $X\times Y$ and the inner product (X, Y) is called the Freudenthal algebra. In $\mathfrak{J}(K)$, we have relations

$$X\circ(X\times X)=(\det X)E, \qquad (X\times X)\times(X\times X)=(\det X)X.$$

An element $X\in\mathfrak{J}(3, \mathfrak{C})$ has the form

$$X=X(\xi, x)=\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix}, \qquad \xi_i\in\boldsymbol{R},\ x_i\in\mathfrak{C}.$$

We correspond such $X\in\mathfrak{J}(3, \mathfrak{C})$ to an element $M+\boldsymbol{a}\in\mathfrak{J}(3, \boldsymbol{H})\oplus\boldsymbol{H}^3$ such that

$$\begin{pmatrix}\xi_1 & m_3 & \bar{m}_2\\ \bar{m}_3 & \xi_2 & m_1\\ m_2 & \bar{m}_1 & \xi_3\end{pmatrix}+(a_1, a_2, a_3)$$

where $x_i=m_i+a_ie\in\boldsymbol{H}\oplus\boldsymbol{H}e=\mathfrak{C}$. Then $\mathfrak{J}(3, \boldsymbol{H})\oplus\boldsymbol{H}^3$ has the multiplication and the inner product

$$(M+\boldsymbol{a})\times(N+\boldsymbol{b})=\Big(M\times N-\frac{1}{2}(\boldsymbol{a}^*\boldsymbol{b}+\boldsymbol{b}^*\boldsymbol{a})\Big)-\frac{1}{2}(\boldsymbol{a}N+\boldsymbol{b}M),$$

$$(M+\boldsymbol{a}, N+\boldsymbol{b})=(M, N)+2(\boldsymbol{a}, \boldsymbol{b})$$

where $(\boldsymbol{a}, \boldsymbol{b})=\frac{1}{2}(\boldsymbol{a}\boldsymbol{b}^*+\boldsymbol{b}\boldsymbol{a}^*)=\frac{1}{2}\mathrm{tr}(\boldsymbol{a}^*\boldsymbol{b}+\boldsymbol{b}^*\boldsymbol{a})$, corresponding those of $\mathfrak{J}(3, \mathfrak{C})$, that is, $\mathfrak{J}(3, \mathfrak{C})$ is isomorphic to $\mathfrak{J}(3, \boldsymbol{H})\oplus\boldsymbol{H}^3$ as Freudenthal algebra. As for $\mathfrak{J}^C=\mathfrak{J}(3, \mathfrak{C}^C)$, the same arguments are valid as above: $\mathfrak{J}(3, \mathfrak{C}^C)=\mathfrak{J}(3, \boldsymbol{H}^C)\oplus(\boldsymbol{H}^C)^3$. In $\mathfrak{J}(3, K)$ we use the following notations.

$$E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad E_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad E_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$F_1(x)=\begin{pmatrix}0&0&0\\0&0&x\\0&\bar{x}&0\end{pmatrix},\quad F_2(x)=\begin{pmatrix}0&0&\bar{x}\\0&0&0\\x&0&0\end{pmatrix},\quad F_3(x)=\begin{pmatrix}0&x&0\\\bar{x}&0&0\\0&0&0\end{pmatrix}.$$

The tables of the Jordan and the Freudenthal multiplications among them are given as follows.

$$\begin{cases} E_i\circ E_i=E_i, & E_i\circ E_j=0, \quad i\neq j, \\ E_i\circ F_i(x)=0, & E_i\circ F_j(x)=\frac{1}{2}F_j(x), \quad i\neq j, \\ F_i(x)\circ F_i(y)=(x,\ y)(E_{i+1}+E_{i+2}), & F_i(x)\circ F_{i+1}(y)=\frac{1}{2}F_{i+2}(\overline{xy}), \end{cases}$$

$$\begin{cases} E_i\times E_i=0, & E_i\times E_{i+1}=\frac{1}{2}E_{i+2}, \\ E_i\times F_i(x)=-\frac{1}{2}F_i(x), & E_i\times F_j(x)=0, \quad i\neq j, \\ F_i(x)\times F_i(y)=-(x,\ y)E_i, & F_i(x)\times F_{i+1}(y)=\frac{1}{2}F_{i+2}(\overline{x}y) \end{cases}$$

where the indexes are considered as mod 3.

The connected linear Lie groups of type F_4 are obtained as the automorphism groups of the Jordan algebras, respectively.

$$F_4{}^C=F_4(\mathfrak{J}(3,\ \mathfrak{C}^C))=\{\alpha\in \mathrm{Iso}_C(\mathfrak{J}(3,\ \mathfrak{C}^C))\mid \alpha(X\circ Y)=\alpha X\circ\alpha Y\},$$

$$F_4=F_4(\mathfrak{J}(3,\ \mathfrak{C}))=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}(3,\ \mathfrak{C}))\mid \alpha(X\circ Y)=\alpha X\circ\alpha Y\},$$

$$F_{4(4)}=F_4(\mathfrak{J}(3,\ \mathfrak{C}'))=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}(3,\ \mathfrak{C}'))\mid \alpha(X\circ Y)=\alpha X\circ\alpha Y\},$$

$$F_{4(-20)}=F_4(\mathfrak{J}(1,\ 2,\ \mathfrak{C}))=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}(1,\ 2,\ \mathfrak{C}))\mid \alpha(X\circ Y)=\alpha X\circ\alpha Y\}.$$

(Similarly the group $F_4(\mathfrak{J}(3,\ \boldsymbol{H}^C))$ is defined). $F_4{}^C$, F_4, $F_{4(-20)}$ are simply connected (see Appendix). The group $F_4{}^C$ naturally contains $G_2{}^C$ as a subgroup, that is, for $\alpha\in G_2{}^C$, define $\tilde{\alpha}:\mathfrak{J}^C\to\mathfrak{J}^C$ by

$$\tilde{\alpha}X(\xi,\ x)=X(\xi,\ \alpha x) \quad \text{where} \quad \alpha x=\alpha(x_1,\ x_2,\ x_3)=(\alpha x_1,\ \alpha x_2,\ \alpha x_3),$$

then $G_2{}^C\cong\{\tilde{\alpha}\mid\alpha\in G_2{}^C\}\subset F_4{}^C$. Similarly $G_2\subset F_4$, $G_{2(2)}\subset F_{4(4)}$, $G_2\subset F_{4(-20)}$.

LEMMA 2.1.1. *For* $\alpha\in \mathrm{Iso}_C(\mathfrak{J}^C)$, *the following three conditions are equivalent.*

$$\det\alpha X=\det X, \quad (\alpha X,\ \alpha Y,\ \alpha Z)=(X,\ Y,\ Z), \quad \alpha X\times\alpha Y={}^t\alpha^{-1}(X\times Y),$$

for $X,\ Y,\ Z\in\mathfrak{J}^C$.

LEMMA 2.1.2. *For* $\alpha\in F_4{}^C$, *we have* $\alpha E=E$ *and* $\mathrm{tr}(\alpha X)=\mathrm{tr}(X)$, $X\in\mathfrak{J}^C$.

PROOF ([4]). $\alpha E=E$ is trivial. Next we use the identity $X\circ(X\times X)=(\det X)E$, that is,

$$X\circ(X\circ X)-\mathrm{tr}(X)X^2+\frac{1}{2}(\mathrm{tr}(X)^2-\mathrm{tr}(X^2))X=(\det X)E. \tag{i}$$

Apply (i) to αX and then operate α^{-1} on it, then

$$X\circ(X\circ X)-\mathrm{tr}(\alpha X)X^2+\frac{1}{2}((\mathrm{tr}(\alpha X))^2-\mathrm{tr}((\alpha X)^2)))X=(\det\alpha X)E. \qquad \text{(ii)}$$

By substruction (i)-(ii) we have

$$(\mathrm{tr}(\alpha X)-\mathrm{tr}(X))X^2+\frac{1}{2}(\mathrm{tr}(X)^2-(\mathrm{tr}(\alpha X))^2+\mathrm{tr}((\alpha X)^2)-\mathrm{tr}(X^2))X$$
$$=(\det X-\det(\alpha X))E.$$

Note that as an additive generator of $\mathfrak{J}^C$ we can choose $\mathfrak{S}=\{E_i,\ F\in\mathfrak{J}^C\,|\,\mathrm{tr}(F)=\det F=0,\ \mathrm{diag}\,F=0,\ F^2=E_i+E_{i+1},\ i=1, 2, 3\}$. Now for $F\in\mathfrak{S}$,

$$\mathrm{tr}(\alpha F)(E_i+E_{i+1})+\frac{1}{2}(-(\mathrm{tr}(\alpha F))^2+\mathrm{tr}((\alpha F)^2)-2)F=-(\det(\alpha F))E.$$

Compare each term of both sides, then we have $\mathrm{tr}(\alpha F)=0$ $(=\mathrm{tr}(F))$ and $\mathrm{tr}((\alpha F)^2)=2$. Hence $\mathrm{tr}(\alpha E_i)=\mathrm{tr}(\alpha(E-F_i(1)^2))=\mathrm{tr}(E)-\mathrm{tr}((\alpha F_i(1))^2)=3-2=1=\mathrm{tr}(E_i)$, $i=1, 2, 3$. Consequently $\mathrm{tr}(\alpha X)=\mathrm{tr}(X)$ for $X\in\mathfrak{J}^C$.

PROPOSITION 2.1.3.

$$F_4{}^C=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C)\mid \alpha(X\circ Y)=\alpha X\circ\alpha Y\} \qquad (1)$$
$$=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C)\mid \mathrm{tr}(\alpha X,\ \alpha Y,\ \alpha Z)=\mathrm{tr}(X,\ Y,\ Z),\ (\alpha X,\ \alpha Y)=(X,\ Y)\} \qquad (2)$$
$$=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C)\mid \det\alpha X=\det X,\ (\alpha X,\ \alpha Y)=(X,\ Y)\} \qquad (3)$$
$$=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C)\mid \det\alpha X=\det X,\ \alpha E=E\} \qquad (4)$$
$$=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C)\mid \alpha(X\times Y)=\alpha X\times\alpha Y\}. \qquad (5)$$

PROOF. (1)→(2) $(\alpha X,\ \alpha Y)=\mathrm{tr}(\alpha X\circ\alpha Y)=\mathrm{tr}(\alpha(X\circ Y))=\mathrm{tr}(X\circ Y)$ (Lemma 2.1.2) $=(X,\ Y)$. $\mathrm{tr}(\alpha X,\ \alpha Y,\ \alpha Z)=(\alpha X,\ \alpha Y\circ\alpha Z)=(\alpha X,\ \alpha(Y\circ Z))=(X,\ Y\circ Z)=\mathrm{tr}(X,\ Y,\ Z)$.

(2)→(1) $(\alpha X\circ\alpha Y,\alpha Z)=\mathrm{tr}(\alpha X,\alpha Y,\alpha Z)=\mathrm{tr}(X,Y,Z)=(X\circ Y,Z)=(\alpha(X\circ Y),\alpha Z)$ holds for all αZ, hence $\alpha X\circ\alpha Y=\alpha(X\circ Y)$.

(2)→(3) Since we have already known (2)→(1), we can use $\mathrm{tr}(\alpha X)=\mathrm{tr}(X)$ (Lemma 2.1.2). Now $3\det\alpha X=\mathrm{tr}(\alpha X,\ \alpha X,\ \alpha X)-\frac{3}{2}\mathrm{tr}(\alpha X)(\alpha X,\ \alpha X)+\frac{1}{2}\mathrm{tr}(\alpha X)^3=\mathrm{tr}(X,\ X,\ X)-\frac{3}{2}\mathrm{tr}(X)(X,\ X)+\frac{1}{2}\mathrm{tr}(X)^3=3\det X$.

(3)→(5) $(\alpha(X\times Y),\ \alpha Z)=(X\times Y,\ Z)=(X,\ Y,\ Z)=(\alpha X,\ \alpha Y,\ \alpha Z)$ (Lemma 2.1.1) $=(\alpha X\times\alpha Y,\ \alpha Z)$ holds for all αZ, hence $\alpha X\times\alpha Y=\alpha(X\times Y)$.

(5)→(4) $(\det\alpha X)\alpha X=(\alpha X\times\alpha X)\times(\alpha X\times\alpha X)=\alpha((X\times X)\times(X\times X))=(\det X)\alpha X$, hence $\det\alpha X=\det X$. Next, in $\alpha X\times\alpha E=\alpha(X\times E)=\frac{1}{2}\alpha(\mathrm{tr}(X)E-X)$, put $\alpha E=P=P(\rho,\ p)$, then

$$\alpha X\times P=\frac{1}{2}\mathrm{tr}(X)P-\frac{1}{2}\alpha X. \qquad \text{(i)}$$

Put $X=\alpha^{-1}E_1$ in (i) and compare each term of both sides, then

$$0=\mu\rho_1-1,\quad \rho_3=\mu\rho_2,\quad \rho_2=\mu\rho_3,\quad -p_1=\mu p_1,\quad 0=\mu p_2,\quad 0=\mu p_3$$

where $\mu=\mathrm{tr}(\alpha^{-1}E_1)$. Consequently we have $p_2=p_3=0$. Similarly $p_1=0$. Again put $X=\alpha^{-1}F_1(1)$ in (i) and compare F_1-parts, then $\rho_1=1$. Similarly $\rho_2=\rho_3=1$. Thus $\alpha E=E$.

(4)→(2) $\mathrm{tr}(\alpha X)=(\alpha X, E, E)=(\alpha X, \alpha E, \alpha E)=(X, E, E)=\mathrm{tr}(X)$, $\frac{1}{2}(\mathrm{tr}(X)\mathrm{tr}(Y)-(X, Y))=(X, Y, E)=(\alpha X, \alpha Y, \alpha E)=(\alpha X, \alpha Y, E)=\frac{1}{2}(\mathrm{tr}(\alpha X)\mathrm{tr}(\alpha Y)-(\alpha X, \alpha Y))=\frac{1}{2}(\mathrm{tr}(X)\mathrm{tr}(Y)-(\alpha X, \alpha Y))$. Hence $(\alpha X, \alpha Y)=(X, Y)$. Finally using $(X, Y, Z)=\mathrm{tr}(X, Y, Z)-\frac{1}{2}\mathrm{tr}(X)(Y, Z)-\frac{1}{2}\mathrm{tr}(Y)(Z, X)-\frac{1}{2}\mathrm{tr}(Z)(X, Y)-\frac{1}{2}\mathrm{tr}(X)\mathrm{tr}(Y)\mathrm{tr}(Z)$, we have $\mathrm{tr}(\alpha X, \alpha Y, \alpha Z)=\mathrm{tr}(X, Y, Z)$.

The Lie algebra $\mathfrak{f}_4{}^C$ of the Lie group $F_4{}^C$ has the following structure.

PROPOSITION 2.1.4 ([2]). $\mathfrak{f}_4{}^C=\mathfrak{d}_4{}^C\oplus(\widetilde{\mathfrak{m}^C})^-$
where $\mathfrak{d}_4{}^C=\{\delta\in\mathfrak{f}_4{}^C\,|\,\delta E_i=0,\ i=1, 2, 3\}$ *is the complex Lie algebra of type* D_4, $(\mathfrak{m}^C)^-=\{A\in M(3, \mathfrak{C}^C)\,|\,A^*=-A\}$ *and for* $A\in(\mathfrak{m}^C)^-$, $\tilde{A}$ *is the C-linear transformation of* $\mathfrak{J}^C$ *defined by* $\tilde{A}X=AX-XA$, $X\in\mathfrak{J}^C$.

2.2. Involutions of Lie groups of type F_4.

We define $\boldsymbol{R}$-linear transformations γ, σ, σ' of $\mathfrak{J}(3, \mathfrak{C})$ by

$$\gamma X=\gamma X(\xi, x)=X(\xi, \gamma x), \qquad X\in\mathfrak{J}(3, \mathfrak{C}),$$

$$\sigma X=\begin{pmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}=I_1XI_1, \qquad \sigma' X=\begin{pmatrix} \xi_1 & x_3 & -\bar{x}_2 \\ \bar{x}_3 & \xi_2 & -x_1 \\ -x_2 & -\bar{x}_1 & \xi_3 \end{pmatrix}=I_2XI_2,$$

respectively. Then $\gamma\in G_2\subset F_4\subset F_4{}^C$, $\sigma, \sigma'\in F_4\subset F_4{}^C$ and $\gamma^2=\sigma^2=\sigma'^2=1$. Let τ be the complex conjugation in $\mathfrak{J}^C$ with respect to $\mathfrak{J}(3, \mathfrak{C})$, then $\tau, \tau\gamma, \tau\sigma$ induce involutive automorphisms $\tilde{\tau}, \widetilde{\tau\gamma}, \widetilde{\tau\sigma}$ of $F_4{}^C$:

$$\tilde{\tau}(\alpha)=\tau\alpha\tau,\quad \widetilde{\tau\gamma}(\alpha)=\tau\gamma\alpha\gamma\tau,\quad \widetilde{\tau\sigma}(\alpha)=\tau\sigma\alpha\sigma\tau,\quad \alpha\in F_4{}^C.$$

LEMMA 2.2.1. $(\mathfrak{J}^C)_\tau=\mathfrak{J}(3, \mathfrak{C})$, $(\mathfrak{J}^C)_{\tau\gamma}\simeq\mathfrak{J}(3, \mathfrak{C}')$, $(\mathfrak{J}^C)_{\tau\sigma}\simeq\mathfrak{J}(1, 2, \mathfrak{C})$.

