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NECESSARY AND SUFFICIENT CONDITIONS FOR THE SOLVABILITY AND MAXIMAL REGULARITY OF ABSTRACT DIFFERENTIAL EQUATIONS OF MIXED TYPE IN UMD SPACES

By

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Abstract. In this paper we give some results on some abstract second order differential elliptic equations with mixed type boundary conditions. The study is performed in UMD spaces. The main purpose of this paper is the study of necessary and sufficient conditions on the data for obtaining existence, uniqueness and maximal regulariy properties of the strict solution. On the other hand, we give some new examples related to traces results.

1. Introduction

Let us recall, for the reader's convenience, some basic and known notions and results to prepare the ground for our work.

Let X be a complex Banach space. We define $L^p(0,1;X)$, $p \in [1,+\infty[$, as:

$$L^p(0,1;X) = \left\{ f \text{ measurable on } (0,1) \text{ and } \left(\int_0^1 \|f(t)\|_X^p \, dt \right)^{1/p} < +\infty \right\}.$$

$W^{2,p}(0,1;X)$ are the well-known Sobolev spaces, i.e.:

$$W^{2,p}(0,1;X) = \{u \in L^p(0,1;X) : u', u'' \in L^p(0,1;X)\}$$

where the derivatives of u are within the meaning of the distributions. We mention also the spaces:

$$(D_A, X)_{\theta,p} = \left\{ \varphi \in X : \left(\int_0^\infty \|x^\theta A G(x)\varphi\|_X^p \frac{dx}{x} \right)^{1/p} < +\infty \right\}$$

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in the case where A is a closed linear operator and infinitesimal generator of an analytic semigroup $G(x)$.

Let us recall that a Banach space X is a UMD space if and only if for some $p > 1$ (and thus for all p), the Hilbert transform is continuous from $L^p(\mathbf{R}; X)$ into itself (see Bourgain [3], Burkholder [4]).

We can give several examples of classic Banach spaces which have UMD property:

1. All Hilbert spaces are UMD.
2. Any space isomorph to a UMD space, is UMD.
3. Any closed subspace of a UMD space, is UMD.
4. If the spaces X and Y are UMD, then interpolated spaces ($(X, Y)_{\theta,p}$ or $[X, Y]_{\theta,p}$) are UMD for $1 < p < \infty$.
5. All spaces constructed on L^p spaces, are UMD for $1 < p < \infty$.

Let us consider, in a complex Banach space X, the second order abstract differential problem

$$u''(x) + Au(x) = f(x), \quad x \in (0, 1) \tag{1}$$

with the Dirichlet-Neumann boundary conditions

$$u(0) = d_0, \quad u'(1) = n_1. \tag{2}$$

Here d_0 and n_1 are given elements in X and A is a closed linear operator of domain $D(A)$ not necessarily dense in X.

We assume in all this work that A verifies the following elliptic hypothesis

$$\forall \lambda \geq 0, \quad \exists (A - \lambda I)^{-1} \in L(X) : \|(A - \lambda I)^{-1}\|_{L(X)} \leq \frac{C}{1 + \lambda}. \tag{3}$$

Our study will treat the existence, uniqueness and regularity of the solutions under assumption (3). Here f belongs to $L^p(0, 1; X)$, $1 < p < \infty$ and

$$X \text{ is a } UMD \text{ space.} \tag{4}$$

Note that in this case our assumptions imply that $\overline{D(A)} = X$.

Also, we suppose that

$$\begin{cases} \forall s \in \mathbf{R},\ (-A)^{is} \in L(X) \text{ and } \exists C \geq 1,\ \alpha \in]0, \pi[: \\ \|(-A)^{is}\|_{L(X)} < Ce^{\alpha|s|}. \end{cases} \tag{5}$$

(For the class of operators verifying (5), see, Prüss-Sohr [18] for more details).

The main result in this section affirms that under assumptions (3), (4) and (5), problem (1)–(2) has a unique strict solution in $L^p(0,1;X)$ i.e. a function u such that

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)),$$

if and only if $d_0 \in (D_A, X)_{1/2p,p}$ and $n_1 \in (D_A, X)_{1/2+1/2p,p}$.

Note that if $u \in W^{2,p}(0,1;X)$ then by J. L. Lions theorem of the traces, $u \in C^1([0,1];X)$ and $u(0)$ and $u'(1)$ are well defined.

