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DENSE CHAOS AND DENSELY CHAOTIC OPERATORS*

By

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Abstract. The aim of this paper is to study dense chaos and densely chaotic operators on Banach spaces. First, we prove that a dynamical system is densely δ-chaotic for some $\delta > 0$ if and only if it is densely chaotic and sensitive. Meanwhile, we also show that for general dynamical systems, Devaney chaos and dense chaos do not imply each other. Then, by using these results, we have that for a operator defined on a Banach space, dense chaos, dense δ-chaos, generic chaos and generic δ-chaos are equivalent and they are all strictly stronger than Li-Yorke chaos.

1. Introduction and Basic Definitions

By a *topological dynamical system* (briefly, dynamical system or system), we mean a pair (X, f), where X is a complete metric space without isolated points and $f : X \to X$ is continuous. The complexity of a dynamical system is a central topic of research since the introduction of the term of chaos in 1975 by Li and Yorke [8], known as Li-Yorke chaos today.

DEFINITION 1.1 ([2, 8]). Let (X, f) be a dynamical system. If $x, y \in X$ and $\delta > 0$, (x, y) is called a *Li-Yorke pair of modulus* δ if

$$\liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} \rho(f^n(x), f^n(y)) \geq \delta.$$

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(1-1) (x, y) is a *Li-Yorke pair* if it is a Li-Yorke pair of modulus δ for some $\delta > 0$;

(1-2) The subset $\Gamma \subset X$ is called a *scrambled set* if for all points $x, y \in \Gamma$ with $x \neq y$, (x, y) is a Li-Yorke pair;

(1-3) (X, f) (or f) is *Li-Yorke chaotic* if X contains an uncountable scrambled set.

The set of Li-Yorke pairs of modulus δ is denoted by $\mathrm{LY}(f,\delta)$ and the set of Li-Yorke pairs by $\mathrm{LY}(f)$.

For each real number ε, we denote

$$\Delta_\varepsilon = \{(x, y) \in X \times X : \rho(x, y) < \varepsilon\}$$

$$\Delta = \{(x, x) \in X \times X : x \in X\}$$

and

$$PR(f) = \left\{(x, y) \in X \times X : \liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0\right\}$$

$$AR_\varepsilon(f) = \left\{(x, y) \in X \times X : \limsup_{n\to\infty} \rho(f^n(x), f^n(y)) < \varepsilon\right\}$$

Clearly $\mathrm{LY}(f,\delta) = PR(f) \backslash AR_\delta(f)$ and $\mathrm{LY}(f) = \bigcup_{\varepsilon>0} PR(f)\backslash AR_\varepsilon(f)$. For any $x \in X$ and any $\varepsilon > 0$, set $B(x,\varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}$.

In 1985, Piórek [12] introduced the concept of generic chaos. Being inspired by this definition, Snoha [14, 15] defined generic δ-chaos, dense chaos and dense δ-chaos in 1990.

Definition 1.2 ([14, 15]). Let (X, f) be a dynamical system.

(2-1) (X, f) is *sensitive* if there exists $\varepsilon > 0$ such that for any $x \in X$ and any $\eta > 0$, there exist $y \in B(x,\eta)$ and $n \in \mathbf{N}$ such that $\rho(f^n(x), f^n(y)) > \varepsilon$. The real number ε is called *the constant of sensitivity*.

(2-2) (X, f) is *infinitely sensitive* if there exists $\varepsilon > 0$ such that for any $x \in X$ and any $\eta > 0$, there is some $y \in B(x,\eta)$ such that $\limsup_{n\to\infty} \rho(f^n(x), f^n(y)) \geq \varepsilon$.

(2-3) (X, f) is *densely chaotic* if $\mathrm{LY}(f)$ is dense in $X \times X$.

(2-4) (X, f) is *densely δ-chaotic for some $\delta > 0$* if $\mathrm{LY}(f,\delta)$ is dense in $X \times X$.

(2-5) (X, f) is *generically chaotic* if $\mathrm{LY}(f)$ is residual in $X \times X$.

(2-6) (X, f) is *generically δ-chaotic for some $\delta > 0$* if $\mathrm{LY}(f,\delta)$ is residual in $X \times X$.

Generic δ-chaos obviously implies both generic chaos and dense δ-chaos, which in turn imply dense chaos.

