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ON SOME CLASSES OF SPECTRAL POSETS

By

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Abstract. This paper deals with sufficient conditions on a poset in order to get it spectral. A motivating question is the following (p. 833 [LO76]): "If X is a height 1 poset such that for all $x \neq y \in X$, $\uparrow x \cap \uparrow y$ and $\downarrow x \cap \downarrow y$ are finite, is X spectral?" We obtain the some sufficient conditions for such a poset X to be spectral. In particular, we prove that either if there is a finite subset $F \subseteq X$ such that $\downarrow F \supseteq \mathrm{Min}\, X$, or if $\mathrm{diam}\, X \leq 2$, then the poset X is spectral.

1. Introduction and Preliminaries

W. J. Lewis and J. Ohm showed the following result [LO76]: An ordered disjoint union X of spectral posets (X_λ), $\lambda \in \Lambda$ is spectral. In the same paper, they also showed that if a height 1 poset X satisfies that for all $x \in X$, $\uparrow x \cap \uparrow y = \emptyset$ and $\downarrow x \cap \downarrow y = \emptyset$ for all but finite many $y \in X$, then X is spectral. Moreover, they asked the following analogous two questions: (1) If a spectral poset X is the ordered disjoint union of posets (X_λ), $\lambda \in \Lambda$, are the X_λ also spectral? (2) If a height 1 poset X satisfies that for all $x \neq y \in X$, $\uparrow x \cap \uparrow y$ and $\downarrow x \cap \downarrow y$ are finite, is X spectral? In [BE04], Belaid and Echi studied the both question. For the second question, several authors contributed to the question (e.g. [BF81], [DFP80], [F79], and [LO76]). The first question was answered negatively in [AZ04]. In particular, M. E. Adams and van der Zypen constructed a negative example (i.e., an example which is not a spectral poset but can be embedded in some spectral poset). Note that there is a non-spectral poset which can not be embedded as a connected component in any spectral poset (see Example 3.3). On the other hand, the second was also answered negatively in [Y09]. In particular, one showed that there are height 1 countable non-spectral posets X with diameter ≥ 3 such that

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for all $x \neq y \in X$, $\uparrow x \cap \uparrow y$ and $\downarrow x \cap \downarrow y$ are finite subsets. In contrast, we consider the sufficient conditions for a height 1 poset to be spectral, which are similar to the condition in the second question.

Recall that a poset $(X, \leq)$ is said to be spectral or representable if there is a commutative ring R with unit such that X is order isomorphic to the set $\mathrm{Spec}(R)$ of its prime ideals with the inclusion order. Define the height of X is the supremum of lengths of chains in X. For an element x of a poset X, $\uparrow x := \{y \in X \mid x \leq y\}$ and $\downarrow x := \{y \in X \mid y \leq x\}$ are called the saturation of x and the cosaturation of x respectively. Note that $\uparrow x$ (resp. $\downarrow x$) is also called the set of generalization (resp. specialization) of x.

For a subset $Y \subseteq X$, $\uparrow Y := \bigcup_{y \in Y} \uparrow y$ and $\downarrow Y := \bigcup_{y \in Y} \downarrow y$ are called the saturation of Y and the cosaturation of Y respectively. A subset $Y \subseteq X$ is called a saturation or a upset if $Y = \uparrow Y$. Similarly a subset $Y \subset X$ is also called a cosaturation or a downset if $Y = \downarrow Y$.

Define the diameter $\mathrm{diam}\, X$ of a poset X as the minimal number n such that there is $x \in X$ such that either $(\uparrow\downarrow)^k x = X$ or $(\downarrow\uparrow)^k x = X$ whenever $n = 2k$ is even, and either $(\uparrow\downarrow)^k \uparrow x = X$ or $\downarrow(\uparrow\downarrow)^k x = X$ whenever $n = 2k+1$ is odd. Here, by induction, we mean that $(\uparrow\downarrow)x = \uparrow(\downarrow x) = \{y \in X \mid y \in \uparrow z \text{ for some } z \in \downarrow x\}$, $\downarrow(\uparrow\downarrow)x = \downarrow(\uparrow(\downarrow x)) = \{y \in X \mid y \in \downarrow z \text{ for some } z \in \uparrow\downarrow x\}$, $(\uparrow\downarrow)^2 x = \uparrow(\downarrow(\uparrow(\downarrow x)))$, and so on. In general, $(\downarrow\uparrow)^k x$ and $(\uparrow\downarrow)^k x$ are different even if $k = 1$ and the height of X is one.

For a subset $Y \subseteq X$, denote by $\mathrm{Min}\, Y$ (resp. $\mathrm{Max}\, Y$) the set of minimal (resp. maximal) elements of Y with respect to the restricted order. The connected component or the order component of X containing an element $x \in X$ is the subset S of X of all elements y which have a path $y = y_0 \leq y_1 \geq y_2 \leq \cdots \geq x$ from y to x. If X has only one component, then X is said to be connected.

A topological space X is said to be spectral if there is a commutative ring R with unit such that X is homeomorphic to the set $\mathrm{Spec}(R)$ of its prime ideals with the Zariski topology.

In [H69], Hochster showed that a topological space X is spectral if and only if X is T_0, sober and compact, and has a compact open basis closed under finite intersections.

Let (X, T) be a topological space and $\leq$ a partial order on X. The topology T is said to be order compatible with $\leq$, if $\overline{\{x\}} = \downarrow x$, for each $x \in X$. One can obviously see that $(X, \leq)$ is spectral if and only if there exists an order compatible spectral topology on X.