This work is based fundamentally on explicit representation of the solution using the square root of $-A$ and the Krein's method see [13]. We then analyze carefully all the components of the solution by using respectively the Dore-Venni method [7], the Lions reiteration theorem, see [15] and [21], the semigroup theory and some techniques applied in [9].

The square root of the operator $-A$ will appear naturally in this paper. When we have to study equation (1) just with Dirichlet's boundary conditions, the use of this square root is not necessary, see Labbas [14].

In the last decades, many researchers have been interested by the resolution of problem (1). Several of them studied problem (1) as an abstract problem of elliptic type, i.e. under assumption (3), with different boundary conditions in both cases f is Hölder continuous or f is in $L^p(0,1;X)$ by using fractional powers of operators or Dunford functional calculus. We cite at first, the pioneer Da Prato and Grisvard theory on the sum of operators [6]. We also find a complete study of problem (1) under Dirichlet's boundary conditions and in variable coefficients operators case, see Labbas [14]. This author has used the Green's kernels techniques.

In a recent work [1] Arendt proved that problem

$$u''(x) + B(x)u'(x) + A(x)u(x) = f(x), \quad x \in (0,\delta) \tag{6}$$

with boundary conditions $u(0) = x$, $u'(0) = y$, has a unique solution u such that

$$u \in W^{2,p}(0,\delta;X) \cap L^p(0,\delta;D(A)) \quad \text{and} \quad u' \in L^p(0,\delta;D(B)),$$

in the case where $D(A)$ and $D(B)$ are Banach spaces which embed continuously and densely into X and f belongs to $L^p(0,\delta;X)$. At last, a new approach based on the semigroup techniques by Krein [13] and fractional powers of operators, has been developed by Favini, Labbas, Maingot, Tanabe and Yagi [8], [10] concerning the complete equation

$$u''(x) + Bu'(x) + Au(x) = f(x), \quad x \in (0,1)$$

under Dirichlet boundary conditions. In our work we have been inspired by these last references.

In this paper, we are interested in the resolution of problem (1) with Dirichlet-Neumann boundary conditions. We give then necessary and sufficient conditions on the data to have existence, uniqueness and maximal regularity of the strict solution. We obtain also some a-priori-estimates. As new applications, we will give some trace results.

The plan of this paper is as follows.

In Sections 2 and 3, we will recall some basic properties of generalized analytic semigroups. We also give some technical Lemmas which are useful to give a precise analysis of the representation of the solution u. Section 4 is devoted to the existence, uniqueness and maximal regularity of the strict solution. In section 5 we give some a-priori-estimates.

Finally, section 6 contains two parts. First, we give abstract trace theorems and then an application to a partial differential equation.

2. Technical Results

We set, in all this paper

$$B = \sqrt{-A}.$$

Remark 1.

1. Hypotheses (3) and (4) imply that X is reflexive, thus $D(A)$ is dense in X (see Haase [12], Proposition 1.1, p. 18).
2. Hypothesis (3) implies that operator $(-\sqrt{-A})$ generates an analytic semigroup noted $(e^{-\sqrt{-A}x})_{x\geq 0}$ on X, see for instance Balakrishnan [2].
3. Hypothesis (5) is equivalent to the following:

$$\exists C \geq 1, \quad \alpha \in]0, \pi[: \forall s \in \mathbf{R}, \quad \|(\sqrt{-A})^{is}\| \leq Ce^{(\alpha/2)|s|},$$

(see Haase [12], Proposition 2.18. p. 64).

We have the following lemmas:

Lemma 2. *Due to assumptions* (3), (4), (5) *and the previous remark*, *for* $f \in L^p(0,1;X)$, $1 < p < \infty$, *we have the assertions*:

1. $x \mapsto L(x, f) = B \int_0^x e^{-(x-s)B} f(s)\, ds \in L^p(0, 1; X)$,
2. $x \mapsto L(1 - x, f(1 - \cdot)) = B \int_x^1 e^{-(s-x)B} f(s)\, ds \in L^p(0, 1; X)$,
3. $x \mapsto \mathscr{L}(x, f) = B \int_0^1 e^{-(x+s)B} f(s)\, ds \in L^p(0, 1; X)$.

Proof. The first and second assertions are a consequence of the Dore-Venni theorem [7].