PROOF. The first two are trivial (Lemma 1.2.1. (2)). The correspondence

$$(\mathfrak{J}^C)_{\tau\sigma} \ni \begin{pmatrix} \xi_1 & ix_3 & i\bar{x}_2 \\ i\bar{x}_3 & \xi_2 & x_1 \\ ix_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \in \mathfrak{J}(1, 2, \mathfrak{C}),$$

$\xi_i \in \boldsymbol{R}$, $x_i \in \mathfrak{C}$, gives an isomorphism as Jordan algebras.

THEOREM 2.2.2. $(F_4{}^C)^\tau = F_4$, $(F_4{}^C)^{\tau\gamma} \cong F_{4(4)}$, $(F_4{}^C)^{\tau\sigma} \cong F_{4(-20)}$.

PROOF. These are direct results of Lemma 2.2.1.

PROPOSITION 2.2.3. (1) *γ and $\gamma\sigma$ are conjugate in F_4: $\delta\gamma = \gamma\sigma\delta$ (moreover under $\delta \in F_4$ such that $\delta\sigma = \sigma\delta$).*

(2) *σ and σ' are conjugate in F_4: $\delta\sigma = \sigma'\delta$ (moreover under $\delta = \delta^{-1} \in F_4$).*

PROOF. (1) Define $\delta : \mathfrak{J}(3, \mathfrak{C}) \to \mathfrak{J}(3, \mathfrak{C})$ by

$$\delta X = \begin{pmatrix} \xi_1 & x_3 e & \bar{x}_2 e \\ -e\bar{x}_3 & \xi_2 & -ex_1 e \\ -xe_2 & -e\bar{x}_1 e & \xi_3 \end{pmatrix} = \bar{D} X D, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}.$$

Then $\delta \in F_4$, $\delta\gamma = \gamma\sigma\delta$ and $\delta\sigma = \sigma\delta$.

(2) Define $\delta : \mathfrak{J}(3, \mathfrak{C}) \to \mathfrak{J}(3, \mathfrak{C})$ by

$$\delta X = \begin{pmatrix} \xi_3 & \bar{x}_1 & x_2 \\ x_1 & \xi_2 & \bar{x}_3 \\ \bar{x}_2 & x_3 & \xi_1 \end{pmatrix} = DXD, \qquad D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\delta = \delta^{-1} \in F_4$ and $\delta\sigma = \sigma'\delta$.

2.3. Subgroups of type $C_1 \oplus C_3$ of Lie groups of type F_4.

LEMMA 2.3.1. *Any element $M \in \mathfrak{J}(3, \boldsymbol{H}^C)$ such that $M^2 = M$, $\mathrm{tr}(M) = 1$ can be transformed to any E_i by a certain $A \in Sp(3, \boldsymbol{H}^C)$: $AMA^* = E_i$ ($i = 1, 2, 3$).*

PROOF. Since $Sp(3, \boldsymbol{H}^C)$ contains the subgroup $Sp(3)$, we may assume

$$M = \begin{pmatrix} \mu_1 & im_3 & i\bar{m}_2 \\ i\bar{m}_3 & \mu_2 & im_1 \\ im_2 & i\bar{m}_1 & \mu_3 \end{pmatrix}, \qquad \begin{matrix} \mu_i \in C,\ m_i \in \boldsymbol{H}, \\ \mu_1 + \mu_2 + \mu_3 = 1. \end{matrix}$$

The condition $M^2 = M$ is

$$\begin{pmatrix} \mu_1{}^2-m_2\bar{m}_2-m_3\bar{m}_3 & -\bar{m}_2\bar{m}_1+i(\mu_1+\mu_2)m_3 & * \\ * & \mu_2{}^2-m_3\bar{m}_3-m_1\bar{m}_1 & -\bar{m}_3\bar{m}_2+i(\mu_2+\mu_3)m_1 \\ -\bar{m}_1\bar{m}_3+i(\mu_3+\mu_1)m_2 & * & \mu_3{}^2-m_1\bar{m}_1-m_2\bar{m}_2 \end{pmatrix}=M.$$

Compare the diagonals, then each μ_i is real. Hence we have

$$m_1m_2=m_2m_3=m_3m_1=0, \qquad \mu_1m_1=\mu_2m_2=\mu_3m_3=0.$$

If $m_1=m_2=m_3=0$ Lemma is clearly valid. Otherwise, for example, in the case $m_3\neq 0$, we have $m_1=m_2=0$, $\mu_1+\mu_2=1$, $\mu_3=0$. Hence M has the form $M=\begin{pmatrix} M & 0 \\ 0 & 0\end{pmatrix}$, $M=\begin{pmatrix} \mu & im \\ i\bar{m} & \nu\end{pmatrix}$, $m\bar{m}=-\mu\nu$, $\mu+\nu=1$, μ, $\nu\in \boldsymbol{R}$, $m\in \boldsymbol{H}$. If $\mu>0$, $\nu<0$, this M can be transformed to E_1 by $\begin{pmatrix} \bar{m}/\sqrt{-\nu} & i\sqrt{-\nu} \\ -i\bar{m}/\sqrt{\mu} & \sqrt{\mu}\end{pmatrix}\in Sp(2, \boldsymbol{H}^C)$. If $\mu<0$, $\nu>0$, then M can be transformed to E_2. Finally note that E_1, E_2, E_3 are transformed by $Sp(3, \boldsymbol{H}^C)$ with one another. Thus Lemma is proved.

PROPOSITION 2.3.2. $F_4(\mathfrak{J}(3, \boldsymbol{H}^C))\cong Sp(3, C)/\boldsymbol{Z}_2$.

PROOF ([6]). We define $\phi: Sp(3, \boldsymbol{H}^C)\to F_4(\mathfrak{J}(3, \boldsymbol{H}^C))$ by

$$\phi(A)M=AMA^*, \qquad M\in\mathfrak{J}(3, \boldsymbol{H}^C).$$

It is clear that ϕ is well-defined and a homomorphism. We shall show ϕ is onto. Let $\alpha\in F_4(\mathfrak{J}(3, \boldsymbol{H}^C))$. Since $\alpha E_i\in\mathfrak{J}(3, \boldsymbol{H}^C)$ satisfies $(\alpha E_i)^2=\alpha E_i$, $\mathrm{tr}(\alpha E_i)=1$, there exists $A_i\in Sp(3, \boldsymbol{H}^C)$ such that

$$\alpha E_i=A_iE_iA_i{}^*, \quad i=1, 2, 3 \quad \text{(Lemma 2.3.1)}.$$

Let $\boldsymbol{a}_i$ be the i-th column vector of A_i and construct a matrix $A=(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$. Then we have $\alpha E_i=AE_iA^*$, $i=1, 2, 3$. Hence $AA^*=A(E_1+E_2+E_3)A^*=\alpha(E_1+E_2+E_3)=\alpha E=E$, that is, $A\in Sp(3, \boldsymbol{H}^C)$. Put $\beta=\phi(A)^{-1}\alpha$, then $\beta\in F_4(\mathfrak{J}(3, \boldsymbol{H}^C))$ and satisfies $\beta E_i=E_i$, $i=1, 2, 3$. β induces C-linear transformations β_i of $\boldsymbol{H}^C$ such that $\beta F_i(m)=F_i(\beta_i m)$, $m\in\boldsymbol{H}^C$ from $2E_j\circ F_i(m)=F_i(m)$, $j\neq i$, moreover β_i are orthogonal: $\beta_i\in O(4, C)=O(\boldsymbol{H}^C)$ from $F_i(m)\circ F_i(n)=(m, n)(E_{i+1}+E_{i+2})$. Furthermore β_1, β_2, β_3 satisfy

$$(\beta_1 m)(\beta_2 n)=\overline{\beta_3(\overline{mn})}, \qquad m, n\in\boldsymbol{H}^C$$

from $2F_1(m)\circ F_2(n)=F_3(\overline{mn})$. Put $p=\beta_1 1$, $q=\beta_2 1$, then p, $q\in Sp(1, \boldsymbol{H}^C)$ and $\beta_2(m)=\bar{p}\beta_1(m)q$, $\beta_3(m)=\overline{\beta_1(\bar{m})}q$, $m\in\boldsymbol{H}^C$. Again put $\beta_1(m)=p\zeta(m)$, then ζ satisfies $(\zeta m)(\zeta n)=\zeta(mn)$, m, $n\in\boldsymbol{H}^C$, that is, ζ is an automorphism of $\boldsymbol{H}^C$. Hence there exists $r\in Sp(1, \boldsymbol{H}^C)$ such that $\zeta(m)=rm\bar{r}$, $m\in\boldsymbol{H}^C$ (Proposition 1.3.1). Therefore

$$\beta_1 m=prm\bar{r}, \quad \beta_2 m=rm\bar{r}q, \quad \beta_3 m=\bar{q}rm\bar{r}\bar{p}$$

Construct a matrix $B=\mathrm{diag}(\bar{q}r, pr, r)\in Sp(3, \boldsymbol{H}^C)$, then $\beta M=BMB^*$, $M\in\mathfrak{J}(3, \boldsymbol{H}^C)$, that is, $\beta=\phi(B)$. Hence $\alpha=\phi(A)\beta=\phi(A)\phi(B)=\phi(AB)$, $AB\in Sp(3, \boldsymbol{H}^C)$. Therefore ϕ is onto. $\mathrm{Ker}\,\phi=\{E, -E\}=\boldsymbol{Z}_2$. Thus $F_4(\mathfrak{J}(3, \boldsymbol{H}^C))\cong Sp(3, \boldsymbol{H}^C)/\boldsymbol{Z}_2\cong Sp(3, C)/\boldsymbol{Z}_2$.

THEOREM 2.3.3. *$(F_4{}^C)^\gamma\cong(Sp(1, C)\times Sp(3, C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(1, E), (-1, -E)\}$.*

PROOF ([6]). We define $\phi: Sp(1, \boldsymbol{H}^C)\times Sp(3, \boldsymbol{H}^C)\to(F_4{}^C)^\gamma$ by

$$\phi(p, A)(M+\boldsymbol{a})=AMA^*+p\boldsymbol{a}A^*, \qquad M+\boldsymbol{a}\in\mathfrak{J}(3, \boldsymbol{H}^C)\oplus(\boldsymbol{H}^C)^3=\mathfrak{J}^C.$$

It is easy to verify that ϕ is well-defined (Proposition 2.1.3. (5)) and a homomorphism. We shall show ϕ is onto. Let $\alpha\in(F_4{}^C)^\gamma$. Since $(\mathfrak{J}^C)_\gamma=\mathfrak{J}(3, \boldsymbol{H}^C)$ is invariant under α, α induces an automorphism of $\mathfrak{J}(3, \boldsymbol{H}^C)$. Hence there exists $A\in Sp(3, \boldsymbol{H}^C)$ such that

$$\alpha M=AMA^*, \quad M\in\mathfrak{J}(3, \boldsymbol{H}^C) \quad \text{(Proposition 2.3.2)}.$$

Put $\beta=\phi(1, A)^{-1}\alpha$, then $\beta|\mathfrak{J}(3, \boldsymbol{H}^C)=1$, hence $\beta\in G_2{}^C=\{\alpha\in F_4{}^C\,|\,\alpha E_i=E_i,\ \alpha F_i(1)=F_i(1),\ i=1, 2, 3\}$, moreover $\beta\in(G_2{}^C)^\gamma$ and $\beta|\boldsymbol{H}^C=1$. By Theorem 1.3.2, there exists $p\in Sp(1, \boldsymbol{H}^C)$ such that $\beta(m+ae)=m+(pa)e$, $m+ae\in\boldsymbol{H}^C\oplus\boldsymbol{H}^Ce=\mathfrak{C}^C$, hence $\beta(M+\boldsymbol{a})=M+p\boldsymbol{a}$, $M+\boldsymbol{a}\in\mathfrak{J}^C$, that is, $\beta=\phi(p, E)$. Hence $\alpha=\phi(1, A)\beta=\phi(1, A)\phi(p, E)=\phi(p, A)$. Therefore ϕ is onto. $\mathrm{Ker}\,\phi=\{(1, E), (-1, -E)\}=\boldsymbol{Z}_2$. Thus we have the required isomorphism.

LEMMA 2.3.4. *$\phi: Sp(1, \boldsymbol{H}^C)\times Sp(1, \boldsymbol{H}^C)\to F_4{}^C$ of Theorem 2.3.3 satisfies*

(1) $\gamma=\phi(-1, E)$, $\gamma_C=\phi(\boldsymbol{j}, \boldsymbol{j}E)$, $\gamma_H=\phi(\boldsymbol{i}, \boldsymbol{i}E)$, $\sigma=\phi(-1, I_1)$.

(2) $\tau\phi(p, A)\tau=\phi(\tau p, \tau A)$, $\gamma_C\phi(p, A)\gamma_C=\phi(\gamma_C p, \gamma_C A)$, $\sigma\phi(p, A)\sigma=\phi(p, I_1AI_1)$.

THEOREM 2.3.5. (1) $(F_4)^\gamma\cong(Sp(1)\times Sp(3))/\boldsymbol{Z}_2\cong(F_{4(4)})^\gamma$.

(2) $(F_{4(4)})^\gamma\sim(\tau\gamma_C)^\gamma\cong(Sp(1, \boldsymbol{R})\times Sp(3, \boldsymbol{R}))/\boldsymbol{Z}_2\times 2$.

(3) $(F_{4(-20)})^\gamma\cong(Sp(1)\times Sp(1, 2))/\boldsymbol{Z}_2\cong(\tau\gamma\sigma)^\gamma\sim(F_{4(4)})^\gamma$.

PROOF. (1) Let $\alpha\in(F_4)^\gamma=((F_4{}^C)^\tau)^\gamma=((F_4{}^C)^\gamma)^\tau\subset(F_4{}^C)^\gamma$. By Theorem 2.3.3, there exist $p\in Sp(1, \boldsymbol{H}^C)$, $A\in Sp(3, \boldsymbol{H}^C)$ such that $\alpha=\phi(p, A)$. From the condition $\tau\alpha=\alpha\tau$, we have $\phi(\tau p, \tau A)=\phi(p, A)$ (Lemma 2.3.4). Hence

$$\tau p=p, \quad \tau A=A \quad \text{or} \quad \tau p=-p, \quad \tau A=-A.$$

The latter case is impossible (cf. Theorem 1.3.4). Therefore $p\in Sp(1)$. $A\in Sp(3)$. Thus $(F_4)^\gamma\cong\phi(Sp(1)\times Sp(3))\cong(Sp(1)\times Sp(3))/\boldsymbol{Z}_2$. $(F_{4(4)})^\gamma=(\tau\gamma)^\gamma=(\tau)^\gamma$.

(2) $$F_{4(4)}=(F_4{}^C)^{\tau\gamma}\cong(F_4{}^C)^{\tau\gamma_C}.$$

In fact, since γ and γ_C are conjugate in $G_2\subset F_4:\delta\gamma=\gamma_C\delta,\ \delta\tau=\tau\delta$ (Proposition 1.2.3), $(F_4{}^C)^{\gamma\tau}\ni\alpha\to\delta\alpha\delta^{-1}\in(F_4{}^C)^{\tau\gamma_C}$ gives an isomorphism. Let $\alpha\in((F_4{}^C)^{\tau\gamma_C})^{\gamma}=(\tau\gamma_C)^{\gamma}$, $\alpha=\varphi(p, A)$, $p\in Sp(1, \boldsymbol{H}^C)$, $A\in Sp(3, \boldsymbol{H}^C)$. From $\tau\gamma_C\alpha=\alpha\tau\gamma_C$, we have $\varphi(\tau\gamma_C p, \tau\gamma_C A)=\varphi(p, A)$. Hence $(\tau\gamma_C)^{\gamma}\cong(Sp(1, \boldsymbol{H}')\times Sp(3, \boldsymbol{H}')\cup \boldsymbol{i}Sp(1, \boldsymbol{H}')\times(\boldsymbol{i}E)Sp(3, \boldsymbol{H}'))/\boldsymbol{Z}_2$ (cf. Theorem 1.3.5)$\cong(Sp(1, \boldsymbol{R})\times Sp(3, \boldsymbol{R}))/\boldsymbol{Z}_2\times 2$. $(\varphi(\boldsymbol{i}, \boldsymbol{i}E)=\gamma_H)$.

(3) Define $\phi: Sp(1, \boldsymbol{H}^C)\times Sp(1, 2, \boldsymbol{H}^C)\to(F_4{}^C)^{\gamma}$ by $\phi(p, A)=\varphi(p, \Gamma_1 A\Gamma_1{}^{-1})$. Let $\alpha\in(F_{4(-20)})^{\gamma}=(\tau\sigma)^{\gamma}$, $\alpha=\phi(p, A)$, $p\in Sp(1, \boldsymbol{H}^C)$, $A\in Sp(1, 2, \boldsymbol{H}^C)$. From $\tau\sigma\alpha=\alpha\tau\sigma$, we have $\phi(\tau p, \tau A)=\phi(p, A)$. Thus, as in (1), $(F_{4(-20)})^{\gamma}\cong(Sp(1)\times Sp(1, 2))/\boldsymbol{Z}_2$.

$$F_{4(4)}=(F_4{}^C)^{\tau\gamma}\cong(F_4{}^C)^{\tau\gamma\sigma}$$

because $\gamma\sim\gamma\sigma$ under $\delta\in F_4:\delta\gamma=\gamma\sigma\delta,\ \delta\tau=\tau\delta$ (Proposition 2.2.3). Now $(F_{4(4)})^{\gamma}\sim(\tau\gamma\sigma)^{\gamma}=(\tau\sigma)^{\gamma}$.

