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ON AXIOM SCHEMATA APPLICABLE TO THE FORMULAE WITH ε-SYMBOLS

(Dedicated to Prof. S. Maehara for his sixtieth birthday)

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§1. Introduction.

As is well known, if a theory T is consistent in the first order predicate calculus, then it is still consistent in the first order predicate calculus with Hilbert's ε-symbol (assumed with equality axioms), which we shall call ε-calculus in short. This fact was given by Hilbert and Bernays [2]. Maehara [3] showed that it is still consistent in ε-calculus extended with additional axioms:

$$(*)\quad \forall x(A(x) \longleftrightarrow B(x)) \longrightarrow \varepsilon xA(x)=\varepsilon xB(x),$$

where $A(a)$ and $B(a)$ are arbitrary formulae with possibly ε-symbols. These theorems however require careful reading, when the theory T has axiom schemata. In applying the theorems, axiom schemata for T may not be applied to formulae containing ε-symbols even in the ε-calculus. For instance, $ZF+\neg AC$, Zermelo-Fraenkel set theroy with the negation of the axiom of choice, yields a contradiction in ε-calculus if the schemata are allowed to apply to all formulae of the language of ε-calculus.

Noting this, we naturally ask the following:

(**) *What kind of axiom schemata can be consistently extended to the ε-calculus in the above sense?*

Even in the model theoretical sense, this is not a trivial question. It is partially answered in [3], where the schema is mathematical induction and the theory in question have certain strong properties. Our purpose here is to answer it in more general cases. The precise description of our result is given in the next section after defining some necessary notions.

§2. Preliminaries.

We use the letters L, L' to denote languages for the first order predicate

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calculus. An *L-formula* is a formula of L, $L(P)$ (respectively $L(\varepsilon)$) is the language obtained from L by adding a new predicate symbol P(respectively Hilbert's ε-symbol). We call shortly ε-calcults for the logical calculus for $L(\varepsilon)$, which has (in addition to the usual ones) the equality axioms for ε-terms and the axioms of the following forms:

$$\exists x B(x) \longrightarrow B(\varepsilon x B(x)),$$

where $B(a)$ is an arbitrary $L(\varepsilon)$-formula (possibly with free variables other than a). To avoid inessential complications, any language is assumed to have no function symbol. For easy readability, we write $\bar{x}$ for a finite sequence of variables $x_1, x_2, \cdots$. An $L(P)$-formula $A(P)$ (with the indicated P standing for all its occurrences in $A(P)$) is called an *L-schema*, and for a language L', *A(L'-formulae)* stands for the set of all formulae of the following form:

$$A(\lambda \bar{x} B(\bar{x})),$$

where $B(\bar{a})$ is an L'-formula.

DEFINITION 2.1. A theory T in L is S-rich, if there is an L-formula $S(a, b, c)$ (containing no other occurrence of free variable) such that the following two sentences are provable in T:

(1) $\exists x \forall y, z \neg S(x, y, z)$,

(2) $\forall u, x, y \exists v \forall z, w(S(v, z, w) \leftrightarrow S(u, z, w) \vee (z = x \wedge w = y))$.

DEFINITION 2.2. (1) An $L(P)$-formula is called a mono-positive L-clause, if it is a subformula of a formula of the following form:

$$A \wedge P(\bar{a}) \wedge \neg P(\bar{a}_1) \wedge \neg P(\bar{a}_2) \wedge \cdots \wedge \neg P(\bar{a}_n),$$

where A is an L-formula,

(2) An $L(P)$-formula is called a mono-negative L-clause, if it is a subformula of a formula of the following form:

$$A \wedge \neg P(\bar{a}) \wedge P(\bar{a}_1) \wedge P(\bar{a}_2) \wedge \cdots \wedge P(\bar{a}_n)$$

where A is an L-formula.

(3) A mono-positive (resp. mono-negative) L-dnf is a finite disjunction of mono-positive (resp. mono-negative) clauses.

(4) A fomula $\exists \bar{x} A$ is called an m.p.Σ (resp. m.n.Σ) L-formula, if A is a mono-positive (resp. mono-negative) L-dnf.

(5) An $L(P)$-formula is called m.p.Π_2 (resp. m.n.Π_2) L-formula if it is of the form $\forall \bar{x} A$ with A a m.p.Σ (resp. m.n.Σ) L-formula.

(6) An L-schema $A(P)$ is called monadic Π_2 if it is a m.p. Π_2 L-formula or a m.n. Π_2 L-formula.