For assertion 3., one writes, for a.e $x \in (0, 1)$

$$
\begin{aligned}
\mathscr{L}(x, f) &= B \int_0^1 e^{-(x+s)B} f(s)\, ds \\
&= B \int_0^x e^{-(x-s)B} e^{-2sB} f(s)\, ds + e^{-2xB} B \int_x^1 e^{-(s-x)B} f(s)\, ds \\
&= L(x, e^{-2\cdot B} f) + e^{-2xB} L(1 - x, f(1 - \cdot)),
\end{aligned}
$$

and we apply the first and second assertions. ■

We also have

Lemma 3. *Assume that hypothesis* (3) *holds. Then we have the assertions*:

1. $B^2 e^{-\cdot B} \varphi \in L^p(0, 1; X)$ *if and only if* $\varphi \in (D(A), X)_{1/2p, p}$.
2. $B e^{-\cdot B} \varphi \in L^p(0, 1; X)$ *if and only if* $\varphi \in (D(A), X)_{1/2p+1/2, p}$.

Proof. We recall that if $m \in \mathbf{N}^*$ and C generates an analytic semigroup then

$$\phi \in (D(C^m), X)_{1/mp, p},$$

if and only if

$$C^m e^{\cdot C} \phi \in L^p(0, 1; X).$$

In fact

$$
\begin{aligned}
\int_0^1 \|C^m e^{xC} \phi\|_X^p\, dx &\le \int_0^\infty \|x^{m(1-(1-1/mp))} C^m e^{xC} \phi\|_X^p \frac{dx}{x} \\
&\le K \|\phi\|_{(D(C^m), X)_{1/mp, p}},
\end{aligned}
$$

(see Triebel [21], Theorem p. 96).

From which

$$B^2 e^{-\cdot B} \varphi \in L^p(0, 1; X)$$

if and only if

$$\varphi \in (D(B^2), X)_{1/2p,p} = (D(A), X)_{1/2p,p}.$$

By the same way

$$Be^{-\cdot B}\varphi \in L^p(0,1;X)$$

if and only if

$$\varphi \in (D(B), X)_{1/p,p}.$$

We conclude by using the reiteration property by Lions-Peetre [15]

$$\begin{aligned}(D(A), X)_{1/2p+1/2,p} &= (X, D(B^2))_{1/2-1/2p,p}\\ &= (X, D(B))_{1-1/p,p}\\ &= (D(B), X)_{1/p,p}. \qquad \blacksquare\end{aligned}$$

Remark 4. Assume that hypotheses (3), (4) and (5) hold. Then

$$-Ae^{-\cdot B}\left(B^{-1}\int_0^1 e^{-sB}f(s)\,ds\right) = Be^{-\cdot B}\int_0^1 e^{-sB}f(s)\,ds = \mathscr{L}(\cdot, f),$$

so, from Lemma 2, we get

$$Ae^{-\cdot B}\left(B^{-1}\int_0^1 e^{-sB}f(s)\,ds\right) \in L^p(0,1;X),$$

and by Lemma 3, we deduce

$$\left(B^{-1}\int_0^1 e^{-sB}f(s)\,ds\right) \in (D(A), X)_{1/2p,p}.$$

Put

$$Z = e^{-2B}.$$

Proposition 5. *Assume that hypothesis* (3) *holds. Then the operator* $I - Z$ *has a bounded inverse and*

$$(I - Z)^{-1} = \frac{1}{2\pi i}\int_{\gamma_\#} \frac{e^{2z}}{1 - e^{2z}}(zI + B)^{-1}\,dz + I,$$

where $\gamma_\#$ *is a suitable curve in the complex plane.*

PROOF. Since the imaginary axis is contained in the resolvent set $\rho(-B)$, we then can adapt the complete proof of Lunardi [16], p. 59 by choosing an appropriated curve $\gamma_{\#}$ which takes into account the fact that $-B$ generates an analytic semigroup. ■

COROLLARY 6. *Under hypothesis* (3), *the operator* $(I + Z)$ *has a bounded inverse.*

PROOF. We have

$$(I - e^{-2B})(I + e^{-2B}) = I - e^{-4B}$$

then

$$(I + e^{-2B}) = (I - e^{-2B})^{-1}(I - e^{-4B}).$$

Consequently

$$(I + e^{-2B})^{-1} = (I - e^{-4B})^{-1}(I - e^{-2B}). \qquad ■$$

3. Representation of the Solution

We assume here that hypotheses (3), (4) and (5) hold.

