TSUKUBA J. MATH.
Vol. 38 No. 2 (2014), 201–206

THE WEIERSTRASS SEMIGROUPS ON DOUBLE COVERS OF GENUS TWO CURVES

By

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Abstract. We show that three numerical semigroups $\langle 5,6,7,8\rangle$, $\langle 3,7,8\rangle$ and $\langle 3,5\rangle$ are of double covering type, i.e., the Weierstrass semigroups of ramification points on double covers of curves. Combining the result with [7] and [4] we can determine the Weierstrass semigroups of the ramification points on double covers of genus two curves.

1. Introduction

Let $\mathbf{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbf{N}_0$ is called a *numerical semigroup* if its complement $\mathbf{N}_0 \backslash H$ is a finite set. The cardinality of $\mathbf{N}_0 \backslash H$ is called the *genus* of H, which is denoted by $g(H)$. For any positive integers $a_1, a_2, \ldots, a_n$ we denote by $\langle a_1, a_2, \ldots, a_n\rangle$ the additive monoid $a_1\mathbf{N}_0 + a_2\mathbf{N}_0 + \cdots + a_n\mathbf{N}_0$ generated by $a_1, a_2, \ldots, a_n$. A numerical semigroup of genus 2 is either $\langle 2,5\rangle$ or $\langle 3,4,5\rangle$, which plays an important role in this article.

Let C be a complete nonsingular irreducible curve over an algebraically closed field of characteristic 0, which is called a *curve* in this paper. For a point P of C, we set

$$H(P) = \{\alpha \in \mathbf{N}_0 \mid \text{there exists a rational function } f \text{ on } C \text{ with } (f)_\infty = \alpha P\},$$

The second author is partially supported by Grant-in-Aid for Scientific Research (24540057), Japan Society for the Promotion Science.
The third author is partially supported by Grant-in-Aid for Scientific Research (24540042), Japan Society for the Promotion Science.

2010 *Mathematics Subject Classification*: 14H55, 14H45, 20M14.
Key words and phrases: Numerical semigroup, Weierstrass semigroup, Double cover of a curve, Curve of genus two.
Received February 6, 2014.
Revised August 18, 2014.

which is called the *Weierstrass semigroup of P*. It is known that the Weierstrass semigroup of a point on a curve of genus g is a numerical semigroup of genus g.

For a numerical semigroup $\tilde{H}$ we denote by $d_2(\tilde{H})$ the set consisting of the elements $\tilde{h}/2$ with even $\tilde{h} \in \tilde{H}$, which becomes a numerical semigroup. A numerical semigroup $\tilde{H}$ is said to be *of double covering type* if there exists a double covering $\pi : \tilde{C} \to C$ of a curve C with a ramification point $\tilde{P} \in \tilde{C}$ over $P \in C$ satisfying $H(\tilde{P}) = \tilde{H}$. In this case we have $d_2(H(\tilde{P})) = H(P)$ (for example, see Lemma 2 in [3]). We are interested in numerical semigroups of double covering type. Let $\tilde{H}_0$ be a numerical semigroup of genus $\tilde{g}$ with $d_2(\tilde{H}_0) = \mathbf{N}_0$ where the genus of $\mathbf{N}_0$ is 0. Then the semigroup $\tilde{H}_0$ is $\langle 2, 2\tilde{g}+1\rangle$, which is the Weierstrass semigroup of a ramification point $\tilde{P}$ on a double cover of the projective line where the covering curve is of genus $\tilde{g}$. Hence, $\tilde{H}_0$ is of double covering type. Let $\tilde{H}_1$ be a numerical semigroup of genus $\tilde{g}$ with $d_2(\tilde{H}_1) = \langle 2, 3\rangle$ where $\langle 2, 3\rangle$ is the only one numerical semigroup of genus 1. Then the semigroup $\tilde{H}_1$ is either $\langle 3, 4, 5\rangle$ or $\langle 3, 4\rangle$ or $\langle 4, 5, 6, 7\rangle$ or $\langle 4, 6, 2\tilde{g}-3\rangle$ with $\tilde{g} \geq 4$ or $\langle 4, 6, 2\tilde{g}-1, 2\tilde{g}+1\rangle$ with $\tilde{g} \geq 4$. We can show that there is a double covering of an elliptic curve with a ramification point whose Weierstrass semigroup is any semigroup in the above ones (for example, see [2], [4]).