Now, our result can be described as follows:

THEOREM 2.3. (main theorem) *Let T be a set of L-formulae and $A(P)$ a monadic Π_2 L-schema. If the theory*

$$T+A(L\text{-}formulae)$$

is S-rich and consistent, then the theory

$$T+A(L(\varepsilon)\text{-}formulae)$$

is consistent in ε-calculus.

Assuming a slightly stronger property for T, we can obtain a similar result for ε-calculus extended with the axioms (*) (see Introduction). But, so far, we have not succeeded in proving it for every S-rich theory T.

As will be stated in the last section, the mathematical induction schema is essentially monadic Π_2, so:

COROLLARY 2.4. (cf. Theorem 3 of [1]) *Let T be an S-rich theory in L, and suppose that there are a unary formula $N(a)$, a unary function $'$, and a constant 1 such that the formula $N(1)$ and the formula*

$$(\#) \quad (A(1)\wedge\forall x(A(x)\longrightarrow A(x'))\longrightarrow\forall x(N(x)\longrightarrow A(x))$$

are provable in T for every L-formula $A(a)$. Then the theory T together with additional axioms of the form (#) for all $L(\varepsilon)$-formulae $A(a)$ is consistent in ε-calculus.

In fact, we shall prove Theorem 2.3. in a stronger form:

THEOREM 2.5. *Suppose T is a set of L-formulae and $A_1(P_1), \cdots, A_n(P_n)$ are monadic Π_2 L-schema. If the theory*

$$T+A_1(L\text{-}formulae)+\cdots+A_n(L\text{-}formulae)$$

is S-rich and conststent, then the theory

$$T+A_1(L(\varepsilon)\text{-}formulae)+\cdots+A_n(L(\varepsilon)\text{-}formulae)$$

is consistent in ε-calculus.

The above theorem is obtained by a routine argument repeatedly using the following lemma:

LEMMA 2.6. *Suppose T is a set of L-formulae, $A_1(P_1), \cdots, A_n(P_n)$ are monadic Π_2 L-schemata, the theory $T+A_1$(L-formulae)$+\cdots+A_n$(L-formulae) is S-rich and consistent, $B(\bar{a}, b)$ is an L-formula (all free variables in which are indicated by $\bar{a}, b$), Q is a predicate symbol not in L and $E(Q)$ is the set of four $L(Q)$-formulae $\forall\bar{x}\exists yQ(\bar{x}, y)$, $\forall\bar{x}\forall y\forall z(Q(\bar{x}, y)\wedge Q(\bar{x}, z)\rightarrow y=z)$, $\forall\bar{w}\forall\bar{x}\forall y\forall z(\bar{w}=\bar{x}\wedge y=z\wedge Q(\bar{w}, y)\rightarrow Q(\bar{x}, z))$ and $\forall\bar{x}\forall y(\exists zB(\bar{x}, z)\wedge Q(\bar{x}, y)\rightarrow B(\bar{x}, y))$. Then the theory $T+E(Q)+A_1(L(Q)$-formulae$)+\cdots+A_n(L(Q)$-formulae$)$ is S-rich and consistent.*

§3. Proof of the theorem.

It suffices to prove Lemma 2.6, so the rest is devoted to the proof of the lemma. We assume the hypothesis of Lemm 2.6 throughout this section. Since the S-richness is obviously preserved by extension, the only problem is in the consistency.

Let T^* stand for the theory $T+A_1$(*L-formulae*)$+\cdots+A_n$(*L-formulae*). By S-richness of T^*, there is an L-formula $S(a, b, c)$ such that the following are provable in T^*:

(1) $\exists x\forall y, z\neg S(x, y, z)$,

(2) $\forall u, x, y\exists v\forall z, w(S(v, z, w)\leftrightarrow S(u, z, w)\vee(z=x\wedge w=y))$.

The proof is done by a so-called forcing argument, for which we prepare some notions.

DEFINITION 3.1. Let n be the length of the sequence $\bar{a}$ used in the L-formula $B(\bar{a}, b)$.