Let us suppose that problem (1)–(2) has a strict solution u and set

$$u(1) = u_1.$$

Then u is the strict solution of the following problem

$$\begin{cases} u''(x) - B^2u(x) = f(x) \\ u(0) = d_0 \\ u(1) = u_1, \end{cases} \tag{7}$$

and therefore, one has the representation

$$u(x) = e^{-xB}\xi_0 + e^{-(1-x)B}\xi_1 - \frac{1}{2}B^{-1}\int_0^x e^{-(x-s)B}f(s)\,ds$$
$$-\frac{1}{2}B^{-1}\int_x^1 e^{-(s-x)B}f(s)\,ds$$

where

$$\xi_0 = (I - Z)^{-1}(d_0 - e^{-B}u_1)$$
$$+\frac{1}{2}(I - Z)^{-1}B^{-1}\left(\int_0^1 e^{-sB}f(s)\,ds - \int_0^1 e^{-(2-s)B}f(s)\,ds\right)$$

$$\xi_1 = (I-Z)^{-1}(-e^{-B}d_0 + u_1)$$
$$+\frac{1}{2}(I-Z)^{-1}B^{-1}\left(\int_0^1 e^{-(1-s)B}f(s)\,ds - \int_0^1 e^{-(1+s)B}f(s)\,ds\right),$$

see [8].

We deduce that

$$n_1 = u'(1) = -2(I-Z)^{-1}Be^{-B}d_0 + (I-Z)^{-1}(I+Z)Bu_1$$
$$-\frac{1}{2}e^{-B}(I-Z)^{-1}\left(\int_0^1 e^{-sB}f(s)\,ds - \int_0^1 e^{-(2-s)B}f(s)\,ds\right)$$
$$+\frac{1}{2}(I-Z)^{-1}\left(-\int_0^1 e^{-(1+s)B}f(s)\,ds + Z\int_0^1 e^{-(1-s)B}f(s)\,ds\right).$$

Then

$$u_1 = (I+Z)^{-1}(2e^{-B}d_0 + (I-Z)B^{-1}n_1)$$
$$+(I+Z)^{-1}B^{-1}\left(\int_0^1 e^{-(1+s)B}f(s)\,ds - \int_0^1 e^{-(1-s)B}f(s)\,ds\right). \tag{8}$$

Therefore our solution u is given formally by

$$u(x) = (I+Z)^{-1}(e^{-xB} + e^{-(2-x)B})d_0$$
$$+(I+Z)^{-1}(e^{-(1-x)B} - e^{-(1+x)B})B^{-1}n_1$$
$$+\frac{1}{2}(I+Z)^{-1}B^{-1}\int_0^1 e^{-(x+s)B}f(s)\,ds$$
$$+\frac{1}{2}(I+Z)^{-1}B^{-1}\int_0^1 e^{-(2-x+s)B}f(s)\,ds$$
$$+\frac{1}{2}(I+Z)^{-1}B^{-1}\int_0^1 e^{-(2+x-s)B}f(s)\,ds$$
$$-\frac{1}{2}(I+Z)^{-1}B^{-1}\int_0^1 e^{-(2-x-s)B}f(s)\,ds$$
$$-\frac{1}{2}B^{-1}\int_0^x e^{-(x-s)B}f(s)\,ds - \frac{1}{2}B^{-1}\int_x^1 e^{-(s-x)B}f(s)\,ds. \tag{9}$$

4. Existence, Uniqueness and Maximal Regularity

The main result in this section is

Theorem 7. *Let* $f \in L^p(0,1;X)$, $1 < p < +\infty$, *and assume that hypotheses* (3), (4) *and* (5) *hold. Then the following assertions are equivalent*

1. *Problem* (1)–(2) *has a unique strict solution* u, *that is*

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)),$$

and satisfies (1)–(2).

2. $d_0 \in (D(A),X)_{1/2p,p}$ *and* $n_1 \in (D(A),X)_{1/2+1/2p,p}$.

Proof. If u is the strict solution of problem (1)–(2) and $d_0 \in (D(A),X)_{1/2p,p}$ and $n_1 \in (D(A),X)_{1/2+1/2p,p}$, then u is given by (9). Let us prove the uniqueness of the solution.