2.4. Subgroups of type B_4 of Lie groups of type F_4.

Hereafter we use the following C-vector subspaces of $\mathfrak{J}^C$.

$$\mathfrak{J}(2, \mathfrak{C}^C)=\{X\in\mathfrak{J}^C\mid E_1\circ X=0\}=\{X\in\mathfrak{J}^C\mid 4E_1\times(E_1\times X)=X\}$$

$$=\left\{\begin{pmatrix}0&0&0\\0&\xi_2&x\\0&\bar{x}&\xi_3\end{pmatrix}\overline{\overline{\text{identify}}}\begin{pmatrix}\xi_2&x\\\bar{x}&\xi_3\end{pmatrix}\middle|\ \xi_2, \xi_3\in C,\ x\in\mathfrak{C}^C\right\},$$

$$\mathfrak{E}_1{}^C=\{\xi E_1\mid\xi\in C\},$$

$$(\mathfrak{J}^C)_{\sigma}=\{X\in\mathfrak{J}^C\mid\sigma X=X\}=\mathfrak{J}(2, \mathfrak{C}^C)\oplus\mathfrak{E}_1{}^C,$$

$$(\mathfrak{J}^C)_{-\sigma}=\{X\in\mathfrak{J}^C\mid\sigma X=-X\}=\{X\in\mathfrak{J}^C\mid 2E_1\circ X=X\}$$

$$=\{X\in\mathfrak{J}^C\mid E_1\times X=0,\ (E_1, X)=0\}$$

$$=\left\{\begin{pmatrix}0&x_3&\bar{x}_2\\\bar{x}_3&0&0\\x_2&0&0\end{pmatrix}\middle|\ x_2, x_3\in\mathfrak{C}^C\right\}$$

and $(\mathfrak{J}^C)_0=\{X\in\mathfrak{J}^C\mid\mathrm{tr}(X)=0\}$, $\mathfrak{J}(2, \mathfrak{C}^C)_0=\{X\in\mathfrak{J}(2, \mathfrak{C}^C)\mid\mathrm{tr}(X)=0\}$. $(\mathfrak{J}^C)_{\sigma}$, $(\mathfrak{J}^C)_{-\sigma}$ are invariant under $\alpha\in(F_4{}^C)^{\sigma}$.

LEMMA 2.4.1. $(F_4{}^C)^{\sigma}=(F_4{}^C)_{E_1}$.

PROOF. Let $\alpha\in(F_4{}^C)^{\sigma}$. Then $\alpha E_2\in\mathfrak{J}(2, \mathfrak{C}^C)$. In fact, $\alpha E_2=\alpha(-F_2(1)\times F_2(1))=-\alpha F_2(1)\times\alpha F_2(1)=-(F_2(x_2)+F_3(x_3))^{\times 2}=x_2\bar{x}_2E_2+x_3\bar{x}_3E_3-F_1(\overline{x_2x_3})\in\mathfrak{J}(2, \mathfrak{C}^C)$. Similarly $\alpha E_3\in\mathfrak{J}(2, \mathfrak{C}^C)$. Therefore $\alpha E_1=E-\alpha E_2-\alpha E_3$ has the form

$E_1+\xi_2E_2+\xi_3E_3+F_1(x)$. Then $0=\alpha(E_1\times E_1)=\alpha E_1\times\alpha E_1=(E_1+\xi_2E_2+\xi_3E_3+F_1(x))^{\times 2}$ $=(\xi_2\xi_3-x\bar{x})E_1+\xi_3E_2+\xi_2E_3-F_1(x)$. This implies $\xi_2=\xi_3=x=0$. Thus we have $\alpha E_1=E_1$. Conversely let $\alpha\in(F_4{}^C)_{E_1}$. Since $\mathfrak{J}^C=(\mathfrak{J}^C)_\sigma\oplus(\mathfrak{J}^C)_{-\sigma}$ and $(\mathfrak{J}^C)_\sigma=$ $\{X\in\mathfrak{J}^C\,|\,E_1\circ X=0\}\oplus\{\xi E_1\,|\,\xi\in C\}$, $(\mathfrak{J}^C)_{-\sigma}=\{X\in\mathfrak{J}^C\,|\,2E_1\circ X=X\}$ are invariant under α, $\alpha\sigma X=\alpha\sigma(X_1+X_2)=\alpha(X_1-X_2)=\alpha X_1-\alpha X_2=\sigma(\alpha X_1)+\sigma(\alpha X_2)=\sigma\alpha(X_1+X_2)=\sigma\alpha X$ for $X=X_1+X_2$, $X_1\in(\mathfrak{J}^C)_\sigma$, $X_2\in(\mathfrak{J}^C)_{-\sigma}$. Hence $\alpha\sigma=\sigma\alpha$, that is, $\alpha\in(F_4{}^C)^\sigma$.

LEMMA 2.4.2. $(F_4{}^C)^\sigma/Spin(8,C)\simeq(S^C)^8$. *In particular, the group* $(F_4{}^C)^\sigma$ *is connected.*

PROOF. We define a complex 8-dimensional sphere $(S^C)^8$ by

$$(S^C)^8=\{X\in\mathfrak{J}^C\,|\,E_1\circ X=0,\ \mathrm{tr}(X)=0,\ (X,X)=2\}$$
$$=\left\{\begin{pmatrix}\xi & x\\ \bar{x} & -\xi\end{pmatrix}\,\middle|\,\xi^2+\bar{x}x=1,\ \xi\in C,\ x\in\mathfrak{C}^C\right\}.$$

The group $(F_4{}^C)^\sigma=(F_4{}^C)_{E_1}$ acts on $(S^C)^8$ (Lemma 2.1.2, Proposition 2.1.3.(3)). We shall show that this action is transitive. To show this we prepare some elements of $(F_4{}^C)^\sigma$.

For $a\in\mathfrak{C}^C$ such that $a\bar{a}\neq 0$, define a C-linear transformation $\alpha(a)$ of $\mathfrak{J}^C$, $\alpha(a)X(\xi,x)=Y(\eta,y)$, by

$$\begin{cases}\eta_1=\xi_1,\\[4pt] \eta_2=\dfrac{1}{2}(\xi_2+\xi_3)+\dfrac{1}{2}(\xi_2-\xi_3)\cos 2\nu+(a,x_1)\dfrac{\sin 2\nu}{\nu},\\[4pt] \eta_3=\dfrac{1}{2}(\xi_2+\xi_3)-\dfrac{1}{2}(\xi_2-\xi_3)\cos 2\nu-(a,x_1)\dfrac{\sin 2\nu}{\nu},\end{cases}$$

$$\begin{cases}y_1=x_1-\dfrac{1}{2}(\xi_2-\xi_3)a\dfrac{\sin 2\nu}{\nu}-2(a,x_1)a\dfrac{\sin^2\nu}{\nu^2},\\[4pt] y_2=x_2\cos\nu-\overline{x_3a}\dfrac{\sin\nu}{\nu},\\[4pt] y_3=x_3\cos\nu+\overline{a x_2}\dfrac{\sin\nu}{\nu}\end{cases}$$

where $\nu\in C$, $\nu^2=a\bar{a}$. Then $\alpha(a)=\exp\tilde{A}(a)$ where $A(a)=\begin{pmatrix}0&0&0\\0&0&a\\0&-\bar{a}&0\end{pmatrix}\in((\mathfrak{m}^C)^-)_{E_1}$ $=\{A\in(\mathfrak{m}^C)^-\,|\,\tilde{A}E_1=0\}$, hence $\tilde{A}(a)\in(\mathfrak{f}_4{}^C)^\sigma=\mathfrak{d}_4{}^C\oplus(\tilde{\mathfrak{m}})^C)^-)_{E_1}$ (Proposition 2.1.4). Therefore $\alpha(a)\in(F_4{}^C)^\sigma$.

Now let $X=\begin{pmatrix}\xi & x\\ \bar{x} & -\xi\end{pmatrix}\in(S^C)^8$. Choose $a\in\mathfrak{C}^C$ such that $(a,x)=0$ and $a\bar{a}=$

$(\pi/4)^2$, then $\alpha(a)X=X_1=\begin{pmatrix} 0 & x_1 \\ \bar{x}_1 & 0 \end{pmatrix}$, $x_1\bar{x}_1=1$. And then $\alpha((\pi/4)x_1)X_1=E_2-E_3$. This shows the transitivity. The isotropy subgroup of $(F_4{}^C)^\sigma$ at E_2-E_3 is $(F_4{}^C)_{E_1,E_2,E_3}=\{\alpha\in F_4{}^C \mid \alpha E_i=E_i,\ i=1,2,3\}$ and we hnow that it is isomorphic to $Spin(8, C)$ as the universal covering group of $SO(8, C)=SO(\mathfrak{C}^C)$ (cf. Principle of triality [8]). Thus we have the homeomorphism $(F_4{}^C)^\sigma/Spin(8, C)\simeq(S^C)^8$.

THEOREM 2.4.3. $(F_4{}^C)^\sigma\cong Spin(9, C)$.

PROOF. Since the group $(F_4{}^C)^\sigma$ is connected (Lemma 2.4.2), we can define a homomorphism $\pi:(F_4{}^C)^\sigma\to SO(9, C)=SO((V^C)^9)$ by $\pi(\alpha)=\alpha\,|\,(V^C)^9$ where

$$(V^C)^9=\mathfrak{J}(2, \mathfrak{C}^C)_0=\left\{X=\begin{pmatrix} \xi & x \\ \bar{x} & -\xi \end{pmatrix} \middle|\, \xi\in C,\ x\in\mathfrak{C}^C\right\}$$

with the norm $(X, X)/2=\xi^2+x\bar{x}$. $\mathrm{Ker}\,\pi=\{1, \sigma\}=Z_2$ (cf. Principle of triality [8]). Hence π induces a monomorphisim $d\pi:(\mathfrak{f}_4{}^C)^\sigma\to\mathfrak{so}(9, C)$. Since $\dim_C(\mathfrak{f}_4{}^C)^\sigma=\dim_C(\mathfrak{d}_4{}^C\oplus((\tilde{\mathfrak{m}}^C)^-)_{E_1})=28+8=36=\dim_C\mathfrak{so}(9, C)$, $d\pi$ is onto, hence π is also onto (Lemma 0.6). Thus $(F_4{}^C)^\sigma/Z_2\cong SO(9, C)$. Therefore $(F_4{}^C)^\sigma$ is isomorphic to $Spin(9, C)$ as the universal covering group of $SO(9, C)$.

THEOREM 2.4.4. (1) $(F_4)^\sigma\cong Spin(9)\cong(F_{4(-20)})^\sigma$.

(2) $(F_{4(4)})^\sigma\cong spin(4, 5)$.

(3) $(F_{4(-20)})^\sigma\sim(\tau\sigma')^\sigma\cong Spin(8, 1)$.

PROOF. (1) $(F_4)^\sigma=((F_4{}^C)^\tau)^\sigma=((F_4{}^C)^\sigma)^\tau$ is connected (Lemma 0.7) because $(F_4{}^C)^\sigma=Spin(9, C)$ (Theorem 2.4.3) is simply connected. Since $(F_4{}^C)^\sigma$ acts on $(V^C)^9$, the group $(F_4)^\sigma=((F_4{}^C)^\sigma)^\tau$ acts on

$$V^9=(\mathfrak{J}(2, \mathfrak{C}^C)_0)_\tau=\left\{X=\begin{pmatrix} \xi & x \\ \bar{x} & -\xi \end{pmatrix} \middle|\, \xi\in \boldsymbol{R},\ x\in\mathfrak{C}\right\}$$

with the norm $(X, X)/2=\xi^2+x\bar{x}$. We can define a homomorphism $\pi:(F_4)^\sigma\to SO(9)=SO(V^9)$ by $\pi(\alpha)=\alpha\,|\,V^9$. $\mathrm{Ker}\,\pi=\{1, \sigma\}=Z_2$. Since $\dim(\mathfrak{f}_4)^\sigma=36=\dim\mathfrak{so}(9)$, π is onto. Thus $(F_4)^\sigma/Z_2\cong SO(9)$. Therefore $(F_4)^\sigma$ is isomorphic to $Spin(9)$ as the uuiversal covering group of $SO(9)$. $(F_{4(-20)})^\sigma=(\tau\sigma)^\sigma=(\tau)^\sigma$.

(REMARK. In the proof of Lemma 2.4.3, if we know that $F_4{}^C$ is simply connected, the connectedness of $(F_4{}^C)^\sigma$ is trivial (Lemma 0.7). But the simply connectedness of $F_4{}^C$ is usually follows from the simply connectedness of F_4 and the fact that $(F_4)^\sigma=Spin(9)$ ([8]). To avoid a circular argument we took the way like Lemma 2.4.2, Theorem 2.4.3).

(2) As in (1), $(F_{4(4)})^{\sigma}=((F_4{}^C)^{\sigma})^{\tau\gamma}$ is connected. The group $(F_{4(4)})^{\sigma}$ acts on

$$V^{4,5}=(\mathfrak{J}(2,\mathfrak{C}^C)_0)_{\tau\gamma}=\left\{X=\begin{pmatrix}\xi & x'\\ \bar{x}' & -\xi\end{pmatrix}\middle|\xi\in\boldsymbol{R},\ x'\in(\mathfrak{C}^C)_{\tau\gamma}=\mathfrak{C}'\right\}$$

with the norm $(X,X)/2=\xi^2+x'\bar{x}'$. We can define a homomorphism $\pi:(F_{4(4)})^{\sigma}\to O(4,5)_0=O(V^{4,5})_0$ by $\pi(\alpha)=\alpha|V^{4,5}$. $\operatorname{Ker}\pi=\{1,\sigma\}=Z_2$. As similar to (1), $(F_{4(4)})^{\sigma}/Z_2\cong O(4,5)_0$. Therefore $(F_{4(4)})^{\sigma}$ is denoted by $spin(4,5)$ (not simply connected) as a double covering group of $O(4,5)_0$.

(3) $$F_{4(-20)}=(F_4{}^C)^{\tau\sigma}\cong(F_4{}^C)^{\tau\sigma'}$$

because $\sigma\sim\sigma'$ under $\delta\in F_4:\delta\sigma=\sigma'\delta,\ \delta\tau=\tau\delta$ (Proposition 2.2.3). As in (1), $((F_4{}^C)^{\tau\sigma'})^{\sigma}=(\tau\sigma')^{\sigma}$ is connected. The group $(\tau\sigma')^{\sigma}$ acts on

$$V^{8,1}=(\mathfrak{J}(2,\mathfrak{C}^C)_0)_{\tau\sigma'}=\left\{X=\begin{pmatrix}\xi & ix\\ i\bar{x} & -\xi\end{pmatrix}\middle|\xi\in\boldsymbol{R},\ x\in\mathfrak{C}\right\}$$

with the norm $(X,X)/2=\xi^2-x\bar{x}$. We can define a homomorphism $\pi:(\tau\sigma')^{\sigma}\to O(8,1)_0=O(V^{8,1})_0$ by $\pi(\alpha)=\alpha|V^{8,1}$. As similar to (1), $(\tau\sigma')/^{\sigma}Z_2\cong O(8,1)_0$. Therefore $(\tau\sigma')^{\sigma}$ is isomorphic to $Spin(8,1)$ as the universal covering group of $O(8,1)_0$.

Group E_6

3.1. Lie groups of type E_6.

The universal connected linear Lie groups of type E_6 are obtained as

$$E_6{}^C=E_6(\mathfrak{J}(3,\mathfrak{C}^C))=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}(3,\mathfrak{C}^C))\,|\,\det\alpha X=\det X\},$$

$$E_6=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}(3,\mathfrak{C}^C))\,|\,\det\alpha X=\det X,\ \langle\alpha X,\alpha Y\rangle=\langle X,Y\rangle\},$$

$$E_{6(6)}=E_6(\mathfrak{J}(3,\mathfrak{C}'))=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}(3,\mathfrak{C}'))\,|\,\det\alpha X=\det X\},$$

$$E_{6(2)}=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}(3,\mathfrak{C}^C))\,|\,\det\alpha X=\det X,\ \langle\alpha X,\alpha Y\rangle_{\gamma}=\langle X,Y\rangle_{\gamma}\},$$

$$E_{6(-14)}=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}(3,\mathfrak{C}^C))\,|\,\det\alpha X=\det X,\ \langle\alpha X,\alpha Y\rangle_{\sigma}=\langle X,Y\rangle_{\sigma}\},$$

$$E_{6(-26)}=E_6(\mathfrak{J}(3,\mathfrak{C}))=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}(3,\mathfrak{C}))\,|\,\det\alpha X=\det X\}$$

where $\langle X,Y\rangle=(\tau X,Y)$, $\langle X,Y\rangle_{\gamma}=(\tau\gamma X,Y)$ and $\langle X,Y\rangle_{\sigma}=(\tau\sigma X,Y)$. (Similarly the group $E_6(\mathfrak{J}(3,\boldsymbol{H}^C))$ is defined). $E_6{}^C$, E_6, $E_{6(-26)}$ are simply connected (see Appendix).

The Lie algebra $\mathfrak{e}_6{}^C$ of the Lie group $E_6{}^C$ has the following structure.

PROPOSITION 3.1.1 ([2]). $\mathfrak{e}_6{}^C=\mathfrak{f}_4{}^C\oplus\tilde{\mathfrak{J}}(3,\mathfrak{C}^C)_0$

where $\mathfrak{J}(3, \mathfrak{C}^C)_0 = \{T \in \mathfrak{J}(3, \mathfrak{C}^C) \mid \mathrm{tr}(T) = 0\}$ *and for* $T \in \mathfrak{J}(3, \mathfrak{C}^C)_0$, $\tilde{T}$ *is the C-linear transformation of* $\mathfrak{J}^C$ *defined by* $\tilde{T}X = T \circ X$.

3.2. Involutions of Lie groups of type E_6.

LEMMA 3.2.1. *If* $\alpha \in E_6{}^C$ *then* ${}^t\alpha^{-1} \in E_6{}^C$.