(1) Tuple $(a, a_1, \cdots, a_n)$ stand for the L-formula $\exists z_1\cdots z_n(a=z_1\wedge z_n=a_n\wedge \bigwedge_{i=1\dots n-1}\forall x, y(S(z_i, x, y)\leftrightarrow x=a_i\wedge y=z_{i+1}))$,

(2) $a\sim b$ stands for the L-formula

$$\exists\bar{x}(\text{Tuple}(a, \bar{x})\wedge\text{Tuple}(b, \bar{x})),$$

(3) Cond(a) stands for the L-formula
$\forall w, x, y, z(S(a, w, x)\wedge S(a, y, z)\wedge w\sim y\rightarrow x=z)\wedge\forall w, \bar{x}, y(S(a, w, y)\wedge\text{Tuple}(w, \bar{x})\wedge\exists zB(\bar{x}, z)\rightarrow B(\bar{x}, y))$,

(4) $a\supset b$ stands for the L-formula

$$\forall x, y(S(b, x, y)\longrightarrow S(a, x, y)).$$

Note that:

LEMMA 3.2. *The following are provable in* T^*:

(1) $\forall x, y \exists z \forall v, w(S(z, v, w) \leftrightarrow v=x \wedge w=y)$,

(2) $\forall \bar{x} \exists y\, Tuple(y, \bar{x})$.

Clearly, the relation $p \supset q$ is shown to be a partial order in T^*, and $\exists x Cond(x)$ follows from $\exists x \forall y, z \neg S(x, y, z)$. We use the letters p, q, and r as variables satisfying $Cond(\cdot)$; thus, $\forall p(\cdots)$ and $\exists p(\cdots)$ stand for $\forall p(Cond(p) \rightarrow \cdots)$ and $\exists p(Cond(p) \wedge \cdots)$ respectively.

DEFINITION 3.3. With a variable p and an arbitrary $L(Q)$-formula C, we associate an L-formula $p \Vdash C$ by the following recursive definition:

(1) $p \Vdash R(\bar{a})$ stands for the formula $R(\bar{a})$, if R is a predicate symbol in L.

(2) $p \Vdash Q(\bar{a}, b)$ stands for $\exists x(\mathrm{Tuple}(x, \bar{a}) \wedge S(p, x, b))$,

(3) $p \Vdash C \wedge D$ stands for $p \Vdash C \wedge p \Vdash D$,

(4) $p \Vdash C \vee D$ stands for $p \Vdash C \vee p \Vdash D$,

(5) $p \Vdash \neg C$ stands for $\forall q(q \supset p \rightarrow \neg q \Vdash C)$,

(6) $p \Vdash C \rightarrow D$ stands for $\forall q \supset p(q \Vdash C \rightarrow q \Vdash D)$,

(7) $p \Vdash \forall x C$ stands for $\forall x \forall q \supset p \exists r \supset q(r \Vdash C)$,

(8) $p \Vdash \exists x C$ stands for $\exists x(p \Vdash C)$.

We let $p \Vdash^* C$ stand for the formula $\forall q \supset p \exists r \supset q(r \Vdash C)$, and $\Vdash^* C$ for $\forall p(p \Vdash^* C)$.

LEMMA 3.4. (1) *If an $L(Q)$-formula C is logically valid, then the formula $p \Vdash^* C$ is provable in T^*,*

(2) *For any $L(Q)$-formulae C and D, the formula $(p \Vdash^* C) \wedge (p \Vdash^* (C \rightarrow D)) \rightarrow (p \Vdash^* D)$ is provable in T^*,*

(3) *If C is an L-formula, then the formula $(p \Vdash^* C) \leftrightarrow C$ is provable in T^*,*

PROOF. The assertions (1) and (2) are proved by routine arguments. (3) is clear from the definition.

By virtue of (1) and (2) of the lemma, the set of all those $L(Q)$-formulae C that $(\Vdash^* C)$ is provable in T^* is closed under logical deductions, and from (3) follows that for all formulae C in T^*, $(\Vdash^* C)$ is provable in T^*. So, in order to show the consistency of the theory in question, we suffices to show that:

LEMMA 3.5. *The following are theorems in* T^*:

(1) $\Vdash^* \forall \bar{x} \exists y Q(\bar{x}, y)$,

(2) $\Vdash^* \forall \bar{x}, y, z(Q(\bar{x}, y) \wedge Q(\bar{x}, z) \rightarrow y=z)$,

(3) $\Vdash^* \forall \bar{x}, y(\exists z B(\bar{x}, z) \wedge Q(\bar{x}, y) \to B(\bar{x}, y))$,

(4) $\Vdash^* \forall \bar{w} \forall \bar{x} \forall y \forall z(\bar{w}=\bar{x} \wedge y=z \wedge Q(\bar{w}, y) \to Q(\bar{x}, z))$

(5) $\Vdash^* A_i(\lambda \bar{x} C(\bar{x}))$,

for every $i=1, \cdots, n$, *and every* $L(Q)$*-formula* $C(\bar{a})$.