Set

$$\begin{aligned}
L(f)(x) = {} & \frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(x+s)B}f(s)\,ds \\
& + \frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2-x+s)B}f(s)\,ds \\
& + \frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2+x-s)B}f(s)\,ds \\
& - \frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2-x-s)B}f(s)\,ds \\
& - \frac{1}{2}\int_0^x B^{-1}e^{-(x-s)B}f(s)\,ds - \frac{1}{2}\int_x^1 B^{-1}e^{-(s-x)B}f(s)\,ds.
\end{aligned}$$

Writing $f(x) = Au(x) + u''(x)$ we obtain

$$\begin{aligned}
L(f)(x) = {} & \frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(x+s)B}Au(s)\,ds \\
& + \frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2-x+s)B}Au(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2+x-s)B}Au(s)\,ds\\
&-\frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2-x-s)B}Au(s)\,ds\\
&-\frac{1}{2}\int_0^x B^{-1}e^{-(x-s)B}Au(s)\,ds-\frac{1}{2}\int_x^1 B^{-1}e^{-(s-x)B}Au(s)\,ds\\
&+\frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(x+s)B}u''(s)\,ds\\
&+\frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2-x+s)B}u''(s)\,ds\\
&+\frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2+x-s)B}u''(s)\,ds\\
&-\frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2-x-s)B}u''(s)\,ds\\
&-\frac{1}{2}\int_0^x B^{-1}e^{-(x-s)B}u''(s)\,ds-\frac{1}{2}\int_x^1 B^{-1}e^{-(s-x)B}u''(s)\,ds\\
=\,&\sum_{i=1}^{6}H_i+\sum_{i=1}^{6}J_i.
\end{aligned}
$$

After integrating by parts we have

$$
\begin{aligned}
J_1=\,&\frac{1}{2}(I+Z)^{-1}B^{-1}(e^{-(1+x)B}u'(1)-e^{-xB}u'(0))\\
&+\frac{1}{2}(I+Z)^{-1}(e^{-(1+x)B}u(1)-e^{-xB}u(0))\\
&+\frac{1}{2}(I+Z)^{-1}\int_0^1 Be^{-(x+s)B}u(s)\,ds,\\
J_2=\,&\frac{1}{2}(I+Z)^{-1}B^{-1}(e^{-(3-x)B}u'(1)-e^{-(2-x)B}u'(0))\\
&+\frac{1}{2}(I+Z)^{-1}(e^{-(3-x)B}u(1)-e^{-(2-x)B}u(0))\\
&+\frac{1}{2}(I+Z)^{-1}\int_0^1 Be^{-(2-x+s)B}u(s)\,ds,
\end{aligned}
$$

$$J_3 = \frac{1}{2}(I+Z)^{-1}B^{-1}(e^{-(1+x)B}u'(1) - e^{-(2+x)B}u'(0))$$
$$- \frac{1}{2}(I+Z)^{-1}(e^{-(1+x)B}u(1) - e^{-(2+x)B}u(0))$$
$$+ \frac{1}{2}(I+Z)^{-1}\int_0^1 Be^{-(2+x-s)B}u(s)\,ds,$$
$$J_4 = -\frac{1}{2}(I+Z)^{-1}B^{-1}(e^{-(1-x)B}u'(1) - e^{-(2-x)B}u'(0))$$
$$+ \frac{1}{2}(I+Z)^{-1}(e^{-(1-x)B}u(1) - e^{-(2-x)B}u(0))$$
$$- \frac{1}{2}(I+Z)^{-1}\int_0^1 Be^{-(2-x-s)B}u(s)\,ds,$$
$$J_5 = -\frac{1}{2}B^{-1}(u'(x) - e^{-xB}u'(0)) + \frac{1}{2}(u(x) - e^{-xB}u(0))$$
$$- \frac{1}{2}\int_0^x Be^{-(x-s)B}u(s)\,ds$$

and

$$J_6 = -\frac{1}{2}B^{-1}(e^{-(1-x)}u'(1) - u'(x)) - \frac{1}{2}(e^{-(1-x)}u(1) - u(x))$$
$$- \frac{1}{2}\int_x^1 Be^{-(s-x)B}u(s)\,ds,$$

The last integral is well defined since $u \in C^1([0,1];X)$.