PROOF. ([4]). ${}^t\alpha^{-1}(Y \times Y) \times {}^t\alpha^{-1}(Y \times Y) = (\alpha Y \times \alpha Y) \times (\alpha Y \times \alpha Y)$ (Lemma 2.1.1) $= (\det \alpha Y)\alpha Y = (\det Y)\alpha Y = \alpha((\det Y)Y) = \alpha((Y \times Y) \times (Y \times Y))$, $Y \in \mathfrak{J}^C$. Put $Y = X \times X$, $X \in \mathfrak{J}^C$, then ${}^t\alpha^{-1}((\det X)X) \times {}^t\alpha^{-1}((\det X)X) = \alpha((\det X)X \times (\det X)X)$.

(1) Case $\det X \neq 0$. We have ${}^t\alpha^{-1}X \times {}^t\alpha^{-1}X = \alpha(X \times X)$. Hence $3\det {}^t\alpha^{-1}X = ({}^t\alpha^{-1}X, {}^t\alpha^{-1}X \times {}^t\alpha^{-1}X) = ({}^t\alpha^{-1}X, \alpha(X \times X)) = (X, X \times X) = 3\det X$. Consider α^{-1} instead of α, then we have also $\det {}^t\alpha X = \det X$.

(2) Case $\det X = 0$. If $\det {}^t\alpha^{-1}X \neq 0$, we can use the result of (1). $0 = \det X = \det {}^t\alpha({}^t\alpha^{-1}X) = \det {}^t\alpha^{-1}X$ (result of (1)) $\neq 0$, a contradiction. Thus $\det {}^t\alpha^{-1}X = 0$, hence $\det {}^t\alpha^{-1}X = \det X$ is also valid.

We define an involutive automorphism λ of $E_6{}^C$ by

$$\lambda(\alpha) = {}^t\alpha^{-1}, \quad \alpha \in E_6{}^C \quad \text{(Lemma 3.2.1)}.$$

Note that λ induces involutive automorphisms of E_6, $E_{6(6)}$, $E_{6(2)}$, $E_{6(-14)}$, $E_{6(-26)}$ and $E_6(\mathfrak{J}(3, \boldsymbol{H}^C))$. As in $F_4{}^C$, $E_6{}^C$ has involutive automorphisms $\tilde{\tau}$, $\widetilde{\tau\gamma}$, and $\widetilde{\tau\sigma}$.

THEOREM 3.2.2. $(E_6{}^C)^{\tau\lambda} = E_6$, $(E_6{}^C)^{\tau\gamma} \cong E_{6(6)}$, $(E_6{}^C)^{\tau\lambda\gamma} = E_{6(2)}$, $(E_6{}^C)^{\tau\lambda\sigma} = E_{6(-14)}$, $(E_6{}^C)^{\tau} = E_{6(-26)}$.

PROOF. As for $E_{6(6)}$, $E_{6(-26)}$, these are direct results of Lemma 1.2.1.(2). E_6, $E_{6(2)}$, $E_{6(-14)}$ are nothing but their definitions.

The Lie algebras of the Lie groups of type E_6 are as follows.

PROPOSITION 3.2.3. (1) $\mathfrak{e}_6 = \{\phi \in \mathfrak{e}_6{}^C \mid -\tau{}^t\phi\tau = \phi\} = \mathfrak{f}_4 \oplus i\tilde{\mathfrak{J}}(3, \mathfrak{C})_0$,

(2) $\mathfrak{e}_{6(6)} = \{\phi \in \mathfrak{e}_6{}^C \mid \tau\gamma\phi\gamma\tau = \phi\} = \mathfrak{f}_{4(4)} \oplus \tilde{\mathfrak{J}}(3, \mathfrak{C}')_0$.

(3) $\mathfrak{e}_{6(2)} = \{\phi \in \mathfrak{e}_6{}^C \mid -\tau\gamma{}^t\phi\gamma\tau = \phi\} = \mathfrak{f}_{4(4)} \oplus i\tilde{\mathfrak{J}}(3, \mathfrak{C}')_0$.

(4) $\mathfrak{e}_{6(-14)} = \{\phi \in \mathfrak{e}_6{}^C \mid -\tau\sigma{}^t\phi\sigma\tau = \phi\} = \mathfrak{f}_{4(-20)} \oplus i\tilde{\mathfrak{J}}(1, 2, \mathfrak{C})_0$.

(5) $\mathfrak{e}_{6(-26)} = \{\phi \in \mathfrak{e}_6{}^C \mid \tau\phi\tau = \phi\} = \mathfrak{f}_4 \oplus \tilde{\mathfrak{J}}(3, \mathfrak{C})_0$.

PROOF. The involutive automorphisms of $\mathfrak{e}_6{}^C$ induced by γ, σ, λ, τ are

$$\gamma\phi\gamma = \gamma\delta\gamma + \widetilde{\gamma T}, \quad \sigma\phi\sigma = \sigma\delta\sigma + \widetilde{\sigma T}, \quad \lambda(\phi) = \delta - \tilde{T}, \quad \tau\phi\tau = \tau\delta\tau + \widetilde{\tau T}$$

for $\delta+\tilde{T}\in\mathfrak{f}_4{}^C\oplus\tilde{\mathfrak{J}}(3, \mathfrak{C}^C)_0=\mathfrak{e}_6{}^C$. From this, Proposition is clear (Lemma 2.2.1).

In addition to γ, γ_C, $\gamma_H\in G_2\subset F_4\subset E_6$, σ, $\sigma'\in F_4\subset E_6$, we define one more involutive element $\rho\in E_6$, $\rho: \mathfrak{J}^C\to\mathfrak{J}^C$ by

$$\rho X=\begin{pmatrix} -\xi_1 & ix_3\boldsymbol{i} & -\boldsymbol{i}i\bar{x}_2 \\ i\bar{x}_3\boldsymbol{i} & -\xi_2 & -\boldsymbol{i}ix_1 \\ ix_2\boldsymbol{i} & i\bar{x}_1\boldsymbol{i} & \xi_3 \end{pmatrix}=\bar{P}XP, \qquad P=\begin{pmatrix} i\boldsymbol{i} & 0 & 0 \\ 0 & i\boldsymbol{i} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

PROPOSITION 3.2.3. (1) *γ and ρ are conjugate in E_6: $\delta\gamma=\rho\delta$, $\delta\in E_6$.*
(2) *σ and $\gamma\rho$ are conjugate in E_6: $\delta\sigma=\gamma\rho\delta$, $\delta\in E_6$.*
(3) *σ and $\gamma_H\rho$ are conjugate in E_6: $\delta\sigma=\gamma_H\rho\delta$, $\delta\in E_6$.*

PROOF will be given in 3.5.12.

3.3. Subgroups of type F_4 of Lie groups of type E_6.

THEOREM 3.3.1. (1) $(E_6{}^C)^\lambda=F_4{}^C$.
(2) $(E_{6(-26)})^\lambda=F_4=(E_6)^\lambda$.
(3) $(E_{6(6)})^\lambda=F_{4(4)}=(E_{6(2)})^\lambda$.
(4) $(E_{6(-14)})^\lambda=F_{4(-20)}\cong(E_{6(-26)})^{\lambda\sigma}$.

PROOF. (1) It is results of Proposition 2.1.3.(1)—(3).

(2) $(E_{6(-26)})^\lambda=(\tau)^\lambda=(\lambda)^\tau=(F_4{}^C)^\tau$ (result of (1))$=F_4$ (Theorem 2.2.2). $(E_6)^\lambda=(\tau\lambda)^\lambda=(\tau)^\lambda=(\lambda)^\tau$.

(3) $(E_{6(6)})^\lambda=(\tau\gamma)^\lambda=(\lambda)^{\tau\gamma}=(F_4{}^C)^{\tau\gamma}=F_{4(4)}$ (Theorem 2.2.2). $(E_{6(2)})^\lambda=(\tau\lambda\gamma)^\lambda=(\tau\gamma)^\lambda=(\lambda)^{\tau\gamma}$.

(4) $(E_{6(-14)})^\lambda=(\tau\lambda\sigma)^\lambda=(\tau\sigma)^\lambda=(\lambda)^{\tau\sigma}=(F_4{}^C)^{\tau\sigma}=F_{4(-20)}$ (Theorem 2.2.2).

$$(E_{6(-26)})^{\lambda\sigma}=(\tau)^{\lambda\sigma}\cong(\tau\sigma)^\lambda.$$

To prove this, define $\delta: \mathfrak{J}^C\to\mathfrak{J}^C$ by

$$\delta X=\begin{pmatrix} \xi_1 & ix_3 & i\bar{x}_2 \\ i\bar{x}_3 & -\xi_2 & -x_1 \\ ix_2 & -\bar{x}_1 & -\xi_3 \end{pmatrix}=DXD, \qquad D=\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}$$

(see Proposition 3.6.5), then $\delta\in E_6$, $\delta^2=\sigma$, $\delta\sigma=\sigma\delta$, $\delta\tau=\tau\delta^{-1}$, ${}^t\delta=\delta$. (Hereafter, this δ will be denoted by $\sqrt{\sigma}$). Now $(\tau)^{\lambda\sigma}\ni\alpha\to\delta^{-1}\alpha\delta\in(\tau\sigma)^\lambda$ gives an isomorphism.

3.4. Subgroups of type C_4 of Lie groups of type E_6.

We consider the Jordan algebra $\mathfrak{J}(4, \boldsymbol{H}^C)=\{P\in M(4, \boldsymbol{H}^C)\,|\,P^*=P\}$ with the Jordan multiplication $P\circ Q=(PQ+QP)/2$ and the inner product $(P, Q)=\mathrm{tr}(P\circ Q)$. We define $g: \mathfrak{J}^C=\mathfrak{J}(3, \boldsymbol{H}^C)\oplus(\boldsymbol{H}^C)^3\to\mathfrak{J}(4, \boldsymbol{H}^C)_0=\{P\in\mathfrak{J}(4, \boldsymbol{H}^C)\,|\,\mathrm{tr}(P)=0\}$ by

$$g(M+\boldsymbol{a})=\begin{pmatrix} \frac{1}{2}\mathrm{tr}(M) & i\boldsymbol{a} \\ i\boldsymbol{a}^* & M-\frac{1}{2}\mathrm{tr}(M)E \end{pmatrix}, \qquad M+\boldsymbol{a}\in\mathfrak{J}^C.$$

LEMMA 3.4.1. *$g: \mathfrak{J}^C\to\mathfrak{J}(4, \boldsymbol{H}^C)_0$ is a C-linear isomorphism and satisfies*

$$\begin{aligned} &gX\circ gY=g(\gamma(X\times Y))+\frac{1}{4}(\gamma X, Y)E, \\ &(gX, gY)=(\gamma X, Y), \end{aligned} \qquad X, Y\in\mathfrak{J}^C.$$

PROOF. $g(\gamma((M+\boldsymbol{a})\times(N+\boldsymbol{b})))=g((M-\boldsymbol{a})\times(N-\boldsymbol{b}))$

$$=g((M\times N-\frac{1}{2}(\boldsymbol{a}^*\boldsymbol{b}+\boldsymbol{b}^*\boldsymbol{a}))+\frac{1}{2}(\boldsymbol{a}N+\boldsymbol{b}M))$$

$$=\begin{pmatrix} \frac{1}{2}\mathrm{tr}(M\times N)-\frac{1}{2}(\boldsymbol{a}, \boldsymbol{b}) & \frac{i}{2}(\boldsymbol{a}N+\boldsymbol{b}M) \\ \frac{i}{2}(\boldsymbol{a}N+\boldsymbol{b}M)^* & M\times N-\frac{1}{2}(\boldsymbol{a}^*\boldsymbol{b}+\boldsymbol{b}^*\boldsymbol{a})-\frac{1}{2}(\mathrm{tr}(M\times N)-(\boldsymbol{a}, \boldsymbol{b}))E \end{pmatrix}$$

$$=g(M+\boldsymbol{a})\circ g(N+\boldsymbol{b})-\Big(\frac{1}{4}(M, N)-\frac{1}{2}(\boldsymbol{a}, \boldsymbol{b})\Big)E$$

$$=g(M+\boldsymbol{a})\circ g(N+\boldsymbol{b})-\frac{1}{4}(\gamma(M+\boldsymbol{a}), N+\boldsymbol{b})E.$$

Thus the first formula is shown. Take the trace of both sides, then we have the second formula.

THEOREM 3.4.2. $(E_6{}^C)^{\lambda\gamma}\cong Sp(4, C)/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2=\{E, -E\}$.

PROOF ([4]). We define $\phi: Sp(4, \boldsymbol{H}^C)\to(E_6{}^C)^{\lambda\gamma}$ by

$$\phi(A)X=g^{-1}(A(gX)A^*), \qquad X\in\mathfrak{J}^C.$$

We have to prove $\phi(A)\in(E_6{}^C)^{\lambda\gamma}$. Denote $\alpha=\phi(A)$ and put $Z=\alpha X$.

$$3\det\alpha X=3\det Z=(Z\times Z, Z)=(g(\gamma(Z\times Z)), gZ)$$

$$=\Big(gZ\circ gZ-\frac{1}{4}(\gamma Z, Z)E, gZ\Big)=\Big(gZ\circ gZ-\frac{1}{4}(gZ, gZ)E, gZ\Big)$$

$$=\Big(A(gX)A^*\circ A(gX)A^*-\frac{1}{4}(A(gX)A^*,\ A(gX)A^*)E,\ A(gX)A^*\Big)$$

$$=\Big(gX\circ gX-\frac{1}{4}(gX,gX)E,gX\Big)=\Big(gX\circ gX-\frac{1}{4}(\gamma X,\ X)E,gX\Big)$$

$$=(g(\gamma(X\times X)),gX)=(X\times X,\ X)=3\det X,$$

$$(\gamma\alpha X,\alpha Y)=(g(\alpha X),g(\alpha Y))=(A(gX)A^*,\ A(gY)A^*)=(gX,gY)=(\gamma X,Y)$$

$$=({}^t\alpha^{-1}\gamma X,\alpha Y),\quad \text{hence } \gamma\alpha={}^t\alpha^{-1}\gamma.$$

Thus $\alpha\in(E_6{}^C)^{\lambda\gamma}$. We shall show ϕ is onto. To show this we prepare

LEMMA 3.4.3. *Any element* $P\in\mathfrak{J}(4,\boldsymbol{H}^C)$ *such that* $P^2=P$, $\mathrm{tr}(P)=1$ *can be transformed to* $E_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}\in\mathfrak{J}(4,\boldsymbol{H}^C)$ *by a certain* $A\in Sp(4,\boldsymbol{H}^C)$: $APA^*=E_1$.

PROOF is similar to Lemma 2.3.1.

Now, for $\alpha\in(E_6{}^C)^{\lambda\gamma}$, $(g(\alpha E))^2=g(\alpha E)+\frac{3}{4}E$. In fact, $(g(\alpha E))^2=g(\gamma(\alpha E\times\alpha E))+\frac{1}{4}(\gamma\alpha E,\ \alpha E)E=g(\gamma{}^t\alpha^{-1}(E\times E))+\frac{1}{4}({}^t\alpha^{-1}\gamma E,\ \alpha E)E=g(\alpha\gamma E)+\frac{1}{4}(\gamma E,\ E)E=g(\alpha E)+\frac{3}{4}E$. Put $P=\frac{1}{4}(2g(\alpha E)+E)$. Then $P\in\mathfrak{J}(4,\boldsymbol{H}^C)$, $P^2=\frac{1}{16}(4(g(\alpha E))^2+4g(\alpha E)+E)=\frac{1}{4}(2g(\alpha E)+E)=P$ and $\mathrm{tr}(P)=1$. Hence there exists $A\in Sp(4,\boldsymbol{H}^C)$ such that

$$P=AE_1A^*\qquad\text{(Lemma 3.4.3).}$$

Then $\phi(A)E=g^{-1}(A(gE)A^*)=g^{-1}\Big(A\Big(2E_1-\frac{1}{2}E\Big)A^*\Big)=g^{-1}\Big(2P-\frac{1}{2}E\Big)=g^{-1}(g(\alpha E))=\alpha E$. Put $\beta=\phi(A)^{-1}\alpha$, then $\beta E=E$, hence $\beta\in F_4{}^C$ (Proposition 2.1.3. (4)), moreover $\beta\in(F_4{}^C)^\gamma$. By Theorem 2.3.3, there exist $p\in Sp(1,\boldsymbol{H}^C)$, $D\in Sp(3,\boldsymbol{H}^C)$ such that

$$\beta(M+\boldsymbol{a})=DMD^*+p\boldsymbol{a}D^*,\qquad M+\boldsymbol{a}\in\mathfrak{J}^C.$$

Put $B=\mathrm{diag}(p,D)\in Sp(4,\boldsymbol{H}^C)$, then $\beta=\phi(B)$. In fact,

$$\phi(B)(M+\boldsymbol{a})=g^{-1}(B(g(M+\boldsymbol{a}))B^*)$$

$$=g^{-1}\left(\begin{pmatrix}p&0\\0&D\end{pmatrix}\begin{pmatrix}\frac{1}{2}\mathrm{tr}(M)&i\boldsymbol{a}\\ i\boldsymbol{a}^*&M-\frac{1}{2}\mathrm{tr}(M)E\end{pmatrix}\begin{pmatrix}\bar{p}&0\\0&D^*\end{pmatrix}\right)$$

$$=g^{-1}\begin{pmatrix}\frac{1}{2}\mathrm{tr}(M)&ip\boldsymbol{a}D^*\\ iD\boldsymbol{a}^*\bar{p}&DMD^*-\frac{1}{2}\mathrm{tr}(M)E\end{pmatrix}=DMD^*+p\boldsymbol{a}D^*=\beta(M+\boldsymbol{a}).$$

Hence $\alpha=\phi(A)\beta=\phi(A)\phi(B)=\phi(AB)$, $AB\in Sp(4, \boldsymbol{H}^C)$. Therefore ϕ is onto. Ker $\phi=\{E, -E\}=\boldsymbol{Z}_2$. Thus we have the required isomorphism.