Snoha [14] proved that for an interval map, generic chaos implies generic δ-chaos for some $\delta > 0$ and the notions of generic δ-chaos and dense δ-chaos coincide, but a densely chaotic interval map may not be generically chaotic. Snoha [15] gave a characterization of densely chaotic interval maps and proved that for a piecewise monotone interval map, dense chaos and generic chaos are consistent. In 2000, Murinová [10] generalized Snoha's work and showed that for a complete metric space, generic δ-chaos and dense δ-chaos are equivalent. She also exhibited a generically chaotic system which is not generically δ-chaotic for any $\delta > 0$.

If X is a complete metric space and $G \subset X \times X$ is a dense G_δ-set, then using Kuratowski's theorem (see, e.g., [11]), one can find an uncountable set S such that $S \times S$ deprived of the diagonal of $X \times X$ is included in G (see, e.g., [6, Lemma 3.1]). Therefore a generically chaotic map on a complete metric space is chaotic in the sense of Li-Yorke. Kuchta and Smítal [7] showed that on the interval the existence of one Li-Yorke pair is enough to imply chaos in the sense of Li-Yorke, and consequently dense chaos implies Li-Yorke chaos for interval maps. However, so far, it is not known whether dense chaos implies Li-Yorke chaos in general.

During the last years many researchers paid attention to the chaotic behavior of orbits governed by linear operators on infinite dimensional spaces (more especially, on Banach or Fréchet spaces) (see e.g., [1, 3, 4, 5]). One of the most significant cases being the hypercyclicity, that is, the existence of vectors $x \in X$ such that the orbit $\mathrm{orb}_T(x) := \{x, T(x), T^2(x), \dots\}$ under a (continuous and linear) operator $T : X \to X$ on a topological vector space (usually, Banach or Fréchet space) X, is dense in X. In our context hypercyclity is equivalent to transitivity. Recently, we proved that for a bounded operator on a Banach space, Li-Yorke chaos, distributional chaos in a sequence, spatio-temporal chaos and Li-Yorke sensitivity are equivalent in [16].

In this paper, we study dense chaos and densely chaotic operators on Banach spaces. First, we characterize densely δ-chaos in terms of sensitivity and dense chaos and prove that a transitive system with a fixed point must be densely chaotic. Besides, some examples are given to show that for general dynamical systems, Devaney chaos and dense chaos do not imply each other. Then, by using these results, the following conclusion is obtained:

(1) For a operator on a Banach space, dense chaos, dense δ-chaos, generic chaos and generic δ-chaos are equivalent and they are all strictly stronger than Li-Yorke chaos.

2. Dense Chaos for Topological Dynamical Systems

In this section, we first show that sensitivity and infinite sensitivity are equivalent. Combining this with Baire category theorem, some characterizations of dense chaos are obtained.

Theorem 2.1. *A dynamical system (X, f) is infinitely sensitive if and only if it is sensitive.*

Proof. Necessity is obvious.

Sufficiency. Assume that (X, f) is sensitive and ε is the constant of sensitivity. Given any $N \in \mathbf{N}$, set $\mathscr{D}_N = \{(x, y) : \rho(f^n(x), f^n(y)) \leq \varepsilon/4, \forall n > N\}$. It is clear that $\mathscr{D}_N$ is a closed set.

Now we assert that for any $N \in \mathbf{N}$, $\operatorname{int} \mathscr{D}_N = \varnothing$.

Indeed, if there exists some $N \in \mathbf{N}$ such that $\operatorname{int} \mathscr{D}_N \neq \varnothing$, there exist non-empty open sets $U, V \subset X$ such that $U \times V \subset \mathscr{D}_N$. This implies that for any pair $(x, y) \in U \times V$, $\rho(f^n(x), f^n(y)) \leq \varepsilon/4$ holds for any $n > N$. So for all points $x_1, x_2 \in U$ and any $n > N$, $\rho(f^n(x_1), f^n(x_2)) \leq \rho(f^n(x_1), f^n(y)) + \rho(f^n(y), f^n(x_2)) \leq \varepsilon/2$. Note that there exists a non-empty open set $U^* \subset U$ such that for any pair $x_1, x_2 \in U^*$ and any $0 \leq i \leq N$, $\rho(f^i(x_1), f^i(x_2)) \leq \varepsilon/2$. So for any pair $x_1, x_2 \in U^*$ and any $n \in \mathbf{N}$, $\rho(f^i(x_1), f^i(x_2)) \leq \varepsilon/2$ which contradicts the sensitivity of (X, f).