We deduce that

$$\sum_{i=1}^{6} H_i + \sum_{i=1}^{6} J_i = -(I+Z)^{-1}(e^{-xB} + e^{-(2-x)B})d_0$$
$$-(I+Z)^{-1}(e^{-(1-x)} - e^{-(1+x)B})B^{-1}n_1 + u(x),$$

from which we obtain formula (9). Thus

$$B^2u(x) = (I+Z)^{-1}B^2(e^{-xB} + e^{-(2-x)B})d_0$$
$$+ (I+Z)^{-1}B^2(e^{-(1-x)B} - e^{-(1+x)B})B^{-1}n_1$$
$$+ \frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(x+s)B}f(s)\,ds$$

$$
\begin{aligned}
&+\frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(2-x+s)B}f(s)\,ds \\
&+\frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(2+x-s)B}f(s)\,ds \\
&-\frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(2-x-s)B}f(s)\,ds \\
&-\frac{1}{2}B\int_0^x e^{-(x-s)B}f(s)\,ds-\frac{1}{2}B\int_x^1 e^{-(s-x)B}f(s)\,ds \\
=&-Au(x).
\end{aligned}
\tag{10}
$$

Therefore, due to (3), (4), (5), lemmas 3 and 2, $Au \in L^p(0,1;X)$. As $f \in L^p(0,1;X)$ then $u'' \in L^p(0,1;X)$, from which we deduce

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D_A)$$

Conversely, let

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A))$$

then

$$Au(.) \in L^p(0,1;X), \quad a.e. \quad x \in (0,1)$$

Using (10) we obtain

$$B^2 e^{-xB} d_0 \in L^p(0,1;X) \quad \text{and} \quad Be^{-xB} n_1 \in L^p(0,1;X)$$

and applying lemma 3 we have

$$d_0 \in (D(A),X)_{1/2p,p} \quad \text{and} \quad n_1 \in (D(A),X)_{1/2+1/2p,p}. \qquad \blacksquare$$

5. A-Priori-Estimates

PROPOSITION 8. *Let $f \in L^p(0,1;X)$, $1 < p < +\infty$, and assume that hypotheses* (3), (4) *and* (5) *hold, and $d_0 \in (D(A),X)_{1/2p,p}$ and $n_1 \in (D(A),X)_{1/2+1/2p,p}$. Then there exists a constant $C > 0$ such that:*

$$\|u''\|_{L^p(X)} + \|Au\|_{L^p(X)}$$
$$\le C(\|f\|_{L^p(X)} + \|d_0\|_{(D(A),X)_{1/2p,p}} + \|n_1\|_{(D(A),X)_{1/2+1/2p,p}}) \tag{11}$$

Proof. From formula (9) we have

$$\begin{aligned}
\int_0^1 \|u''(x)\|_X^p \, dx \le & \int_0^1 \|(I+Z)^{-1}(I+e^{-2(1-x)B})Ae^{-xB}d_0\|_X^p \, dx \\
& + \int_0^1 \|(I+Z)^{-1}(I-e^{-2xB})Be^{-(1-x)B}n_1\|_X^p \, dx \\
& + \int_0^1 \left\|\frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(x+s)B}f(s)\, ds\right\|_X^p dx \\
& + \int_0^1 \left\|\frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(2-x+s)B}f(s)\, ds\right\|_X^p dx \\
& + \int_0^1 \left\|\frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(2+x-s)B}f(s)\, ds\right\|_X^p dx \\
& + \int_0^1 \left\|\frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(2-x-s)B}f(s)\, ds\right\|_X^p dx \\
& + \int_0^1 \left\|\frac{1}{2}B\int_0^x e^{-(x-s)B}f(s)\, ds\right\|_X^p dx \\
& + \int_0^1 \left\|\frac{1}{2}B\int_x^1 e^{-(s-x)B}f(s)\, ds\right\|_X^p dx \\
& + \int_0^1 \|f(x)\|_X^p \, dx \\
= & \sum_{j=1}^{9} I_j.
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_0^1 \|(I+Z)^{-1}(I+e^{-2(1-x)B})B^2e^{-xB}d_0\|_X^p \, dx \\
&\le C\int_0^\infty \|x^{2(1/2p)}B^2e^{-xB}d_0\|_X^p \frac{dx}{x} \\
&\le C\|d_0\|_{(D(B^2),X)_{1/2p,p}} \\
&\le C\|d_0\|_{(D(A),X)_{1/2p,p}}.
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_0^1 \|(I+Z)^{-1}(I-e^{-2xB})Be^{-(1-x)B}n_1\|_X^p \, dx \\
&\leq C \int_0^1 t\|Be^{-tB}n_1\|_X^p \frac{dt}{t} \\
&\leq C \int_0^\infty \|t^{1/p}Be^{-tB}n_1\|_X^p \frac{dt}{t} \\
&\leq C\|n_1\|_{(D(B),X)_{1/p,p}} \leq C\|n_1\|_{(D(A),X)_{1/2+1/2p,p}}.
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_0^1 \left\| \frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(x+s)B}f(s)\, ds \right\|_X^p dx \\
&\leq \int_0^1 \left\| \frac{1}{2}(I+Z)^{-1}\int_0^x Be^{-(x-s)B}e^{-2sB}f(s)\, ds \right\|_X^p \\
&\quad + \int_0^1 \left\| \frac{1}{2}(I+Z)^{-1}e^{-2xB}\int_x^1 Be^{-(s-x)B}f(s)\, ds \right\|_X^p dx \\
&\leq C\int_0^1 \|e^{-xB}-e^{-(1+x)B}\|_X^p\|f(x)\|_X^p \, dx \\
&\quad + C\int_0^1 \|I-e^{-(1-x)B}\|_X^p\|f(x)\|_X^p \, dx \\
&\leq C\int_0^1 \|f(x)\|_X^p \, dx.
\end{aligned}$$