LEMMA 3.4.4. *ϕ: $Sp(4, \boldsymbol{H}^C)\to E_6{}^C$ of Theorem 3.4.2 satisfies*

(1) $\gamma=\phi(I_1)$, $\gamma_C=\phi(\boldsymbol{j}E)$, $\gamma_H=\phi(\boldsymbol{i}E)$, $\sigma=\phi(I_2)$.

(2) $\tau\phi(A)\tau=\gamma\phi(\tau A)\gamma=\phi(I_1(\tau A)I_1)$, ${}^t\phi(A)^{-1}=\gamma\phi(A)\gamma=\phi(I_1AI_1)$, $\gamma_C\phi(A)\gamma_C=\phi(\gamma_C A)$, $\sigma\phi(A)\sigma=\phi(I_2AI_2)$.

PROOF. It follows from $\tau g(\tau X)=g(\gamma X)=I_1(gA)I_1$, $g(\gamma_C X)=\gamma_C(gX)=\boldsymbol{j}(gX)\bar{\boldsymbol{j}}$, $g(\gamma_H X)=\gamma_H(gX)=\boldsymbol{i}(gX)\bar{\boldsymbol{i}}$, $g(\sigma X)=I_2(gX)I_2$, $X\in\mathfrak{J}^C$.

THEOREM 3.4.5. (1) $(E_6)^{\lambda\gamma}\cong Sp(4)/\boldsymbol{Z}_2\cong(E_{6(6)})^{\lambda\gamma}$.

(2) $(E_{6(6)})^{\lambda\gamma}\sim(\tau\gamma\gamma_C)^{\lambda\gamma}\cong Sp(4, \boldsymbol{R})/\boldsymbol{Z}_2\times 2\cong(\tau\lambda\gamma_C)^{\lambda\gamma}\sim(E_{6(2)})^{\lambda\gamma}$.

(3) $(E_{6(-26)})^{\lambda\gamma}\cong Sp(1,3)/\boldsymbol{Z}_2\cong(E_{6(2)})^{\lambda\gamma}$.

(4) $(E_{6(-14)})^{\lambda\gamma}\cong Sp(2,2)/\boldsymbol{Z}_2\times 2\cong(\tau\gamma\sigma)^{\lambda\gamma}\sim(E_{6(6)})^{\lambda\gamma}$.

PROOF. (1) Let $\alpha\in(E_6)^{\lambda\gamma}=((E_6{}^C)^{\tau\lambda})^{\lambda\gamma}$, $\alpha=\phi(A)$, $A\in Sp(4, \boldsymbol{H}^C)$ (Theorem 3.4.2). From $\gamma{}^t\alpha^{-1}\gamma=\alpha$, we have $\phi(\tau A)=\phi(A)$ (Lemma 3.4.4). Hence $\tau A=A$ or $\tau A=-A$. The latter case is impossible. In fact, put $A=iB$, then $BB^*=-E$, $B\in M(4, \boldsymbol{H})$, a contradiction. Therefore $A\in Sp(4)$. Thus $(E_6)^{\lambda\gamma}\cong Sp(4)/\boldsymbol{Z}_2$. $(E_{6(6)})^{\lambda\gamma}=(\tau\gamma)^{\lambda\gamma}=(\tau\lambda)^{\lambda\gamma}$.

(2)
$$E_{6(6)}=(E_6{}^C)^{\tau\gamma}\cong(E^{9C})^{\tau\gamma\gamma_C}$$

because $\gamma\sim\gamma\gamma_C$ under $\delta\in G_2\subset F_4\subset E_6$: $\delta\gamma=\gamma\gamma_C\delta, \delta\tau=\tau\delta$ (Proposition 1.2.3). Let $\alpha\in((E_6{}^C)^{\tau\gamma\gamma_C})^{\lambda\gamma}=(\tau\gamma\gamma_C)^{\lambda\gamma}$, $\alpha=\phi(A)$, $A\in Sp(4, \boldsymbol{H}^C)$. From $\tau\gamma\gamma_C{}^t\alpha^{-1}\gamma_C\gamma\tau=\alpha$, we have $\phi(\tau\gamma_C A)=\phi(A)$. Thus $(\tau\gamma\gamma_C)^{\lambda\gamma}=(Sp(4, \boldsymbol{H}')\cup(\boldsymbol{i}E)Sp(4, \boldsymbol{H}'))/\boldsymbol{Z}_2$ (cf. Theorem 1.3.5) $\cong Sp(4, \boldsymbol{R})/\boldsymbol{Z}_2\times 2$. ($\phi(\boldsymbol{i}E)=\gamma_H$).

$$E_{6(2)}=(E_6{}^C)^{\tau\lambda\gamma}=(E_6{}^C)^{\tau\lambda\gamma_C}$$

because $\gamma\sim\gamma_C$ under $\delta\in G_2\subset F_4\subset E_6$: $\delta\gamma=\gamma_C\delta, \delta\tau\lambda=\tau\lambda\gamma$ (Proposition 1.2.3). Now $(E_{6(2)})^{\lambda\gamma}\sim(\tau\lambda\gamma_C)^{\lambda\gamma}=(\tau\gamma\gamma_C)^{\lambda\gamma}$.

(3) Define ϕ: $Sp(1, 3, \boldsymbol{H}^C)\to(E_6{}^C)^{\lambda\gamma}$ by $\phi(A)=\phi(\Gamma_1A\Gamma_1{}^{-1})$. Let $\alpha\in(E_{6(-26)})^{\lambda\gamma}=(\tau)^{\lambda\gamma}$, $\alpha=\phi(A)$, $A\in Sp(1, 3, \boldsymbol{H}^C)$. From $\tau\alpha=\alpha\tau$, we have $\phi(\tau A)=\phi(A)$. Hence $\tau A=A$ or $\tau A=-A$. The latter case is impossible. In fact, there exists no $A\in M(4, \boldsymbol{H})$ such that $A^*I_1A=-I_1$ because the signature of both sides are different. Therefore $A\in Sp(1, 3)$. Thus $(E_{6(-26)})^{\lambda\gamma}\cong Sp(1, 3)/\boldsymbol{Z}_2$. $(E_{6(2)})^{\lambda\gamma}=(\tau\lambda\gamma)^{\lambda\gamma}=(\tau)^{\lambda\gamma}$.

(4) Define ϕ: $Sp(2, 2, \boldsymbol{H}^C)\to(E_6{}^C)^{\lambda\gamma}$ by $\phi(A)=\phi(\Gamma_2A\Gamma_2{}^{-1})$. Let $\alpha\in(E_{6(-14)})^{\lambda\gamma}=(\tau\lambda\sigma)^{\lambda\gamma}$, $\alpha=\phi(A)$, $A\in Sp(2, 2, \boldsymbol{H}^C)$. From $\tau\sigma{}^t\alpha^{-1}\sigma\tau=\alpha$, we have $\phi(\tau A)=\phi(A)$.

Hence $(E_{6(-14)})^{\lambda\gamma} \cong (Sp(2,2) \cup i\begin{pmatrix} 0 & J' \\ J' & 0 \end{pmatrix} Sp(2,2))/\boldsymbol{Z}_2 = Sp(2,2)/\boldsymbol{Z}_2 \times 2$. (The explicit from of $\rho_e = \phi\left(i\begin{pmatrix} 0 & J' \\ J' & 0 \end{pmatrix}\right): \mathfrak{J}^C \to \mathfrak{J}^C$ is

$$\rho_e X = \begin{pmatrix} -\xi_1 & e x_3 e & -ie\bar{x}_1 \\ e\bar{x}_3 e & -\xi_1 & -iex_1 \\ ix_1 e & i\bar{x}_1 e & \xi_3 \end{pmatrix} = \bar{P}_e X P_e, \quad P_e = \begin{pmatrix} ie & 0 & 0 \\ 0 & ie & 0 \\ 0 & 0 & 1 \end{pmatrix}).$$

$$E_{6(6)} = (E_6{}^C)^{\tau\gamma} \cong (E_6{}^C)^{\tau\gamma\sigma}$$

because $\gamma \sim \gamma\sigma$ under $\delta \in F_4 \subset E_6: \delta\gamma = \gamma\sigma\delta, \delta\tau = \tau\delta$ (Proposition 2.3.3). Now $(E_{6(6)})^{\lambda\gamma} \sim (\tau\gamma\sigma)^{\lambda\gamma} = (\tau\lambda\sigma)^{\lambda\gamma}$.

3.5. Subgroups of type $C_1 \oplus A_5$ of Lie groups of type E_6.

Let $k: M(3, \boldsymbol{H}^C) \to \{P \in M(6, \boldsymbol{H}^C) \mid JP = \bar{P}J\}$ be the algebraic C-isomorphism (resp. $k: (\boldsymbol{H}^C)^3 \to \{P \in M(2, 6, \boldsymbol{H}^C) \mid JP = \bar{P}J\}$ be the C-linear isomorphism) defined by

$$k\big((a+bj)\big) = \left(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\right), \qquad a, b \in \boldsymbol{C}^C$$

and denote the inverse k^{-1} of k by h.

LEMMA 3.5.1. $\det(kM) = (\det M)^2$, $M \in \mathfrak{J}(3, \boldsymbol{H}^C)$.

PROOF. Since we know that the determinant of a skew-symmetric matrix S is square of a polynomial with respect to its components s_{ij}, we can easily calculate as

$$\det\begin{pmatrix} 0 & s_{13} & s_{13} & s_{14} & s_{15} & s_{16} \\ -s_{12} & 0 & s_{23} & s_{24} & s_{25} & s_{26} \\ -s_{13} & -s_{23} & 0 & s_{34} & s_{35} & s_{36} \\ -s_{14} & -s_{24} & -s_{34} & 0 & s_{45} & s_{46} \\ -s_{15} & -s_{25} & -s_{35} & -s_{45} & 0 & s_{56} \\ -s_{16} & -s_{26} & -s_{36} & -s_{46} & -s_{56} & 0 \end{pmatrix} = \begin{array}{l} (s_{12}s_{34}s_{56} - s_{11}s_{35}s_{46} + s_{12}s_{36}s_{45} \\ -s_{13}s_{24}s_{56} + s_{13}s_{25}s_{46} - s_{13}s_{26}s_{45} \\ +s_{14}s_{23}s_{56} - s_{14}s_{25}s_{36} + s_{14}s_{26}s_{35} \\ -s_{15}s_{23}s_{46} + s_{15}s_{24}s_{36} - s_{15}s_{26}s_{35} \\ +s_{16}s_{23}s_{45} - s_{16}s_{24}s_{35} + s_{16}s_{25}s_{34})^2 \end{array}$$

Note that $(kM)J \in M(6, \boldsymbol{C}^C)$ is skew-symmetric and use the above formula, then

$$\det(kM) = \det((kM)J) \quad (m_i, n_i \in \boldsymbol{C}^C)$$

$$=\det\begin{pmatrix} 0 & \xi_1 & -n_3 & m_3 & n_2 & \bar{m}_2 \\ -\xi_1 & 0 & -\bar{m}_3 & -\bar{n}_3 & -m_2 & -\bar{n}_2 \\ n_3 & \bar{m}_3 & 0 & \xi_2 & n_1 & m_1 \\ -m_3 & \bar{n}_3 & -\xi_2 & 0 & -\bar{m}_1 & -\bar{n}_1 \\ -n_2 & m_2 & n_1 & \bar{m}_1 & 0 & \xi_3 \\ -\bar{m}_2 & -\bar{n}_2 & -m_1 & -\bar{n}_1 & -\xi_2 & 0 \end{pmatrix} = \begin{matrix}(\xi_1\xi_2\xi_3-\xi_1n_1\bar{n}_1-\xi_1m_1\bar{m}_1 \\ -n_3\bar{n}_3\xi_3-n_3m_2\bar{n}_1-n_3\bar{n}_2\bar{m}_1 \\ -m_3\bar{m}_3\xi_3+m_3m_2m_1-m_3\bar{n}_2n_1 \\ -n_2\bar{m}_3\bar{n}_1-n_2\bar{n}_3m_1-n_2\bar{n}_2\xi_2 \\ +\bar{m}_2\bar{m}_3\bar{m}_1-\bar{m}_2\bar{n}_3n_1-\bar{m}_2m_2\xi_2)^2\end{matrix}$$

On the other hand, $\det M$ is

$$\det\begin{pmatrix} \xi_1 & m_3+n_3\boldsymbol{j} & \overline{m_2+n_2\boldsymbol{j}} \\ \overline{m_3+n_3\boldsymbol{j}} & \xi_2 & m_1+n_1\boldsymbol{j} \\ m_2+n_2\boldsymbol{j} & \overline{m_1+n_1\boldsymbol{j}} & \xi_3 \end{pmatrix} = \begin{matrix}\xi_3\xi_2\xi_3+(m_1+n_1\boldsymbol{j})(m_2+n_2\boldsymbol{j})(m_3+n_3\boldsymbol{j}) \\ +\overline{(m_1+n_1\boldsymbol{j})(m_2+n_2\boldsymbol{j})(m_3+n_3\boldsymbol{j})} \\ -\sum_{i=1}^{3}\xi_i(m_i+n_i\boldsymbol{j})\overline{(m_i+n_i\boldsymbol{j})}\end{matrix}$$

=the interior part of the above bracket.

LEMMA 3.5.2. *The group $E_6(\mathfrak{J}(3, \boldsymbol{H}^C))$ is connected.*

PROOF. The group $(E_6(\mathfrak{J}(3, \boldsymbol{H}^C)))^{\tau\lambda}=\{\alpha\in E_6(\mathfrak{J}(3, \boldsymbol{H}^C))\,|\,\langle\alpha M, \alpha N\rangle=\langle M, N\rangle\}$ is connected. The outline of the proof is as follows (see [7]). In the homogeneous space $(E_6(\mathfrak{J}(3, \boldsymbol{H}^C)))^{\tau\lambda}/F_4(\mathfrak{J}(3, \boldsymbol{H}))\cong EIV_{\boldsymbol{H}}=\{X\in\mathfrak{J}(3, \boldsymbol{H}^C)\,|\,\det M=1, \langle M, M\rangle=3\}$, $F_4(\mathfrak{J}(3, \boldsymbol{H}))=Sp(3)/\boldsymbol{Z}_2$ and $EIV_{\boldsymbol{H}}$ are connected, hence $(E_6(\mathfrak{J}(3, \boldsymbol{H}^C)))^{\tau\lambda}$ is also connected. (In reality, $(E_6(\mathfrak{J}(3, \boldsymbol{H}^C)))^{\tau\lambda}=SU(6)/\boldsymbol{Z}_2$). And $(E_6(\mathfrak{J}(3, \boldsymbol{H}^C)))^{\tau\lambda}$ is a maximal compact subgroup of $E_6(\mathfrak{J}(3, \boldsymbol{H}^C))$. Therefore the group $E_6(\mathfrak{J}(3, \boldsymbol{H}^C))$ is connected.

PROPOSITION 3.5.3. $E_6(\mathfrak{J}(3, \boldsymbol{H}^C))\cong SU^*(6, \boldsymbol{C}^C)/\boldsymbol{Z}_2$.

PROOF. We define ϕ: $SU^*(6, \boldsymbol{H}^C)\to E_6(\mathfrak{J}(3, \boldsymbol{H}^C))$ by

$$\phi(A)M=k^{-1}(A(kM)A^*)=(hA)M(hA)^*, \qquad M\in\mathfrak{J}(3, \boldsymbol{H}^C).$$

We have to prove $\phi(A)\in E_6(\mathfrak{J}(3, \boldsymbol{H}^C))$. In fact, $(\det(\phi(A)M))^2=\det(k(\phi(A)M))$ (Lemma 3.5.1) $=\det(A(kM)A^*)=\det(kM)=(\det M)^2$ (Lemma 3.5.1). Therefore $\det(\phi(A)M)=\pm\det M$. Since $SU^*(6, \boldsymbol{C}^C)$ is connected (Proposition 0.2), the sign of $\det(\phi(A)M)$ is constant with respect to A. Hence $\det(\phi(A)M)=\det M$, that is, ϕ is well-defined. $\operatorname{Ker}\phi=\{E, -E\}=\boldsymbol{Z}_2$. Hence ϕ induces a monomorphism $d\phi$: $\mathfrak{su}^*(6, \boldsymbol{C}^C)\to\mathfrak{e}_6(\mathfrak{J}(3, \boldsymbol{H}^C))$. Since the Lie algebra $\mathfrak{e}_6(\mathfrak{J}(3, \boldsymbol{H}^C))$ has the structure $\mathfrak{e}_6(\mathfrak{J}(3, \boldsymbol{H}^C))=\mathfrak{f}_4(\mathfrak{J}(3, \boldsymbol{H}^C))\oplus\tilde{\mathfrak{J}}(3, \boldsymbol{H}^C)_0$ (cf. Proposition 3.1.1) and $\dim_C\mathfrak{e}_6(\mathfrak{J}(3, \boldsymbol{H}^C))=21+14=35=\dim_C\mathfrak{su}^*(6, \boldsymbol{H}^C)$, $d\phi$ is onto, hence ϕ is also onto (Lemma 0.6) because $E_6(\mathfrak{J}(3, \boldsymbol{H}^C))$ is connected (Lemma 3.5.2). Thus we have the required isomorphism.

PROPOSITION 3.5.4. $(E_6{}^C)^\gamma \cong (Sp(1, C)\times SU^*(6, \boldsymbol{C}^C))/\boldsymbol{Z}_2$.