It follows that the set $\mathscr{D} = \bigcup_{N \in \mathbf{N}} \mathscr{D}_N$ is a first category set in $X \times X$. This implies that the set $X \times X \backslash \mathscr{D} = \{(x, y) : \forall N \in \mathbf{N}, \exists n > N$ such that $\rho(f^n(x), f^n(y)) > \varepsilon/4\}$ is residual in $X \times X$.

Suppose that (X, f) is not infinitely sensitive, then there exist $x_0 \in X$ and $\eta > 0$ such that for any $y \in B(x_0, \eta)$, $\limsup_{n \to \infty} \rho(f^n(x_0), f^n(y)) \leq \varepsilon/16$. Noting the fact that $X \times X \backslash \mathscr{D}$ is residual in $X \times X$, it follows that there exists a pair $(y_1, y_2) \in [B(x_0, \eta) \times B(x_0, \eta)] \cap (X \times X \backslash \mathscr{D})$. As for any $n \in \mathbf{N}$, $\rho(f^n(y_1), f^n(y_2)) \leq \rho(f^n(y_1), f^n(x_0)) + \rho(f^n(x_0), f^n(y_2))$, we have $\limsup_{n \to \infty} \rho(f^n(y_1), f^n(y_2)) \leq \varepsilon/8$ which contradicts to $(y_1, y_2) \in X \times X \backslash \mathscr{D}$.

Hence (X, f) is infinitely sensitive. $\square$

Theorem 2.2. *For a dynamical system (X, f), the following statements are equivalent*:

(1) (X, f) *is densely δ-chaotic for some $\delta > 0$.*

(2) (X, f) *is generically δ-chaotic for some $\delta > 0$.*

(3) (X, f) *is generically chaotic and sensitive.*

(4) (X, f) *is generically chaotic and infinitely sensitive.*
(5) (X, f) *is densely chaotic and sensitive.*
(6) (X, f) *is densely chaotic and infinitely sensitive.*
(7) $\overline{PR(f)} = X \times X$ *and* (X, f) *is sensitive.*
(8) $\overline{PR(f)} = X \times X$ *and* (X, f) *is infinitely sensitive.*

Proof. According to [10, Theorem A], (1) $\Leftrightarrow$ (2) holds. Applying Theorem 2.1, (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Leftrightarrow$ (6) $\Rightarrow$ (7) $\Leftrightarrow$ (8) holds trivially.

(8) $\Rightarrow$ (1) It is easy to see that

$$PR(f) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} ((f \times f)^{-m}(\Delta_{1/n})) \tag{1}$$

Then $PR(f)$ is a dense G_δ-subset of $X \times X$ since $\overline{PR(f)} = X \times X$.

Since f is infinitely sensitive, there exists $\varepsilon > 0$ such that for all $x \in X$ and all $\delta > 0$, there exists some $y \in B(x, \delta)$ satisfying $\limsup_{n\to\infty} \rho(f^n(x), f^n(y)) > \varepsilon$.

Claim: $\operatorname{int}(\bigcap_{k=1}^{\infty}(f \times f)^{-(m+k)}(\overline{\Delta_{\varepsilon/2-1/n}})) = \varnothing$ for any pair $m, n \in \mathbf{N}$.

Proof of Claim. If there exist $m, n \in \mathbf{N}$ such that $\bigcap_{k=1}^{\infty}(f \times f)^{-(m+k)} \cdot (\overline{\Delta_{\varepsilon/2-1/n}})$ has non-empty interior, then there exist some $x, y \in X$ and $r > 0$ such that $B(x, r) \times B(y, r) \subset \bigcap_{k=1}^{\infty}(f \times f)^{-(m+k)}(\overline{\Delta_{\varepsilon/2-1/n}})$. So for any pair $z_1, z_2 \in B(x, r)$ and any $k \in \mathbf{N}$, $\rho(f^{m+k}(z_1), f^{m+k}(z_2)) \le \rho(f^{m+k}(z_1), f^{m+k}(y)) + \rho(f^{m+k}(y), f^{m+k}(z_2)) \le 2(\varepsilon/2 - 1/n) = \varepsilon - 2/n$.