We treat the terms I_4, I_5, I_6, I_7, I_8 and I_9 in the same way as I_3.

Then we deduce that

$$\int_0^1 \|u''(x)\|_X^p \, dx \leq C(\|d_0\|_{(D(A),X)_{1/2p,p}} + \|n_1\|_{(D(A),X)_{1/2+1/2p,p}} + \|f\|_{L^p(X)}).$$

In the same way, we prove that

$$\int_0^1 \|Au(x)\|_X^p \, dx \leq C(\|d_0\|_{(D(A),X)_{1/2p,p}} + \|n_1\|_{(D(A),X)_{1/2+1/2p,p}} + \|f\|_{L^p(X)}). \qquad \blacksquare$$

6. Concrete Applications

6.1. A Trace Result in L^p-Case

Here X is a UMD space and A is a linear closed densely defined operator with domain $D(A) \subset X$, satisfying hypotheses (3) and (5).

We will establish a trace result in relation to this operator for functions in L^p. We set

$$\begin{cases} \mathscr{F}_p = (D_A, X)_{1/2p,p} \times (D_A, X)_{1/2+1/2p,p} \\ \mathscr{V}_p = W^{2,p}((0,1); X) \cap L^p((0,1); D(A)) \quad \text{where } 1 < p < +\infty \end{cases}$$

Let us consider the mapping defined by

$$\begin{aligned} G : \mathscr{F}_p \qquad & \to \mathscr{V}_p \\ (u(0), u'(1)) & \mapsto u \end{aligned}$$

Proposition 9. *The mapping G is linear continuous and bijective.*

Proof. It is easy to see that G is linear. Using the main result in the fourth section we have

$$u \in W^{2,p}((0,1); X) \cup L^p((0,1); D(A))$$

if and only if

$$d_0 \in (D_A, X)_{1/2p,p} \quad \text{and} \quad n_1 \in (D_A, X)_{1/2+1/2p,p}$$

so the mapping G is well defined and bijective. More, estimate (11) in previous section proves that G is continuous. Which implies the following result. ■

Corollary 10. *The mapping T defined by*

$$\begin{aligned} T : \mathscr{V}_p & \to \mathscr{F}_p \\ u \quad & \mapsto (u(0), u'(1)) \end{aligned}$$

is linear continuous and bijective.

Proof. It is a deduction of previous proposition. ■

Example 11. Let $X = L^2(\mathbf{R})$. We define operator A as follows

$$\begin{cases} D(A) = H^2(\mathbf{R}) \\ Au = u''. \end{cases}$$

As X is a Hilbert space, $D(A)$ is dense in X, moreover $(-A)$ is a self-adjoint operator, then

$$D(\sqrt{-A}) = (D_A, X)_{1/2,2} = (H^2(\mathbf{R}), L^2(\mathbf{R}))_{1/2,2} = H^1(\mathbf{R})$$

see [15]. Using Fourier transformation we prove that A verifies 3 and 5.