PROOF. We define ϕ: $Sp(1, \boldsymbol{H}^C)\times SU^*(6, \boldsymbol{C}^C)\to(E_6{}^C)^\gamma$ by

$$\phi(p, A)(M+\boldsymbol{a})=k^{-1}(A(kM)A^*)+pk^{-1}((k\boldsymbol{a})A^{-1})$$
$$=(hA)M(hA)^*+p\boldsymbol{a}(hA)^{-1}, \qquad M+\boldsymbol{a}\in\mathfrak{J}(3, \boldsymbol{H}^C)\oplus(\boldsymbol{H}^C)^3=\mathfrak{J}^C.$$

We have to prove $\phi(p, A)\in(E_6{}^C)^\gamma$.

ASSERTION 3.5.5. ${}^t\phi(p, A)^{-1}=\phi(p, A^{*-1})$.

PROOF. $2({}^t\phi(p, A)(M+\boldsymbol{a}), N+\boldsymbol{b}) \qquad M+\boldsymbol{a}, N+\boldsymbol{b}\in\mathfrak{J}^C$

$=2(M+\boldsymbol{a}, \phi(p, A)(N+\boldsymbol{b}))=(k(M+\boldsymbol{a}), k(\phi(p, A)(N+\boldsymbol{b})))$

(where the inner product (X, Y) in $M(6, \boldsymbol{C}^C)$ (resp. $M(2, 6, \boldsymbol{C}^C)$) is defined by $\frac{1}{2}\mathrm{tr}(X^*Y+Y^*X)$)

$=(kM+k\boldsymbol{a}, A(kN)A^*+(k(p\boldsymbol{b}))A^{-1})=(kM, A(kN)A^*)+2(k\boldsymbol{a}, (k(p\boldsymbol{b}))A^{-1})$

$=(A^*(kM)A, kN)+2((k(\bar{p}\boldsymbol{a}))A^{*^{-1}}, k\boldsymbol{b})=(A^*(kM)A+(k(\bar{p}\boldsymbol{a}))A^{*^{-1}}, kN+k\boldsymbol{b})$

$=(k(\phi(\bar{p}, A^*)(M+\boldsymbol{a})), k(N+\boldsymbol{b}))=2(\phi(\bar{p}, A^*)(M+\boldsymbol{a}), N+\boldsymbol{b})$.

This shows ${}^t\phi(p, A)=\phi(\bar{p}, A^*)$, hence ${}^t\phi(p, A)^{-1}=\phi(p, A^{*-1})$.

ASSERTION 3.5.6. $\phi(p, A)\in(E_6{}^C)^\gamma$.

PROOF. Put $\alpha=\phi(p, A)$ and we shall show ${}^t\alpha^{-1}(X\times Y)=\alpha X, \alpha Y$, $X, Y\in\mathfrak{J}^C$. Recall

$$(M+\boldsymbol{a})\times(N+\boldsymbol{b})=\Big(M\times N-\frac{1}{2}(\boldsymbol{a}^*\boldsymbol{b}+\boldsymbol{b}^*\boldsymbol{a})\Big)-\frac{1}{2}(\boldsymbol{a}N+\boldsymbol{b}M)$$

Now ${}^t\alpha^{-1}(M\times N)=\alpha M\times\alpha N$ is nothing but $\det\alpha M=\det M$ (Lemma 2.1.2, Proposition 3.5.3).

$$(\alpha\boldsymbol{a})^*(\alpha\boldsymbol{b})=(p\boldsymbol{a}(hA)^{-1})^*(p\boldsymbol{b}(hA)^{-1})=(hA)^{*^{-1}}\boldsymbol{a}^*\boldsymbol{b}(hA)^{-1}=\phi(p, A^{*^{-1}})(\boldsymbol{a}^*\boldsymbol{b})$$
$$={}^t\phi(p, A)^{-1}(\boldsymbol{a}^*\boldsymbol{b}) \text{ (Assertion 3.5.5)}={}^t\alpha^{-1}(\boldsymbol{a}^*\boldsymbol{b}),$$
$$(\alpha\boldsymbol{a})(\alpha N)=(p\boldsymbol{a}(hA)^{-1})((hA)N(hA)^*)=p\boldsymbol{a}N(hA)^*=\phi(p, A^{*^{-1}})(\boldsymbol{a}N)$$
$$={}^t\phi(p, A)^{-1}(\boldsymbol{a}N) \text{ (Assertion 3.5.5)}={}^t\alpha^{-1}(\boldsymbol{a}N).$$

This shows $\alpha\in E_6{}^C$. Clearly $\gamma\phi(p, A)=\phi(p, A)\gamma$. Thus Assertion 3.5.6 is shown.

We return to the proof of Proposition 3.5.4. Obviously ϕ is a homomorphism. We shall show ϕ is onto. Let $\alpha\in(E_6{}^C)^\gamma$. Since the restriction of α to $(\mathfrak{J}^C)_\gamma=\mathfrak{J}(3, \boldsymbol{H}^C)$ belongs to $E_6(\mathfrak{J}(3, \boldsymbol{H}^C))$, there exists $A\in SU^*(6, \boldsymbol{C}^C)$ such that

$$\alpha M=k^{-1}(A(kM)A^*), \quad M\in\mathfrak{J}(3, \boldsymbol{H}^C) \quad \text{(Proposition 3.5.3)}.$$

Put $\beta=\phi(1, A)^{-1}\alpha$, then $\beta|\mathfrak{J}(3, \boldsymbol{H}^C)=1$. Hence $\beta\in(G_2{}^C)^\gamma$ and $\beta|\boldsymbol{H}^C=1$. By Theorem 1.3.2, there exists $p\in Sp(1, \boldsymbol{H}^C)$ such that $\beta=\phi(p, E)$. Hence $\alpha=\phi(1, A)\beta=\phi(1, A)\phi(p, E)=\phi(p, A)$. Therefore ϕ is onto. Ker $\phi=\{(1, E), (-1, -E)\}=\boldsymbol{Z}_2$. Thus we have the required isomorphism.

LEMMA 3.5.7. *ϕ: $Sp(1, \boldsymbol{H}^C)\times SU^*(6, \boldsymbol{C}^C)\to E_6{}^C$ of Proposition 3.5.4 satisfies*

(1) $\gamma=\phi(-1, E)$, $\gamma_C=\phi(\boldsymbol{j}, J)$, $\gamma_H=\phi(\boldsymbol{i}, \boldsymbol{i}I)$, $\sigma=\phi(-1, I_2)$.

(2) $\tau\phi(p, A)\tau=\phi(\tau p, \tau A)$, $\gamma_C\phi(p, A)\gamma_C=\phi(\gamma_C p, -JAJ)$.

THEOREM 3.5.8. $(E_{6(-26)})^\gamma\cong(Sp(1)\times SU^*(6))/\boldsymbol{Z}_2\cong(E_{6(6)})^\gamma$.

PROOF. Let $\alpha\in(E_{6(-26)})^\gamma=(\tau)^\gamma$, $\alpha=\phi(p, A)$, $p\in Sp(1, \boldsymbol{H}^C)$, $A\in SU^*(6, \boldsymbol{C}^C)$ (Proposition 3.5.4). From $\tau\alpha=\alpha\tau$, we have $\phi(\tau p, \tau A)=\phi(p, A)$ (Lemma 3.5.7). Hence $(E_{6(6)})^\gamma\cong(Sp(1)\cong(SU^*(6))/\boldsymbol{Z}_2$ (cf. Theorem 1.3.4). $(E_{6(6)})^\gamma=(\tau\gamma)^\gamma=(\tau)^\gamma$.

TEEOREM 3.5.9. (1) $(E_6{}^C)^\gamma\cong(Sp(1, C)\times SL(6, C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(1, E), (-1, -E)\}$.

(2) $(E_{6(6)})^\gamma\sim(\tau\gamma_C)^\gamma\cong(Sp(1, \boldsymbol{R})\times SL(6, \boldsymbol{R}))/\boldsymbol{Z}_2\times 2$.

PROOF. (1) Since f': $SL(6, C)\to SU^*(6, \boldsymbol{C}^C)$, $f'(A)=\varepsilon A-\bar{\varepsilon}JAJ$ where $\varepsilon=\frac{1}{2}(1+\boldsymbol{i}i)$, is an isomorphism (Proposition 0.2), ϕ': $Sp(1, \boldsymbol{H}^C)\times SL(6, C)\to(E_6{}^C)^\gamma$, $\phi'(p, A)=\phi(p, f'A)$ induces the required isomorphism (Proposition 3.5.4).

$$E_{6(6)}=(E_6{}^C)^{\tau\gamma}\cong(E_6{}^C)^{\tau\gamma_C} \tag{3}$$

because $\gamma\sim\gamma_C$ under $\delta\in G_2\subset F_4\subset E_6$: $\delta\gamma=\gamma_C\delta$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 1.2.3). Let $\alpha\in(\tau\gamma_C)^\gamma$, $\alpha=\phi'(p, A)$, $p\in Sp(1, \boldsymbol{H}^C)$, $A\in SL(6, C)$. From $\tau\gamma_C\alpha=\alpha\tau\gamma_C$, we have $\phi'(\tau\gamma_C p, \tau A)=\phi'(p, A)$ $(\tau(f'A)=f'(-J(\tau A)J)$ and Lemma 3.5.7). Hence $(E_{6(6)})^\gamma\sim(\tau\gamma_C)^\gamma\cong(Sp(1, \boldsymbol{H}')\times SL(6, \boldsymbol{R})\cup \boldsymbol{i}Sp(1, \boldsymbol{H}')\times(-iI)SL(6, \boldsymbol{R}))/\boldsymbol{Z}_2$ (cf. Theorem 1.3.5) $\cong(Sp(1, \boldsymbol{R})\times SL(6, \boldsymbol{R}))/\boldsymbol{Z}_2\times 2$. $(\phi'(\boldsymbol{i}, -iI)=\gamma_H)$.

LEMMA 3.5.10. *Since f: $SU(6, \boldsymbol{C}^C)\to SU^*(6, \boldsymbol{C}^C)$, $f(A)=\varepsilon A-\bar{\varepsilon}J\bar{A}J$ where $\varepsilon=\frac{1}{2}(1+\boldsymbol{i}i)$, is an isomorphism* (Proposition 0.2), *ϕ: $Sp(1, \boldsymbol{H}^C)\times SU(6, \boldsymbol{C}^C)\to(E_6{}^C)^\gamma$, $\phi(p, A)=\phi(p, fA)$ is also an isomorphism. Now this ϕ satisfies*

(1) $\gamma=\phi(-1, E)$, $\gamma_C=\phi(\boldsymbol{j}, J)$, $\gamma_H=\phi(\boldsymbol{i}, \boldsymbol{i}I)$, $\rho=\phi(1, I_2')$ *where* $I_2'=\mathrm{diag}(-1, 1, -1, 1, 1, 1)$, $\sigma=\phi(-1, I_2)$.

(2) $\tau\phi(p, A)\tau=\phi(\tau p, -J\overline{\tau A}J)$, ${}^t\phi(p, A)^{-1}=\phi(p, -J\bar{A}J)$, $\gamma_C\phi(p, A)\gamma_C=\phi(\gamma_C p, -JAJ)$, $\gamma_H\phi(p, A)\gamma_H=\phi(\gamma_H p, IAI)$, $\sigma\phi(p, A)\sigma=\phi(p, I_2AI_2)$.

PROOF. It is clear from $\tau f(A)=f(-J\overline{\tau A}J)$, $(fA)^*=f(-JA^*J)$, $f(J)=J$, $f(I_2)=I_2$ and Lemma 3.5.7. $\rho=\phi(1, I_2')$ is obtained by the direct calculation.

THEOREM 3.5.11. (1) $(E_6)^\gamma\cong(Sp(1)\times SU(6))/\boldsymbol{Z}_2\cong(E_{6(2)})^\gamma$.
(2) $(E_{6(-14)})^\gamma\cong(Sp(1)\times SU(2,4))/\boldsymbol{Z}_2\cong(\tau\lambda\gamma\sigma)^\gamma\sim(E_{6(2)})^\gamma$.
(3) $(E_{6(2)})^\gamma\sim(\tau\lambda\gamma_H)^\gamma\cong(Sp(1,\boldsymbol{R})\times SU(3,3))/\boldsymbol{Z}_2\times 2$.

PROOF. (1) Let $\alpha\in(E_6)^\gamma=(\tau\lambda)^\gamma$, $\alpha=\phi(p, A)$, $p\in Sp(1,\boldsymbol{H}^C)$, $A\in SU(6,\boldsymbol{C}^C)$. From $\tau\lambda\alpha=\alpha\tau\lambda$, we have $\phi(\tau p, \tau A)=\phi(p, A)$ (Lemma 3.5.10). Thus $(E_6{}^C)^\gamma\cong(Sp(1)\times SU(6))/\boldsymbol{Z}_2$ (cf. Theorem 1.3.4). $(E_{6(2)})^\gamma=(\tau\lambda\gamma)^\gamma=(\tau\lambda)^\gamma$.

(2) Define $\varphi: Sp(1,\boldsymbol{H}^C)\times SU(2,4,\boldsymbol{C}^C)\to(E_6{}^C)^\gamma$ by $\varphi(p, A)=\phi(p, \Gamma_2A\Gamma_2{}^{-1})$. Let $\alpha\in(E_{6(-14)})^\gamma=(\tau\lambda\sigma)^\gamma$, $\alpha=\varphi(p, A)$, $p\in Sp(1,\boldsymbol{H}^C)$, $A\in SU(2,4,\boldsymbol{C}^C)$. From $\tau\sigma{}^t\alpha^{-1}\sigma\tau=\alpha$, we have $\varphi(\tau p, \tau A)=\varphi(p, A)$. Thus $(E_{6(-14)})^\gamma\cong(Sp(1)\times SU(2,4))/\boldsymbol{Z}_2$ (cf. Theorem 1.3.4).

$$E_{6(2)}=(E_6{}^C)^{\tau\lambda\gamma}\cong(E_6{}^C)^{\tau\lambda\gamma\sigma}$$

because $\gamma\sim\gamma\sigma$ under $\delta\in F_4\subset E_6: \delta\gamma=\gamma\sigma\delta$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 2.2.3). Now $(E_{6(2)})^\gamma\sim(\tau\lambda\gamma\sigma)^\gamma=(\tau\lambda\sigma)^\gamma$.

(3) $$E_{6(2)}=(E_6{}^C)^{\tau\lambda\gamma}\cong(E_6{}^C)^{\tau\lambda\gamma_H}$$

because $\gamma\sim\gamma_H$ under $\delta\in G_2\subset F_4\subset E_6: \delta\gamma=\gamma_H\delta$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 1.2.3). Let $SU(3,3,K)=\{A\in M(6,K)\,|\,A^*IA=I,\ \det A=1\}$, $I=\mathrm{diag}(1,-1,1,-1,1,-1)$, $K=\boldsymbol{C}$, $\boldsymbol{C}^C$ and define $\varphi: Sp(1,\boldsymbol{H}^C)\times SU(3,3,\boldsymbol{C}^C)\to(E_6{}^C)^\gamma$ by $\varphi(p, A)=\phi(p, \Gamma_3'A\Gamma_3'^{-1})$ where $\Gamma_3'=\mathrm{diag}(1,i,1,i,1,i)$. Let $\alpha\in(\tau\lambda\gamma_H)^\gamma$, $\alpha=\varphi(p, A,)$ $p\in Sp(1,\boldsymbol{H}^C)$, $A\in SU(3,3,\boldsymbol{C}^C)$. From $\tau\gamma_H{}^t\alpha^{-1}\gamma_H\tau=\alpha$, we have $\varphi(\tau\gamma_H p, \tau A)=\varphi(p, A)$. Thus $(E_{6(2)})^\gamma\sim(\tau\lambda\gamma_H)^\gamma\cong(Sp(1,'\boldsymbol{H})\times SU(3,3)\cup \boldsymbol{j}Sp(1,'\boldsymbol{H})\times(iJ')SU(3,3))/\boldsymbol{Z}_2\cong(Sp(1,\boldsymbol{R})\times SU(3,3))/\boldsymbol{Z}_2\times 2$ (cf. Theorem 1.3.5). $(\varphi(\boldsymbol{j}, iJ')=\gamma_C)$.

3.5.12. PROPOSITION 3.2.3. (1) $\gamma\sim\rho$. (2) $\sigma\sim\gamma\rho$. (3) $\sigma\sim\gamma_H\rho$.

PROOF. (1) Since $I_2'\sim I_2$ under a certain $D_1\in SU(6)$, $\rho=\phi(1, I_2')\sim\gamma\sigma=\phi(1, I_2)$ under $\delta_1=\phi(1, D_1)\in(E_6)^\gamma$ (Theorem 3.5.11.(1)). Furthermore $\gamma\sigma\sim\gamma$ in $F_4\subset E_6$ (Proposition 2.2.3.(1)). Consequently $\rho\sim\gamma$ in E_6.

(2) As is shown in (1), $\rho\sim\gamma\sigma$ under $\delta_1\in(E_6)^\gamma$, hence $\gamma\rho\sim\gamma\gamma\sigma=\sigma$ in E_6.

(3) $\gamma_H\sim\gamma$ under $\delta\in G_2\subset F_4\subset E_6$ (Proposition 1.2.3). This δ satisfies $\delta(\boldsymbol{i})=\boldsymbol{i}$, hence $\delta\rho=\rho\delta$. Therefore $\gamma_H\rho\sim\gamma\rho$ under $\delta\in E_6$. Thus $\gamma_H\rho\sim\gamma\rho\sim\gamma$ (result of (1)) in E_6.