Let $\tilde{H}_2$ be a numerical semigroup of genus $\tilde{g}$ with $g(d_2(\tilde{H}_2)) = 2$. Oliveira and Pimentel [7] studied the semigroup $\tilde{H}_2 = \langle 6, 8, 10, n\rangle$ with an odd number $n \geq 11$. They showed that the semigroup $\tilde{H}_2$ is of double covering type. In this case we have $d_2(\tilde{H}_2) = \langle 3, 4, 5\rangle$. Moreover, in [4] we proved that any numerical semigroup $\tilde{H}_2$ with $d_2(\tilde{H}_2) = \langle 3, 4, 5\rangle$ except $\tilde{H}_2 = \langle 5, 6, 7, 8\rangle$, $\langle 3, 7, 8\rangle$, $\langle 3, 5\rangle$ and $\langle 3, 5, 7\rangle$ is of double covering type. The semigroup $\langle 3, 5, 7\rangle$ is not of double covering type, because of the fact that $g(\langle 3, 5, 7\rangle) = 3 < 2 \cdot 2$. Using the result of Main Theorem in [6] every numerical semigroup $\tilde{H}_2$ with $d_2(\tilde{H}_2) = \langle 2, 5\rangle$ is of double covering type. In this paper we will study the remaining three numerical semigroups. Namely we prove the following:

THEOREM 1.1. *The three numerical semigroups* $\langle 5, 6, 7, 8\rangle$, $\langle 3, 7, 8\rangle$ *and* $\langle 3, 5\rangle$ *are of double covering type.*

Combining this theorem with the results in [7] and [4], we have the following conclusion:

THEOREM 1.2. *Let* $\tilde{H}$ *be a numerical semigroup with* $g(d_2(\tilde{H})) = 2$. *If* $\tilde{H} \neq \langle 3, 5, 7\rangle$, *then it is of double covering type, and vice versa.*

2. Proof of Theorem 1.1

To prove that the three numerical semigroups are of double covering type we use the following theorem which is stated in Theorem 2.2 of [5].

Theorem 2.1. *Let $\tilde{H}$ be a numerical semigroup. We set*

$$n = \min\{\tilde{h} \in \tilde{H} \mid \tilde{h} \text{ is odd}\}.$$

Then we get

$$g(\tilde{H}) = 2g(d_2(\tilde{H})) + (n-1)/2 - r$$

with some non-negative integer r (for example, see Lemma 3.1 in [1]). Assume that $H = d_2(\tilde{H})$ is Weierstrass. Take a pointed curve (C, P) with $H(P) = H$. Let $Q_1, \dots, Q_r$ be points of C different from P with $h^0(Q_1 + \cdots + Q_r) = 1$. Moreover, assume that $\tilde{H}$ has an expression

$$\tilde{H} = 2H + \langle n, n+2l_1, \dots, n+2l_s\rangle$$

of generators with positive integers $l_1, \dots, l_s$ such that

$$h^0(l_iP + Q_1 + \cdots + Q_r) = h^0((l_i - 1)P + Q_1 + \cdots + Q_r) + 1$$

for all i. If the divisor $nP - 2Q_1 - \cdots - 2Q_r$ is linearly equivalent to some reduced divisor not containing P, then there is a double covering $\pi : \tilde{C} \to C$ with a ramification point $\tilde{P}$ over P satisfying $H(\tilde{P}) = \tilde{H}$, hence $\tilde{H}$ is of double covering type.

Proof. By seeing the proof of Theorem 2.2 in [5] we may replace the assumption in Theorem 2.2 in [5] that the complete linear system $|nP - 2Q_1 - \cdots - 2Q_r|$ is base point free by the above assumption that the divisor $nP - 2Q_1 - \cdots 2Q_r$ is linearly equivalent to some reduced divisor not containing P. $\square$

Now we prove Theorem 1.1 in each case.

Case 1. Let $\tilde{H} = \langle 5, 6, 7, 8\rangle$. Then we have $H = d_2(\tilde{H}) = \langle 3, 4, 5\rangle$ and $g(\tilde{H}) = 5 = 2 \cdot 2 + (5-1)/2 - 1$. Moreover, we have $\tilde{H} = 2H + \langle 5, 5 + 2 \cdot 1\rangle$. Let C be a curve of genus 2 and ι the hyperelliptic involution on C. If we take a general point P of C with $H(P) = \langle 3, 4, 5\rangle$, then we may assume that $3(P - \iota(P)) \not\sim 0$. Indeed, assume that $3(P - \iota(P)) \sim 0$ for all point P with $H(P) = \langle 3, 4, 5\rangle$. Then there are distinct points P_1 and P_2 with $H(P_i) = \langle 3, 4, 5\rangle$, $i = 1, 2$ such that $P_1 - \iota(P_1) \sim P_2 - \iota(P_2)$, because the number of the linearly equivalent classes of the divisors D of degree 0 satisfying $3D \sim 0$ is finite. Hence, we get