Let us consider the mapping defined by

$$\begin{aligned} F: H^2(0,1;L^2(\mathbf{R})) \cap L^2(0,1;H^2(\mathbf{R})) &\to H^{3/2}(\mathbf{R}) \times H^{1/2}(\mathbf{R}) \\ u &\mapsto (u(0), u'(1)) \end{aligned}$$

applying Theorem 10 we have

PROPOSITION 12. *The mapping F is linear continuous and bijective.*

EXAMPLE 13. Let $X = L^q(\mathbf{R})$. We define an operator A as follows

$$\begin{cases} D(A) = W^{2,q}(\mathbf{R}) \\ Au = u''. \end{cases}$$

$D(A)$ is dense in X and $(-A)$ is a self-adjoint operator, then

$$D(\sqrt{-A}) = (D_A, X)_{1/2,2} = (W^{2,q}(\mathbf{R}), L^q(\mathbf{R}))_{1/2,2} = W^{1,q}(\mathbf{R})$$

see [15]. Using Fourier transformation we prove that A verifies (3) and (5).

Let us consider the mapping $\mathscr{I}$ defined by

$$\begin{aligned} \mathrm{E}^{p,q} &\to (W^{2,q}(\mathbf{R}), L^q(\mathbf{R}))_{1/2p,p} \times (W^{2,q}(\mathbf{R}), L^q(\mathbf{R}))_{1/2p+1/2,p} \\ u &\mapsto (u(0), u'(1)) \end{aligned}$$

where

$$\mathrm{E}^{p,q} = W^{2,p}(0,1;L^q(\mathbf{R})) \cap L^p(0,1;W^{2,q}(\mathbf{R})).$$

Applying Theorem 10 we have

PROPOSITION 14. *The mapping $\mathscr{I}$ is linear continuous and bijective.*

Note that the interpolation spaces

$$(W^{2,q}(\mathbf{R}), L^q(\mathbf{R}))_{1/2p,p}, \quad (W^{2,q}(\mathbf{R}), L^q(\mathbf{R}))_{1/2p+1/2,p}$$

coincide respectively with the following well known Besov spaces

$$B_{q,p}^{2(1-1/2p-1/2)}(\mathbf{R}) = B_{q,p}^{1-1/p}(\mathbf{R}), \quad B_{q,p}^{2(1-1/2p)}(\mathbf{R}) = B_{q,p}^{2-1/p}(\mathbf{R}),$$

which are completely described in Grisvard [11] p. 680.

6.2. Application to PDE

EXAMPLE 15. Let $X = L^p(\mathbf{R})$ and $f \in L^p(0,1;L^p(\mathbf{R}))$, $1 < p < \infty$. Consider the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \dfrac{\partial^2 u}{\partial y^2}(x,y) = f(x,y), \quad (x,y) \in]0,1[\times \mathbf{R}, \\ u(0,y) = d_0(y), \quad y \in \mathbf{R}, \\ \dfrac{\partial u}{\partial x} u(1,y) = n_1(y), \quad y \in \mathbf{R}, \end{cases} \tag{12}$$

We define the operator A as follows

$$\begin{cases} D(A) = W^{2,p}(\mathbf{R}), \\ Au = u'', \end{cases}$$

then problem (12) is equivalent to the abstract problem (1)–(2). $D(A)$ is dense in $L^p(\mathbf{R})$ and A verifies (3) and (5), moreover

$$D(\sqrt{-A}) = (W^{2,p}(\mathbf{R}), L^p(\mathbf{R}))_{1/2,2} = W^{1,p}(\mathbf{R})$$

see Lions-Peetre [15]. We obtain the following result

PROPOSITION 16. *Problem* (12) *has a unique strict solution* u, *such that*

$$u \in W^{2,p}(0,1;L^p(\mathbf{R})) \cap L^p(0,1;W^{2,p}(\mathbf{R}))$$

if and only if

$$\begin{cases} d_0 \in (W^{2,p}(\mathbf{R}), L^p(\mathbf{R}))_{1/2p+1/2,p} = W^{1-1/p,p}(\mathbf{R}) \\ u_1 \in (W^{2,p}(\mathbf{R}), L^p(\mathbf{R}))_{1/2p,p} = W^{2-1/p,p}(\mathbf{R}). \end{cases}$$

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