THEOREM 3.5.13. $(E_{6(-14)})^{\gamma}\sim(\tau\lambda\gamma_H\rho)^{\gamma}\cong(Sp(1,\boldsymbol{R})\times SU(5,1))/\boldsymbol{Z}_2$.

PROOF. $$E_{6(-14)}=(E_6{}^C)^{\tau\lambda\sigma}\cong(E_6{}^C)^{\tau\lambda\gamma_H\rho}$$

because $\sigma\sim\gamma_H\rho$ under $\delta\in E_6:\delta\sigma=\gamma_H\rho\delta$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 3.2.3). Put $I_5'=I_2'I=\mathrm{diag}(-1,-1,-1,-1,1,-1)$ and $SU(5,1,K)=\{A\in M(6,K)\,|\,A^*I_5'A=I_5',$ $\det A=1\}$, $K=\boldsymbol{C}$, $\boldsymbol{C}^C$. Define $\varphi:Sp(1,\boldsymbol{H}^C)\times SU(5,1,\boldsymbol{C}^C)\to(E_6{}^C)^{\gamma}$ by $\varphi(p,A)=\phi(p,\Gamma_5'A\Gamma_5'^{-1})$ where $\Gamma_5'=\mathrm{diag}(i,i,i,i,1,i)$. Let $\alpha\in(\tau\lambda\gamma_H\rho)^{\gamma}$, $\alpha=\varphi(p,A)$, $p\in Sp(1,\boldsymbol{H}^C)$, $A\in SU(5,1,\boldsymbol{C}^C)$. From $\gamma_H\rho\tau{}^t\alpha^{-1}\tau\rho\gamma_H=\alpha$, we have $\varphi(\tau\gamma_Hp,\tau A)=\varphi(p,A)$. Thus $(E_{6(-14)})^{\gamma}\sim(\tau\lambda\gamma_H\rho)^{\gamma}\cong(Sp(1,{}'\boldsymbol{H})\times SU(5,1))/\boldsymbol{Z}_2$ (cf. Theorem 3.4.5.(3)) $\cong(Sp(1,\boldsymbol{R})\times SU(5,1))/\boldsymbol{Z}_2$.

3.6. Subgroups of type $C\oplus D_5$ of Lie groups of type E_6.

LEMMA 3.6.1. *For $\alpha\in(E_6{}^C)^{\sigma}$, there exists $\xi\in C^*=C-\{0\}$ such that $\alpha E_1=\xi E_1$.*

PROOF. Note that for $\alpha\in(E_6{}^C)^{\sigma}$ we have ${}^t\alpha$, ${}^t\alpha^{-1}\in(E_6{}^C)^{\sigma}$. As in Section 2.4, $(\mathfrak{J}^C)_{\sigma}$, $(\mathfrak{J}^C)_{-\sigma}$ are invariant under $\alpha\in(E_6{}^C)^{\sigma}$, hence αE_2, ${}^t\alpha E_2$, ${}^t\alpha^{-1}E_2$, αE_3, ${}^t\alpha E_3$, ${}^t\alpha^{-1}E_3\in\mathfrak{J}(2,\mathfrak{C}^C)$ as in Lemma 2.4.1. Suppose that αE_1 and ${}^t\alpha^{-1}E_1\in\mathfrak{J}(2,\mathfrak{C}^C)$. Then αE and ${}^t\alpha^{-1}E\in\mathfrak{J}(2,\mathfrak{C}^C)$, and $\xi_2E_2+\xi_3E_3+F_1(x)=\alpha E=\alpha(E\times E)={}^t\alpha^{-1}E\times{}^t\alpha^{-1}E$ $=(\eta_2E_2+\eta_3E_3+F_1(y))^{\times2}=(\eta_2\eta_3-y\bar{y})E_1$ for some ξ_i, $\eta_i\in C$, x, $y\in\mathfrak{C}^C$. This implies $\xi_2=\xi_3=x=0$. Hence $\alpha E=0$, a contradiction. Therefore $\alpha E_1\notin\mathfrak{J}(2,\mathfrak{C}^C)$ or ${}^t\alpha^{-1}E_1\notin\mathfrak{J}(2,\mathfrak{C}^C)$.

(1) Case $\alpha E_1\notin\mathfrak{J}(2,\mathfrak{C}^C)$. We can put $\alpha E_1=\xi E_1+\xi_2E_2+\xi_3E_3+F_1(X)$, $\xi\neq0$. Then $0={}^t\alpha^{-1}(E_1\times E_1)=\alpha E_1\times\alpha E_1=(\xi E_1+\xi_2E_2+\xi_3E_3+F_1(x))^{\times2}=(\xi_2\xi_3-x\bar{x})E_1+\xi\xi_3E_2$ $\xi\xi_2E_3-\xi F_1(x)$. This implies $\xi_2=\xi_3=x=0$. Hence $\alpha E_1=\xi E_1$, $\xi\neq0$.

(2) Case ${}^t\alpha^{-1}E_1\notin\mathfrak{J}(2,\mathfrak{C}^C)$. Similarly as above, there exists $\eta\in C^*$ such that ${}^t\alpha^{-1}E_1=\eta E_1$, Then ${}^t\alpha E_1=\eta^{-1}E_1$ (put $\xi=\eta^{-1}$).

$$(\alpha E_1,E_1)=(E_1,{}^t\alpha E_1)=(E_1,\xi E_1)=\xi\,,$$

$$(\alpha E_1,E_i)=(E_1,{}^t\alpha E_i)=0\ (\text{because } {}^t\alpha E_i\in\mathfrak{J}(2,\mathfrak{C}^C)),\ i=2,3\,.$$

Hence αE_1 has the form $\xi E_1+F_1(x)$, $\xi\neq0$. From $0={}^t\alpha^{-1}(E_1\times E_1)=\alpha E_1\times\alpha E_1=(\xi E_1+F_1(x))^{\times2}=-x\bar{x}E_1-\xi F_1(x)$, we have $x=0$. Hence $\alpha E_1=\xi E_1$. Thus the proof of Lemma is completed.

LEMMA 3.6.2. *If $\alpha\in((E_6{}^C)^{\sigma})_{E_1}$ then ${}^t\alpha$, ${}^t\alpha^{-1}\in((E_6{}^C)^{\sigma})_{E_1}$.*

PROOF. Put ${}^t\alpha E_1=\xi E_1$, $\xi\in C^*$ (Lemma 3.6.1). Then $\xi=(\xi E_1,E_1)=({}^t\alpha E_1,E_1)$

$=(E_1, \alpha E_1)=(E_1, E_1)=1.$

LEMMA 3.6.3. $((E_6{}^C)^\sigma)_{E_1}/Spin(9, C)\simeq(S^C)^9$. *In particular, the group* $((E_6{}^C)^\sigma)_{E_1}$ *is connected.*

PROOF. We define a complex 9-dimensional sphere $(S^C)^9$ by

$$(S^C)^9=\{X\in\mathfrak{J}^C\,|\,4E_1\times(E_1\times X)=X,\ (E_1, X, X)=1\}$$
$$=\left\{\begin{pmatrix}\xi & x\\ \bar{x} & \eta\end{pmatrix}\,\middle|\,\xi\eta-x\bar{x}=1,\ \xi,\ \eta\in C,\ x\in\mathfrak{C}^C\right\}.$$

The group $((E_6{}^C)^\sigma)_{E_1}$ acts on $(S^C)^9$ (Lemma 3.6.2). We shall show that this action is transitive. To show this we prepare some elements of $((E_6{}^C)^\sigma)_{E_1}$.

(1) For $d\in\mathfrak{C}^C$, put $D_{32}=\begin{pmatrix}1&0&0\\0&1&0\\0&d&1\end{pmatrix}$ and define a C-linear transformation $\delta_{32}(d)$ of $\mathfrak{J}^C$ by $\delta_{32}(d)X=D_{32}XD_{32}{}^*$, $X=X(\xi, x)\in\mathfrak{J}^C$, explicitly

$$\delta_{32}(d)X=\begin{pmatrix}\xi_1 & x_3 & x_3\bar{d}+\bar{x}_2\\ \bar{x}_3 & \xi_2 & \xi_2\bar{d}+x_1\\ d\bar{x}_3+x_2 & \xi_2 d+\bar{x}_1 & \xi_2 d\bar{d}+2(d, \bar{x}_1)+\xi_3\end{pmatrix}.$$

Then $\delta_{32}(d)\in((E_6{}^C)^\sigma)_{E_1}$. Similarly $\delta_{32}(d)\in((E_6{}^C)^\sigma)_{E_1}$ can be defined.

(2) For $\theta\in C^*$, define a C-linear transformation $\delta(\theta)$ of $\mathfrak{J}^C$ by

$$\delta(\theta)X_\circ\begin{pmatrix}\xi_1 & \theta x_3 & \theta^{-1}\bar{x}_2\\ \theta\bar{x}_3 & \theta^2\xi_2 & x_1\\ \theta^{-1}x_2 & \bar{x}_1 & \theta^{-2}\xi_3\end{pmatrix}=D_\theta XD_\theta,\qquad D_\theta=\begin{pmatrix}1&0&0\\0&\theta&0\\0&0&\theta^{-1}\end{pmatrix}.$$

Then $\delta(\theta)\in((E_6{}^C)^\sigma)_{E_1}$.

Now let $X=\begin{pmatrix}\xi & x\\ \bar{x} & \eta\end{pmatrix}\in(S^C)^9$. If $\xi\neq0$ (resp. $\eta\neq0$), operate $\delta_{32}(-\bar{x}/\xi)$ (resp. $\delta_{23}(-x/\eta)$) on X, then X is transformed to a diagonal form. In the case of $\xi=\eta=0$, choose $d\in\mathfrak{C}^C$ such that $(d, \bar{x})\neq0$, then $\delta_{32}(d)\begin{pmatrix}0 & x\\ \bar{x} & 0\end{pmatrix}=\begin{pmatrix}0 & x\\ \bar{x} & 2(d, \bar{x})\end{pmatrix}$, hence it is reduced to the first case. Thus X is transformed to a diagonal form $\xi E_2+\eta E_3$, $\xi\eta=1$. Moreover choose $\theta\in C$ such that $\theta^2=\eta$ and operate $\delta(\theta)$, then it can be transformed to E_2+E_3. This shows the transitivity. The isotropy subgroup of $((E_6{}^C)^\sigma)_{E_1}$ at E_2+E_3 is $((E_6{}^C)^\sigma)_E=(F_4{}^C)^\sigma$ (Proposition 2.1.3.(4))$=Spin(9, C)$ (Theorem 2.4.3). Thus we have the hemeomorphism $((E_6{}^C)^\sigma)_{E_1}/Spin(9, C)\simeq(S^C)^9$.

PROPOSITION 3.6.4. $((E_6{}^C)^\sigma)_{E_1}\cong Spin(10, C)$.

PROOF. Since the group $((E_6{}^C)^\sigma)_{E_1}$ is connected (Lemma 3.6.3), we can define a homomorphism $\pi:((E_6{}^C)^\sigma)_{E_1}\to SO(10, C)=SO((V^C)^{10})$ by $\pi(\alpha)=\alpha|(V^C)^{10}$ where

$$(V^C)^{10}=\mathfrak{J}(2, \mathfrak{C}^C)=\{X\in\mathfrak{J}(3, \mathfrak{C}^C)\mid 4E_1\times(E_1\times X)=X\}$$

with the norm (E_1, X, X). $\operatorname{Ker}\pi=\{1, \sigma\}=Z_2$. Hence π induces a monomorphism $d\pi:((\mathfrak{e}_6{}^C)^\sigma)_{E_1}\to\mathfrak{so}(10, C)$. Since $((\mathfrak{e}_6{}^C)^\sigma)_{E_1}=(\mathfrak{f}_4{}^C\oplus(\tilde{\mathfrak{J}}^C)_0)_{E_1}=(\mathfrak{f}_4{}^C)_{E_1}\oplus\tilde{\mathfrak{J}}(2, \mathfrak{C}^C)$ (Proposition 3.1.1) and $\dim_C((\mathfrak{e}_6{}^C)^\sigma)_{E_1})=36+9$ (Theorem 2.4.3)$=45=\dim_C\mathfrak{so}(10, C)$, $d\pi$ is onto, hence π is onto. Thus $((E_6{}^C)^\sigma)_{E_1}/Z_2\cong SO(10, C)$. Therefore $((E_6{}^C)^\sigma)_{E_1}$ is isomorphic to $Spin(10, C)$ as the universal covering group of $SO(10, C)$.

PROPOSITION 3.6.5. *$(E_6{}^C)^\sigma$ has a subgroup $\phi(C^*)$ which is isomorphic to the group C^*. Where $\phi(\theta)$, $\theta\in C^*$, is the C-linear transformation of $\mathfrak{J}^C$ defined by*

$$\phi(\theta)X=\begin{pmatrix}\theta^4\xi_1 & \theta x_3 & \theta\bar{x}_2\\ \theta\bar{x}_3 & \theta^{-2}\xi_2 & \theta^{-2}x_1\\ \theta x_2 & \theta^{-2}\bar{x}_1 & \theta^{-2}\xi_3\end{pmatrix}=S_\theta XS_\theta,\quad S_\theta=\begin{pmatrix}\theta^2 & 0 & 0\\ 0 & \theta^{-1} & 0\\ 0 & 0 & \theta^{-1}\end{pmatrix}.$$

LEMMA 3.6.6. *The groups $\phi(C^*)$ and $Spin(10, C)$ commute in $(E_6{}^C)^\sigma$ elementwisely.*

PROOF. The restrictions of $\phi(\theta)$, $\theta\in C^*$, to $\mathfrak{C}_1{}^C$, $\mathfrak{J}(2, \mathfrak{C}^C)$, $(\mathfrak{J}^C)_{-\sigma}$ are constant mappings and $\beta\in Spin(10, C)$ leaves invariant these spaces. From this we see that $\phi(\theta)$ and β are commutative.

THEOREM 3.6.7. $(E_6{}^C)^\sigma\cong(C^*\times Spin(10, C))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4=\{(1,1), (-1, \sigma), (i, \sigma\sqrt{\sigma}), (-i, \sqrt{\sigma})\}$.

PROOF ([7]). We define $\psi: C^*\times Spin(10, C)\to(E_6{}^C)^\sigma$ by

$$\psi(\theta, \beta)=\phi(\theta)\beta.$$

Then ψ is a homomorphism (Lemma 3.6.6). We shall show ψ is onto. For $\alpha\in(E_6{}^C)^\sigma$, there exists $\theta\in C^*$ such that

$$\alpha E_1=\theta^4E_1 \qquad \text{(Lemma 3.6.1)}.$$

Put $\beta=\phi(\theta)^{-1}\alpha$, then $\beta E_1=E_1$, hence $\beta\in Spin(10, C)$ (Proposition 3.6.4). Therefore $\alpha=\phi(\theta)\beta=\psi(\theta, \beta)$, that is, ψ is onto. $\operatorname{Ker}\psi=\{(1, 1), (-1, \phi(-1)), (i, \phi(-i)), (-i, \phi(i))\}=\boldsymbol{Z}_4$. ($\phi(-1)=\sigma$, $\phi(i)=\sqrt{\sigma}$ (Proposition 3.2.3.(1)). In fact, let $\phi(\theta)\beta=1$, $\theta\in C^*$, $\beta\in Spin(10, C)$. Operate on E_1, then $\theta^4=1$, hence $\theta=\pm1$, $\pm i$.

Therefore Ker $\psi = \boldsymbol{Z}_4$. Thus we have the required isomorphism.

THEOREM 3.6.8. (1) $(E_{6(-26)})^\sigma \cong \boldsymbol{R}^+ \times Spin(9, 1)$.
(2) $(E_{6(6)})^\sigma \cong (\boldsymbol{R}^+ \times spin(5, 5)) \times 2$.

PROOF. (1) For $\alpha \in (E_{6(-26)})^\sigma$ there exists $\xi \in \boldsymbol{R}^+ = \{\xi \in \boldsymbol{R} \mid \xi > 0\}$ such that $\alpha E_1 = \xi E_1$. In fact, $\xi E = \alpha_1 E_1$ ($\xi \in C^*$ (Lemma 3.6.1)) $= \tau\alpha\tau E_1 = \tau\xi E_1$, hence $\tau\xi = \xi$, that is, $\xi \in \boldsymbol{R}^* = \boldsymbol{R} - \{0\}$. Moreover $\xi > 0$. (Although it follows from the connectedness of $(E_{6(-26)})^\sigma$ (Lemma 0.7) we will give here a direct proof). As in Lemma 2.4.1 we have

$$\alpha E_2 = \eta_2 E_2 + \eta_3 E_3 + F_1(y), \qquad \eta_2, \eta_3 \geqq 0, \ y \in \mathfrak{C},$$

$$^t\alpha^{-1} E_3 = \zeta_2 E_2 + \zeta_3 E_3 + F_1(z), \qquad \zeta_2, \zeta_3 \geqq 0, \ z \in \mathfrak{C}.$$

Suppose $\xi < 0$. Then from $\zeta_2 E_2 + \zeta_3 E_3 + F_1(z) = {}^t\alpha^{-1} E_3 = 2\,{}^t\alpha^{-1}(E_1 \times E_2) = 2\alpha E_1 \times \alpha E_2 = 2\xi E_1 \times (\eta_2 E_2 + \eta_3 E_3 + F_1(y)) = \xi\eta_3 E_2 + \xi\eta_2 E_3 - \xi F_1(y)$ we have $\eta_2 = \eta_3 = 0$. Hence $\alpha E_2 = F_1(y)$. Again from $0 = {}^t\alpha^{-1}(E_2 \times E_2) = \alpha E_2 \times \alpha E_2 = F_1(y) \times F_1(y) = -y\bar{y} E_1$ we have $y = 0$, a contradiction.