$P_1 + \imath(P_2) \sim P_2 + \imath(P_1)$, which implies that $P_1 + \imath(P_2) \sim P_1 + \imath(P_1)$. This is a contradiction. Now we have $h^0(P + \imath(P)) = 2 = h^0(\imath(P)) + 1$. Moreover, if the complete linear system $|5P - 2\imath(P)|$ has a base point R, then we have $R \neq P$. Indeed, we assume that $R = P$. Then we have

$$h^0(5P - 2\imath(P) - P) = h^0(5P - 2\imath(P)) = 3 + 1 - 2 = 2,$$

which implies that

$$4P - 2\imath(P) \sim g_2^1 \sim P + \imath(P).$$

Hence, we get $3(P - \imath(P)) \sim 0$. This is a contradiction. We assume that $|5P - 2\imath(P)|$ has a base point R. Then we get $5P - 2\imath(P) \sim R + E$, where E is an effective divisor of degree 2 with projective dimension 1. In this case the complete linear system $|E|$ is base point free. Therefore, the divisor $5P - 2\imath(P)$ is linearly equivalent to some reduced divisor not containing P. If $|5P - 2\imath(P)|$ is base point free, then the divisor $5P - 2\imath(P)$ satisfies the above condition. By Theorem 2.1 the semigroup $\tilde{H} = \langle 5, 6, 7, 8 \rangle$ is of double covering type.

Case 2. Let $\tilde{H} = \langle 3, 7, 8 \rangle$. Then we have $H = d_2(\tilde{H}) = \langle 3, 4, 5 \rangle$ and $g(\tilde{H}) = 4 = 2 \cdot 2 + (3 - 1)/2 - 1$. Moreover, we have $\tilde{H} = 2H + \langle 3, 3 + 2 \cdot 2 \rangle$. We may take a pointed curve (C, P) with $H(P) = \langle 3, 4, 5 \rangle$ such that the covering $\varphi : C \to \mathbf{P}^1$ corresponding to the complete linear system $|3P|$ has a simple ramification point Q. Then there is another simple ramification point of φ by Riemann-Hurwitz formula. Hence, we may assume that $\imath P \neq Q$, which implies that $P + Q \not\sim g_2^1$. Thus, we get $h^0(2P + Q) = 2 = h^0(P + Q) + 1$. Let R be the point satisfying $2Q + R \sim 3P$. Then we have $R \neq P$ and $3P - 2Q \sim R$. By Theorem 2.1 the semigroup $\tilde{H} = \langle 3, 7, 8 \rangle$ is of double covering type.

Case 3. Let $\tilde{H} = \langle 3, 5 \rangle$. Then we have $H = d_2(\tilde{H}) = \langle 3, 4, 5 \rangle$ and $g(\tilde{H}) = 4 = 2 \cdot 2 + (3 - 1)/2 - 1$. Moreover, we have $\tilde{H} = 2H + \langle 3, 3 + 2 \cdot 1 \rangle$. Let C be a curve whose function field is $k(x, y)$ with an equation $y^3 = (x - c_1)(x - c_2)(x - c_3)^2$, where c_1, c_2 and c_3 are distinct elements of k. Let $\pi : C \to \mathbf{P}^1$ be the morphism corresponding to the inclusion $k(x) \subset k(x, y)$. Then C is of genus 2 by Riemann-Hurwitz formula. Let $P = P_1$, P_2, P_3 and P_4 be the ramification points of π. Since π is a cyclic covering, it induces an automorphism σ of C with $C/\langle \sigma \rangle \cong \mathbf{P}^1$. Let $\imath$ be the hyperelliptic involution on C. Then we have $\sigma \circ \imath = \imath \circ \sigma$. Indeed, we have

$$(\sigma \circ \imath \circ \sigma^{-1}) \circ (\sigma \circ \imath \circ \sigma^{-1}) = \sigma \circ \imath \circ \imath \circ \sigma^{-1} = \sigma \circ \sigma^{-1} = id.$$

Hence, the automorphism $\sigma \circ \imath \circ \sigma^{-1}$ is an involution. Moreover, we have a bijective correspondence between the sets $\mathrm{Fix}(\imath)$ and $\mathrm{Fix}(\sigma \circ \imath \circ \sigma^{-1})$ sending Q

to $\sigma(Q)$, where $\mathrm{Fix}(\imath)$ and $\mathrm{Fix}(\sigma \circ \imath \circ \sigma^{-1})$ are the sets of the fixed points by $\imath$ and $\sigma \circ \imath \circ \sigma^{-1}$ respectively. Hence, $\sigma \circ \imath \circ \sigma^{-1}$ is also the hyperelliptic involution. Thus, we have $\sigma \circ \imath \circ \sigma^{-1} = \imath$. Since $\sigma(\imath(P)) = \imath(\sigma(P)) = \imath(P)$, the point $\imath(P)$ is a fixed point of σ. Moreover, we have $H(P) \ni 3$, which implies that $\imath P \neq P$. Hence, we have $\imath P = P_i$ for some $i \in \{2, 3, 4\}$. Then we obtain $h^0(P + P_i) = 2 = h^0(P_i) + 1$. Moreover, we have

$$3P - 2P_i \sim 3P_i - 2P_i = P_i \neq P.$$

By Theorem 2.1 the semigroup $\langle 3, 5 \rangle$ is of double covering type. $\square$

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