PROOF. (1) Note that in general, $p \Vdash^* \forall \bar{x} C \leftrightarrow \forall \bar{x}(p \Vdash^* C)$. Thus:

$$\Vdash^* \forall \bar{x} \exists y Q(\bar{x}, y)$$

$$\longleftrightarrow \forall \bar{x} \Vdash^* \exists y Q(\bar{x}, y)$$

$$\longleftrightarrow \forall \bar{x} \forall p \exists q \supset p \exists y(q \Vdash Q(\bar{x}, y))$$

Suppose (in T^*) that $\bar{x}$ and p are given. If $\exists v, w(Tuple(v, \bar{x}) \wedge S(p, v, w))$, then by such v and w, $p \Vdash Q(\bar{x}, w)$. So, suppose not. By lemma 3.2.(2), there is v_1 such that $Tuple(v_1, \bar{x})$. Take y_1 such that $(\exists z B(\bar{x}, z) \to B(\bar{x}, y_1))$. By the property of the formula S, there is q such that

$$\forall z, w(S(q, z, w) \longleftrightarrow S(p, z, w) \vee (z=v_1 \wedge w=y_1)).$$

Then, $Cond(q)$, $q \supset p$, and $Tuple(v_1, \bar{x})$ and $S(q, v_1, y_1)$, which implies $q \Vdash Q(\bar{x}, y_1)$, q. e. d.

(2) It suffices to show that for any $\bar{x}, y, z$, $\forall p(p \Vdash Q(\bar{x}, y) \wedge Q(\bar{x}, y) \to y=z)$. Suppose $p \Vdash Q(\bar{x}, y)$ and $p \Vdash Q(\bar{x}, z)$. Take v, w such that $Tuple(v, \bar{x})$, $Tuple(w, \bar{x})$, and $S(p, v, y) \wedge S(p, w, y)$. Then $v \sim w$, which implies $y=z$ by $Cond(p)$, q. e. d.

(3) Let $\bar{x}$, y, and p be given, and suppose that $p \Vdash Q(\bar{x}, y)$ and $\exists z B(\bar{x}, z)$. Then we can take w such that $Tuple(w, \bar{x})$ and $S(p, w, y)$. Hence by $Cond(p)$, $B(\bar{x}, y)$, q. e. d.

(4) Follows from the equality axioms in T^*.

(5) We use Sublemma 3.6 below. By the assumption, each $A_i(P_i)$ is monadic Π_2 L-schema. Hence, for every $L(Q)$-formula $C_i(\bar{a})$, by Sublemma 3.6. (5), there is an L-formula $C_i^*(p, \bar{a})$ such that $\forall p A_i(\lambda \bar{x} C_i^*(p, \bar{x})) \to (\Vdash^* A_i(\lambda \bar{x} C_i(\bar{x}))$ is provable in T^*. Since the left hand of the implication is an axiom of T^*, $\Vdash^* A_i(\lambda \bar{x} C_i(\bar{x}))$ is provable in T^*, q. e. d.

SUBLEMMA 3.6. (1) *If an* $L(P)$*-formula* $C(P)$ *is a mono-positive L-clause, then the following is provable in* T^* *for any* $L(Q)$*-formula* $D(\bar{a})$:

$$C(\lambda \bar{x}(\exists q \supset p(\Vdash q^* D(\bar{x})))) \longrightarrow (\exists q \supset p)(q \Vdash^* C(\lambda \bar{x} D(\bar{x}))),$$

(2) *If an* $L(P)$*-formula* $C(P)$ *is a mono-negative L-clause, then the following is provable in* T^* *for any* $L(Q)$*-formula* $D(\bar{a})$:

$$C(\lambda \bar{x}(p \Vdash^* D(\bar{x}))) \longrightarrow (\exists q \supset p)(q \Vdash^* C(\lambda \bar{x} D(\bar{x}))),$$

(3) *If an $L(P)$-formula $C(P)$ is a mono-positive or mono-negative L-dnf, then for any $L(Q)$-formula $D(\bar{a})$, there is an L-formula $D^*(p, \bar{a})$ such that the following is provable in T^*:*

$$C(\lambda\bar{x}D^*(p, \bar{x})) \longrightarrow (\exists q \supset p)(q \Vdash^* C(\lambda\bar{x}D(\bar{x}))),$$

(4) *If an $L(P)$-formula $C(P)$ is m.p. Σ or m.n. Σ, then for and $L(Q)$-formula $D(\bar{a})$, there is an L-formula $D^*(p, \bar{a})$ such that the following is provable in T^*:*

$$C(\lambda\bar{x}D^*(p, \bar{x})) \longrightarrow (\exists q \supset p)(q \Vdash^* C(\lambda\bar{x}D(\bar{x}))),$$