This implies that $\limsup_{k\to\infty} \rho(f^k(z_1), f^k(z_2)) \le \varepsilon - 2/n < \varepsilon$ for any pair $z_1, z_2 \in B(x, r)$ which is a contradiction.

Meanwhile, we have

$$\begin{aligned} AR_{\varepsilon/2}(f) &= \left\{(x, y) \in X \times X : \limsup_{n\to\infty} \rho(f^n(x), f^n(y)) < \frac{\varepsilon}{2}\right\} \\ &= \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \left\{(x, y) \in X \times X : \rho(f^{m+k}(x), f^{m+k}(y)) \le \frac{\varepsilon}{2} - \frac{1}{n}\right\} \\ &= \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \left(\bigcap_{k=1}^{\infty} (f \times f)^{-(m+k)}(\overline{\Delta_{\varepsilon/2-1/n}})\right) \end{aligned}$$

Combining this with the Claim, it follows that $AR_{\varepsilon/2}(f)$ is of first category in $X \times X$. This leads with (1) to that $PR(f) \backslash AR_{\varepsilon/2}(f)$ is dense in $X \times X$ by the Baire category theorem. Thus (X, f) is densely $\varepsilon/2$-chaotic. $\square$

The following proposition is a direct corollary of [9, Theorem 3.5].

PROPOSITION 2.3. *If* (X, f) *is a topological transitive system with a fixed point, then it is densely chaotic.*

PROPOSITION 2.4. *Assume that* $f : X \to X$ *is uniformly continuous. If* f *is densely chaotic, then* f^n *is too for any* $n \in \mathbf{N}$.

PROOF. It holds trivially since $\mathrm{LY}(f) = \mathrm{LY}(f^n)$ holds for any $n \in \mathbf{N}$. □

EXAMPLE 2.5. Take $X = [0,1] \cup [2,3]$ and define the map $f : X \to X$ by

$$f(x) = \begin{cases} \Lambda(x) + 2, & x \in [0,1], \\ \Lambda(x-2), & x \in [2,3], \end{cases}$$

where $\Lambda(x) = 1 - |1 - 2x|$, $x \in [0,1]$. By simple calculation, we know

$$f^2(x) = \begin{cases} \Lambda^2(x), & x \in [0,1], \\ \Lambda^2(x-2) + 2, & x \in [2,3], \end{cases}$$

It is not difficult to check that f is Devaney chaotic. For any pair $(x, y) \in [0,1] \times [2,3]$, we have $\inf\{|f^{2n}(x) - f^{2n}(y)| : n \in \mathbf{N}\} \geq 1$, so $[0,1] \times [2,3] \cap \mathrm{LY}(f^2) = \varnothing$. This implies that f^2 is not densely chaotic. Applying Proposition 2.4, it follows that f is not densely chaotic.

Combining this Example with [13, Example 6.3.16], we have that for general dynamical systems, Devaney chaos and dense chaos do not imply each other.

3. Dense Chaos for Bounded Operators

In this section, X denotes a Banach space over $\mathbf{C}$ (or $\mathbf{R}$) and $T : X \to X$ a bounded operator. In this case, the associated metric is $\rho(x, y) = \|x - y\|$ for any pair $x, y \in X$, where $\|\cdot\|$ is the norm of X. A vector $x \in X$ is said to be *irregular* for T if $\liminf_{n\to\infty} \|T^n(x)\| = 0$ and $\limsup_{n\to\infty} \|T^n(x)\| = \infty$. In [2], Bermúdez et al. proved the following conclusion, which is important for our discussion.

THEOREM 3.1 ([2, Theorem 5]). *Let* $T : X \to X$ *be a bounded operator. The following assertions are equivalent*:

(1) (X, T) *is Li-Yorke chaotic.*

(2) (X, T) *admits a Li-Yorke pair.*

(3) (X, T) *admits an irregular vector.*

Applying this theorem, it is obvious that every densely chaotic operator is Li-Yorke chaotic. In comparison with the result in section 2, we have the following results on bounded operators.

Proposition 3.2. *If a bounded operator $T : X \to X$ is transitive, then it is densely chaotic. In particular, every Devaney chaotic operator is densely chaotic.*

Proof. It is clear that 0 is a fixed point of T. This leads with Proposition 2.3 to that T is densely chaotic. $\square$

Theorem 3.3. *Assume that T is a bounded operator defined on a Banach space X. Then T is densely δ-chaotic for some $\delta > 0$ if and only if it is densely chaotic.*

Proof. Necessity is obvious.