Now $((E_{6(-26)})^\sigma)_{E_1} = (((E_6{}^C)^\tau)_{E_1})^\sigma = (((E_6{}^C)^\sigma)_{E_1})^\tau$ is connected (Lemma 0.7) because $((E_6{}^C)^\sigma)_{E_1} \cong Spin(10, C)$ (Proposition 3.6.4) is simply connected. The group $((E_{6(-26)})^\sigma)_{E_1}$ acts on

$$V^{9,1} = (\mathfrak{J}(2, \mathfrak{C}^C))_\tau = \left\{ X = \begin{pmatrix} \xi & x \\ \bar{x} & \eta \end{pmatrix} \middle| \ \xi, \eta \in \boldsymbol{R}, \ x \in \mathfrak{C} \right\}$$

with the norm $(E_1, X, X) = \xi\eta - x\bar{x}$. We can define a homomorphism $\pi : ((E_{6(-26)})^\sigma)_{E_1} \to O(9, 1)_0 = O(V^{9,1})_0$. Ker $\pi = \{1, \sigma\} = Z_2$. As similar to Proposition 3.6.4, π is onto. Thus $((E_{6(-26)})^\sigma)_{E_1}/Z_2 \cong O(9, 1)_0$. Therefore $((E_{6(-26)})^\sigma)_{E_1}$ is isomorphic to $Spin(9, 1)$ as the universal covering group of $O(9, 1)_0$. Let $\phi : \boldsymbol{R}^+ \to (E_{6(-26)})^\sigma$ be the restriction of $\phi : C^* \to (E_6{}^C)^\sigma$ defined in Lemma 3.6.5. Now $\psi : \boldsymbol{R}^+ \times Spin(9, 1) \to (E_{6(-26)})^\sigma$, $\psi(\theta, \beta) = \phi(\theta)\beta$, gives the required isomorphism (cf. Theorem 3.6.7).

(2) As in (1), for $\alpha \in (E_{6(6)})^\sigma$, $\alpha E_1 = \xi E_1$, $\xi \in \boldsymbol{R}^*$. In this case there exists surely $\alpha \in (E_{6(6)})^\sigma$ such that $\alpha E_1 = -E_1$. In fact, ρ_e in Theorem 3.4.5. (4) is the required one. Now the connected group $((E_{(6)})^\sigma)_{E_1}$, as in (1), acts on

$$V^{5,5} = (\mathfrak{J}(3, \mathfrak{C}^C))_{\tau\gamma} = \left\{ X = \begin{pmatrix} \xi & x' \\ \bar{x}' & \eta \end{pmatrix} \middle| \ \xi, \eta \in \boldsymbol{R}, \ x' \in (\mathfrak{C}^C)_{\tau\gamma} = \mathfrak{C}' \right\}$$

with the norm $(E_1, X, X) = \xi\eta - x'\bar{x}'$. We can define a homomorphism $\pi : ((E_{6(6)})^\sigma)_{E_1} \to O(5, 5)_0 = O(V^{5,5})_0$. Ker $\pi = \{1, \sigma\} = Z_2$. As similar to (1) we have $((E_{6(6)}{}^\sigma)_{E_1}/Z_2 \cong O(5, 5)_0$. Therefore $((E_{6(6)}{}^\sigma)_{E_1}$ is denoted by $spin(5, 5)$ (not simply connected) as a double covering group of $O(5,5)_0$. Put $((E_{6(6)})^\sigma)_0=\{\alpha\in(E_{6(6)})^\sigma \mid \alpha E_1=\xi E_1,\ \xi>0\}$. By the use of ϕ in (1), we see that $\psi: \boldsymbol{R}^+\times spin(5,5)\to((E_{6(6)})^\sigma)_0$, $\psi(\theta,\beta)=\phi(\theta)\beta$, is an isomorphism (cf. Theorem 3.6.7). Thus $(E_{6(6)})^\sigma=((E_{6(6)})^\sigma)_0\cup\rho_e((E_{6(6)})^\sigma)_0\cong(\boldsymbol{R}^+\times spin(5,5))\times 2$.

THEOREM 3.6.9. (1) $(E_6)^\sigma\cong(U(1)\times Spin(10))/\boldsymbol{Z}_4\cong(E_{6(-14)})^\sigma$.

(2) $(E_{6(2)})^\sigma\cong(U(1)\times spin(6,4))/\boldsymbol{Z}_4$.

(3) $(E_{6(-14)})^\sigma\sim(\tau\lambda\sigma')^\sigma\cong(U(1)\times spin(8,2))/\boldsymbol{Z}_4$.

PROOF. (1) For $\alpha\in(E_6)^\sigma$, $\xi E_1=\alpha E_1$ ($\xi\in C^*$ (Lemma 3.6.1))$=\tau{}^t\alpha^{-1}\tau E_1=(\tau\xi)^{-1}E_1$, hence $\xi(\tau\xi)=1$, that is, $\xi\in U(1)=\{\xi\in C \mid \xi(\tau\xi)=1\}$. $((E_6)^\sigma)_{E_1}=(((E_6{}^C)^\sigma)_{E_1})^{\tau\lambda}$ is connected as in Theorem 3.6.8. (1). The group $((E_6)^\sigma)_{E_1}$ acts on

$$V^{10}=\{X\in\mathfrak{J}^C \mid 2E_1\times X=-\tau X\}=\left\{\begin{pmatrix}\xi & x\\ \bar{x} & -\tau\xi\end{pmatrix} \mid \xi\in C,\ x\in\mathfrak{C}\right\}$$

with the norm $\langle X,X\rangle/2=\xi(\tau\xi)+x\bar{x}$. We can define a homomorphism $\pi: ((E_6)^\sigma)_{E_1}\to SO(10)=SO(V^{10})$. $\operatorname{Ker}\pi=\{1,\sigma\}=Z_2$. Since $(\mathfrak{e}_6)^\sigma=(\mathfrak{f}_4)^\sigma\oplus i(\tilde{\mathfrak{J}}(3,\mathfrak{C})_0)^\sigma$ (Proposition 3.1.1) and $((\mathfrak{e}_6)^\sigma)_{E_1}=(\mathfrak{f}_4)^\sigma\oplus i\tilde{\mathfrak{J}}(2,\mathfrak{C})_0$, $\dim((\mathfrak{e}_6)^\sigma)_{E_1}=36+9=45=\dim\mathfrak{so}(10)$, hence π is onto. Thus $((E_6)^\sigma)_{E_1}/Z_2\cong SO(10)$. Therefore $((E_6)^\sigma)_{E_1}$ is isomorphic to $Spin(10)$ as the universal covering group of $SO(10)$. (In reality $((E_6)^\sigma)_{E_1}=(E_6)_{E_1}$). Thus $\psi: U(1)\times Spin(10)\to(E_6)^\sigma$, $\psi(\theta,\beta)=\phi(\theta)\beta$ where $\phi(\theta)$ is one defined in Lemma 3.6.5, induces the required isomorphism. $(E_{6(-14)})^\sigma=(\tau\lambda\sigma)^\sigma=(\tau\lambda)^\sigma$.

(2) For $\alpha\in(E_{6(2)})^\sigma=(\tau\lambda\gamma)^\sigma$, $\alpha E_1=\xi E_1$, $\xi\in U(1)$. The connected group $((E_{6(2)})^\sigma)_{E_1}$, as in Theorem 3.6.8. (1), acts on

$$V^{6,4}=\{X\in\mathfrak{J}^C \mid 2E_1\times X=-\tau\gamma X\}=\left\{\begin{pmatrix}\xi & x'\\ \bar{x}' & -\tau\xi\end{pmatrix} \mid \xi\in C,\ x'\in(\mathfrak{C}^C)_{\tau\gamma}=\mathfrak{C}'\right\}$$

with the norm $\langle X,X\rangle_\gamma/2=\xi(\tau\xi)+x'\bar{x}'$. As in (1), we have $((E_{6(2)})^\sigma)_{E_1}/Z_2\cong O(6,4)_0=O(V^{6,4})_0$. Therefore $((E_{6(2)})^\sigma)_{E_1}$ is denoted by $spin(6,4)$ (not simply connected) as a double covering group of $O(6,4)_0$. Thus $\psi: U(1)\times spin(6,4)\to(E_{6(2)})^\sigma$, $\psi(\theta,\beta)=\phi(\theta)\beta$, induces the required isomorphism.

(3) $$E_{6(-14)}=(E_6{}^C)^{\tau\lambda\sigma}\cong(E_6{}^C)^{\tau\lambda\sigma'}$$

because $\sigma\sim\sigma'$ under $\delta\in F_4\subset E_6$: $\delta\sigma=\sigma'\delta$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 2.2.3). For $\alpha\in((E_6{}^C)^{\tau\lambda\sigma'})^\sigma=(\tau\lambda\sigma')^\sigma$, $\alpha E_1=\xi E_1$, $\xi\in U(1)$. The connected group $((\tau\lambda\sigma')^\sigma)_{E_1}$ acts on

$$V^{8,2}=\{X\in\mathfrak{J}^C \mid 2E_1\times X=\tau\sigma' X\}=\left\{\begin{pmatrix}\xi & x\\ \bar{x} & \tau\xi\end{pmatrix} \mid \xi\in C,\ x\in\mathfrak{C}\right\}$$

with the norm $\langle X,X\rangle_{\sigma'}/2=\xi(\tau\xi)-x\bar{x}$. As in (1), we have $((\tau\lambda\sigma')^\sigma)_{E_1}/Z_2\cong$

$O(8,2)_0=O(V^{8,2})_0$. Therefore $((\tau\lambda\sigma')^\sigma)_{E_1}$ is denoted by $spin(8,2)$ (not simply connected) as a double covering group of $O(8,2)_0$. Thus $\psi: U(1)\times spin(8,2)\to(\tau\lambda\sigma')^\sigma$, $\psi(\theta,\beta)=\phi(\theta)\beta$, induces the required isomorphism.

THEOREM 3.6.10. $(E_{6(2)})^\sigma\sim(\tau\lambda\rho)^\sigma\cong(U(1)\times spin^*(10))/\boldsymbol{Z}_4\cong(\tau\lambda\gamma\rho)^\sigma\sim(E_{6(-14)})^\sigma$.

PROOF.
$$E_{6(2)}=(E_6{}^C)^{\tau\lambda\gamma}\cong(E_6{}^C)^{\tau\lambda\rho}$$
because $\gamma\sim\rho$ under $\delta\in E_6$: $\delta\gamma=\rho\delta$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 3.2.3). As in Theorem 3.6.9, for $\alpha\in((E_6{}^C)^{\tau\lambda\rho})^\sigma=(\tau\lambda\rho)^\sigma$, $\alpha E_1=\xi E_1$, $\xi\in U(1)$ and the group $((\tau\lambda\rho)^\sigma)_{E_1}$ is connected. The group $((\tau\lambda\rho)^\sigma)_{E_1}$ acts on
$$(V^C)^{10}=\mathfrak{J}(2,\mathfrak{C}^C)=\left\{X=\begin{pmatrix}\xi_2 & x\\ \bar{x} & \xi_3\end{pmatrix}\,\middle|\,\xi_2,\xi_3\in C,\ x\in\mathfrak{C}^C\right\}$$
with the norm $(E_1,X,X)=\xi_2\xi_3-x\bar{x}$ and the inner product $\langle X,Y\rangle_\rho$. Here
$$\langle X,Y\rangle_\rho=\left\langle\begin{pmatrix}\xi_2 & x\\ \bar{x} & \xi_3\end{pmatrix},\begin{pmatrix}\eta_2 & y\\ \bar{y} & \eta_3\end{pmatrix}\right\rangle_\rho=\left(\tau\begin{pmatrix}-\xi_2 & -\boldsymbol{i}x\\ \boldsymbol{i}\bar{x}\boldsymbol{i} & \xi_3\end{pmatrix},\begin{pmatrix}\eta_2 & y\\ \bar{y} & \eta_3\end{pmatrix}\right)$$
$$=-(\tau\xi_2)\eta_2+(\tau\xi_3)\eta_3+2(\boldsymbol{i}\boldsymbol{i}\tau x,y)=(\tau\xi,\tau x)S\begin{pmatrix}\eta\\ y\end{pmatrix}$$
where $\xi=(\xi_2,\xi_3)$, $\eta=(\eta_2,\eta_3)$ and $S=\mathrm{diag}(-1,1,2iJ,2iJ,2iJ,2iJ)\in M(10,C)$. By the following coordinate transformation
$$\xi_2=is_2+s_3,\quad \xi_3=is_2-s_3,\quad \eta_2=it_2+t_3,\quad \eta_3=it_2-t_3,$$
we have $\xi_2\xi_3=-s_2{}^2-s_3{}^2$, $-(\tau\xi_2)\eta_2+(\tau\xi_3)\eta_3=2i(-(\tau s_3)t_2+(\tau s_2)t_3)$. Hence $(E_1,X,X)=-(s,x)E\begin{pmatrix}s\\ x\end{pmatrix}$ and $\langle X,Y\rangle_\rho=(\tau s,\tau t)(2iJ)\begin{pmatrix}t\\ y\end{pmatrix}$ where $s=(s_2,s_3)$, $t=(t_2,t_3)$. This shows that we have an isomorphism
$$\{\alpha\in\mathrm{Iso}_C((V^C)^{10})\mid(E_1,\alpha X,\alpha X)=(E_1,X,X),\ \langle\alpha X,\alpha Y\rangle_\rho=\langle X,Y\rangle_\rho\}$$
$$\cong\{A\in M(10,C)\mid{}^tAA=E,\ JA=(\tau A)J\}=O^*(10)=O^*((V^C)^{10}).$$
Thus we can define a homomorphism $\pi:((\tau\lambda\rho)^\sigma)_{E_1}\to SO^*(10)=(O^*(10))_0$ by $\pi(\alpha)=\alpha\mid(V^C)^{10}$. $\mathrm{Ker}\,\pi=\{1,\sigma\}=Z_2$. As similar to Theorem 3.6.9 $((\tau\lambda\rho)^\sigma)_{E_1}/Z_2\cong SO^*(10)$. Therefore $((\tau\lambda\rho)^\sigma)_{E_1}$ is denoted by $spin^*(10)$ (not simply connected) as a double covering group of $SO^*(10)$. And $\psi: U(1)\times spin^*(10)\to(\tau\lambda\rho)^\sigma$, $\psi(\theta,\beta)=\eta(\theta)\beta$, induces the required isomorphism as in Theorem 3.6.9.
$$E_{6(-14)}=(E_6{}^C)^{\tau\lambda\sigma}\cong(E_6{}^C)^{\tau\lambda\gamma\rho}$$
because $\sigma\sim\gamma\rho$ under $\delta\in E_6$: $\delta\sigma=\gamma\rho\delta$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 3.2.3). To determine the group $(((E_6{}^C)^{\tau\lambda\gamma\rho})^\sigma)_{E_1}=((\tau\lambda\gamma\rho)^\sigma)_{E_1}$, consider the space $(V^C)^{10}=\mathfrak{J}(2,\mathfrak{C}^C)$ with

the norm $(E_1, X, X)=\xi_2\xi_3-x\bar{x}$ and the inner product $\langle X, Y\rangle_{\gamma\rho}=(\tau\gamma\rho X, Y)=(\tau\xi, \tau\gamma x)S\binom{\eta}{y}=(\tau\xi, \tau x)S'\binom{\eta}{y}$ as above, where $S'=\mathrm{diag}(-1, 1, 2iJ, 2iJ, -2iJ, -2iJ)$ $\in M(10, C)$. Since J and $-J$ are conjugate in $O(2)$ (see Proposition 0.4), by a suitable coordinate transformation, we have $\langle X, Y\rangle_{\gamma\rho}=(\tau s, \tau x')(2iJ)\binom{t}{y'}$. This shows

$$\{\alpha\in \mathrm{Iso}_C((V^C)^{10}) \mid (E_1, \alpha X, \alpha X)=(E_1, X, X), \langle\alpha X, \alpha Y\rangle_{\gamma\rho}=\langle X, Y\rangle_{\gamma\rho}\}\cong O^*(10).$$

Hence by the same arguments just as before we have the isomorphism $(\tau\lambda\gamma\rho)^\sigma \cong(U(1)\times spin^*(10))/\boldsymbol{Z}_4$.

Appendix

The Cartan decompositions of the exceptional universal linear Lie groups of type G_2, F_4 and E_6 are given as follows.

G_2: simply connected compact Lie group of type G_2,
$G_2{}^C\simeq G_2\times\boldsymbol{R}^{14}$,
$G_{2(2)}\simeq(Sp(1)\times Sp(1))/\boldsymbol{Z}_2\times\boldsymbol{R}^8$,
F_4: simply connected compact Lie group of type F_4,
$F_4{}^C\simeq F_4\times\boldsymbol{R}^{52}$,
$F_{4(4)}\simeq(Sp(1)\times Sp(3))/\boldsymbol{Z}_2\times\boldsymbol{R}^{28}$,
$F_{4(-20)}\simeq Spin(9)\times\boldsymbol{R}^{16}$,
E_6: simply connected compact Lie group of type E_6,
$E_6{}^C\simeq E_6\times\boldsymbol{R}^{78}$,
$E_{6(6)}\simeq Sp(4)/\boldsymbol{Z}_2\times\boldsymbol{R}^{42}$,
$E_{6(2)}\simeq(Sp(1)\times SU(6))/\boldsymbol{Z}_2\times\boldsymbol{R}^{40}$,
$E_{6(-14)}\simeq(U(1)\times Spin(10))/\boldsymbol{Z}_4\times\boldsymbol{R}^{32}$,
$E_{6(-26)}\simeq F_4\times\boldsymbol{R}^{26}$.

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