(5) *If an L-schema $C(P)$ is monadic Π_2, then for any $L(Q)$-formula $D(\bar{a})$, there is an L-formula $D^*(p, \bar{a})$ such that the following is provable in T^*:*

$$\forall p C(\lambda\bar{x}D^*(p, \bar{x})) \longrightarrow (\Vdash^* C(\lambda\bar{x}D(\bar{x}))).$$

PROOF. (1) If $C(P)$ is of the form $A \wedge \neg P(\bar{a}_1) \wedge \cdots \wedge \neg P(\bar{a}_n) \wedge P(\bar{a})$, the left hand of the arrow is of the form

$$A \wedge \neg \exists q \supset p(q \Vdash^* D(\bar{a}_1)) \wedge \cdots \wedge \neg \exists q \supset p(q \Vdash^* D(\bar{a}_n) \wedge \exists q \supset p(q \Vdash^* D(\bar{a})),$$

which implies $A \wedge p \Vdash^* \neg D(\bar{a}_1) \wedge \cdots p \Vdash^* \neg D(\bar{a}_n) \wedge \exists q \supset p(q \Vdash^* D(\bar{a}))$, which implies $(\exists q \supset p)(q \Vdash^* A \wedge \neg D(\bar{a}_1) \wedge \cdots \wedge \neg D(\bar{a}_n) \wedge D(\bar{a}))$, which is the right hand of the arrow.

(2) Note that $\neg(p \Vdash^* D(\bar{a}))$ implies $(\exists q \supset p)(q \Vdash^* \neg D(\bar{a}))$. The rest is similar to the above.

(3) Let $D^*(p, \bar{a})$ be $(\exists q \supset p)(q \Vdash^* D(\bar{a}))$ or $(p \Vdash^* D(\bar{a}))$ according whether $C(P)$ is mono-positive or mono-negative. The assertion comes from the fact that $(\exists q \supset p)(q \Vdash^* C_1) \vee (\exists q \supset p)(q \Vdash^* C_2)$ implies $(\exists q \supset p)(q \Vdash^* C_1 \vee C_2)$.

(4) $C(P)$ is of the form $\exists \bar{z} C'(P, \bar{z})$. By the above, for any $L(Q)$-formula $D(\bar{a})$ there is an L-formula $D^*(p, \bar{a})$ such that (in T^*) $C'(\lambda\bar{x}D^*(p, \bar{x}), \bar{b}) \rightarrow (\exists q \supset p)(q \Vdash^* C'(\lambda\bar{x}D(\bar{x}), \bar{b})$. Thus the assertion immediately follows from the fact that $(\exists \bar{x})(\exists q \supset p)(q \Vdash^* E(\bar{x}))$ implies $(\exists q \supset p)(q \Vdash^* \exists \bar{x} E(\bar{x}))$

(5) $C(P)$ is of the form $\forall \bar{y} C'(P, \bar{y})$. By the above,

$$C'(\lambda\bar{x}D^*(p, \bar{x}), \bar{b}) \longrightarrow (\exists q \supset p)(q \Vdash^* C'(\lambda\bar{x}D(\bar{x}, \bar{b})).$$

Hence, from $\forall p C(\lambda\bar{x}D^*(p, \bar{x}))$ follows $\forall \bar{y} \forall p(\exists q \supset p)(q \Vdash^* C'(\lambda\bar{x}D(\bar{x}), \bar{y}))$,
which implies $\forall \bar{y}(\Vdash^* C'(\lambda\bar{x}D(\bar{x}), \bar{y}))$,
which implies $\Vdash^* \forall \bar{y} C'(\lambda\bar{x}D(\bar{x}), \bar{y}))$, q.e.d.

§4. Remarks.

As an example of monadic Π_2 schemata is a schema of mathematical induction, whose standard form is

$$P(1)\wedge\forall x(N(x)\wedge P(x)\longrightarrow P(x'))\longrightarrow\forall x(N(x)\longrightarrow P(x)),$$

which can be written equivalently as

$$\forall x\exists y,\ z(\neg P(1)\vee(N(y)\wedge z=y'\wedge P(y)\wedge\neg P(z))\vee\neg(N(x)\vee P(x)).$$

On the other hand, there is a monadic Π_3 schema that holds in $ZF+\neg AC$ but yields a contradiction in ε-calculus. The following is such one:

$$\forall x\exists y\forall v,\ w,\ z(z=\langle v,\ w\rangle\longrightarrow(z\in y\longleftrightarrow w\in x\wedge v\in w\wedge P(v,\ w)).$$

The gap between the forms of these two schemata is rather wide. Natural questions arising from this are left open.

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