Sufficiency. By Theorem 2.2, it is sufficient to show that T is infinitely sensitive.

According to Theorem 3.1, it follows that T admits an irregular vector v. For any $x \in X$ and any $\delta > 0$, let us take $y = x + v/\|v\| \cdot \delta/2$. Clearly, $y \in B(x, \delta)$. Furthermore, we have

$$\limsup_{n\to\infty} \rho(T^n(x), T^n(y)) = \limsup_{n\to\infty} \|T^n(x) - T^n(y)\| = \limsup_{n\to\infty} \left\| T^n\left(\frac{v}{\|v\|} \cdot \frac{\delta}{2}\right) \right\| = \infty$$

Thus T is infinitely sensitive. $\square$

Being the end of this section, we shall construct a bounded operator which is not densely chaotic but Li-Yorke chaotic.

Example 3.4. Let $\mathbf{H}$ be an infinite dimensional separable Hilbert space and $x_0 \in \mathbf{H}$ $(x_0 \neq 0)$. Take $V_1 = \mathrm{span}\{x_0\}$ and $V_2 = V_1^{\perp} := \{x \in \mathbf{H} : x \perp V_1\}$. According to [2, Theorem 35], there exists a bounded operator $T_2 : V_2 \to V_2$ such that T_2 is chaotic in the sense of Li-Yorke. Define the operator $T_1 : V_1 \to V_1$ by $T_1(x) = 2x$ for each $x \in V_1$. According to projection theorem, we know that every $x \in \mathbf{H}$ can be uniquely written as $x = x_1 + x_2$ with $x_i \in V_i$, $i = 1, 2$. Let $T : \mathbf{H} \to \mathbf{H}$ be a operator given by

$$T(x) = T(x_1 + x_2) = T_1(x_1) + T_2(x_2).$$

Clearly, T is continuous. Now we assert that T is Li-Yorke chaotic but not densely chaotic.

Indeed, as $T|_{V_2} = T_2$ is Li-Yorke chaotic, it is clear that T is Li-Yorke chaotic. It remains to show that T is not densely chaotic. Let us choose $W_1 = B(0,1)$ and $W_2 = B(4(x_0/\|x_0\|),1)$. Clearly, $W = W_2 - W_1 := \{x - y : x \in W_2, y \in W_1\} \subset B(4(x_0/\|x_0\|),2)$. For any pair $(x,y) \in W_2 \times W_1$, there exist $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$ such that $x_1 + x_2 = x - 4(x_0/\|x_0\|)$ and $y_1 + y_2 = y$. Then $x - y = (4(x_0/\|x_0\|) + x_1 - y_1) + (x_2 - y_2) \in W \subset B(4(x_0/\|x_0\|),2)$. So $\|x_1 - y_1\| < 2$. This implies that $\|4(x_0/\|x_0\|) + x_1 - y_1\| > 2$. Thus for any $n \in \mathbf{N}$,

$$\begin{aligned}\|T^n(x) - T^n(y)\| = \|T^n(x-y)\| &= \left\|T_1^n\left(4\frac{x_0}{\|x_0\|} + x_1 - y_1\right) + T_2^n(x_2 - y_2)\right\| \\ &\geq \left\|T_1^n\left(4\frac{x_0}{\|x_0\|} + x_1 - y_1\right)\right\| = 2^n \cdot \left\|\left(4\frac{x_0}{\|x_0\|} + x_1 - y_1\right)\right\| \\ &\geq 2^{n+1} \to \infty \quad (n \to \infty)\end{aligned}$$

as $T_1^n(4(x_0/\|x_0\|) + x_1 - y_1) \perp T_2^n(x_2 - y_2)$. Therefore, $\limsup_{n\to\infty}\|T^n(x) - T^n(y)\| = \infty$.

Hence, $W_2 \times W_1 \cap \mathrm{LY}(T) = \emptyset$, i.e., T is not densely chaotic.

We deduce from Theorem 3.3, Example 3.4 and [10, Theorem A] that for a bounded operator, dense chaos, generic chaos, dense δ-chaos and generic δ-chaos are equivalent and they are all strictly stronger than Li-Yorke chaos.

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