A poset $(X, \leq)$ with an order compatible topology is called a CTOD (or Priestley) space if X is compact and is totally order-disconnected in the sense that, given $y \not\leq x \in X$, there exists a clopen downset U such that $x \in U$, $y \notin U$. By the results in [S37] and [P94], it is shown that a poset X is spectral if and only if X has a CTOD-topology. Note that a poset $(X, \leq)$ is spectral if and only if the poset $(X, \geq)$ with the opposite order is spectral.

We obtain the following result, which is a generalization of Corollary (p. 166 [BF81]).

Theorem 1.1. *Let $(X, \leq)$ be a height 1 connected poset. Suppose that $|{\downarrow}x \cap {\downarrow}y| < \infty$ for any elements $x \neq y$ of X. If there is a finite subset $F \subseteq X$ such that ${\downarrow}F \supseteq \mathrm{Min}\, X$, then X is a spectral poset. In particular, if either $\mathrm{Max}\, X$ or $\mathrm{Min}\, X$ is finite, then X is spectral.*

By the well-known fact that for a spectral poset $(X, \leq)$ the set $(X, \geq)$ with the reverse order is spectral, the dual statement of the above result holds.

Because any height 1 poset X with diameter ≤ 2 has an element $x \in X$ such that either ${\uparrow}x \supseteq \mathrm{Max}\, X$ or ${\downarrow}x \supseteq \mathrm{Min}\, X$, the poset X satisfies the conditions in the above theorem or the dual statement. The following corollary is induced.

Corollary 1.2. *Any height 1 poset X with diameter ≤ 2 and with $|{\uparrow}x \cap {\uparrow}y| + |{\downarrow}x \cap {\downarrow}y| < \infty$ for any distinct elements $x \neq y \in X$ is spectral.*

This result is in stark contrast to the existence of non-spectral height 1 poset with diameter 3 satisfying the finiteness condition in the above corollary. We will show the following corollary in the next section.

Corollary 1.3. *Let $(X, \leq)$ be a height 1 poset with connected components X_i, $i \in I$. Suppose that $|{\downarrow}x \cap {\downarrow}y| < \infty$ for any elements $x \neq y$ of X. If there are finite subsets $F_i \subseteq X$ for all $i \in I$ such that $\bigcup_{i \in I} {\downarrow}F_i \supseteq \{x \in X : |{\downarrow}x| + |{\uparrow}x| = \infty\}$, then X is spectral.*

2. Proofs of Results

In this section, we show Theorem 1.1 and Corollary 1.3.

Proof of Theorem 1.1. Let $w_1, \ldots, w_n$ be finitely many elements of X such that $\bigcup_{i=1}^n {\downarrow}w_i \supseteq \mathrm{Min}\, X$. Let $Y = X - \bigcup_{i=1}^n {\downarrow}w_i = \mathrm{Max}\, X - \{w_1, \ldots, w_n\}$. Since ${\downarrow}y \cap {\downarrow}w_i$ for any $y \in Y$ and any $i = 1, \ldots, n$ is finite, this implies that ${\downarrow}y \cap \mathrm{Min}\, X = \bigcup_{i=i}^n ({\downarrow}y \cap {\downarrow}w_i)$ is finite. Thus ${\downarrow}y$ is finite for any element $y \in Y$. Let

$W = \bigcup_{i \neq j} \downarrow w_i \cap \downarrow w_j$. Since any intersection of cosaturation of two distinct elements is finite, W is finite. Define an order compatible topology T of X by the closed subbasis $\mathscr{F}_X = \{\downarrow F : F \subseteq X \text{ is finite}\} \cup \{X - S : S \subseteq Y\}$.

Claim 2.1. *$\mathscr{F}_X$ is the set of all closed subsets.*

Indeed, put $\mathscr{F}_0 := \{\downarrow F : F \subseteq X \text{ is finite}\}$ and $\mathscr{F}_1 := \bigcup\{X - S : S \subseteq Y\}$. For $C \in \mathscr{F}_0$, there are $L \subseteq \{1, \dots, n\}$ and a finite downset $F \subseteq X - \{w_1, \dots, w_n\}$ such that $C = \bigcup_{i \in L} \downarrow w_i \cup F$. For $C_1, \dots, C_n \in \mathscr{F}_X$, if there is $i \in \{1, \dots, n\}$ such that $C_i \in \mathscr{F}_1$, then $\bigcup_{i=1}^n C_i \in \mathscr{F}_1$. Otherwise $C_1, \dots, C_n \in \mathscr{F}_0$ and so there are $L \subseteq \{1, \dots, n\}$ and a finite downset $F \subseteq X - \{w_1, \dots, w_n\}$ such that $\bigcup_{i=1}^n C_i = \bigcup_{i \in L} \downarrow w_i \cup F \in \mathscr{F}_0$. Thus $\mathscr{F}_X$ is closed under finite unions. Therefore it suffices to show that $\mathscr{F}_X$ is closed under arbitrary intersections. For $\{C_\lambda\}_{\lambda \in \Lambda} \subseteq \mathscr{F}_X$, if $\{C_\lambda\}_{\lambda \in \Lambda} \subseteq \mathscr{F}_1$ then $\bigcap_{\lambda \in \Lambda} C_\lambda \in \mathscr{F}_1$. Replacing $\{C_\lambda\}_{\lambda \in \Lambda} \cap \mathscr{F}_0$ by $\bigcap\{C_\lambda \mid C_\lambda \in \mathscr{F}_0, \lambda \in \Lambda\}$, we may assume that $|\{C_\lambda\}_{\lambda \in \Lambda} \cap \mathscr{F}_1| \leq 1$. If there is a unique element $C \in \mathscr{F}_1$, then either $\{C_\lambda\}_{\lambda \in \Lambda}$ consists of exactly a single element C or there is some $C_\lambda \in \mathscr{F}_0 \cap \{C_\lambda\}_{\lambda \in \Lambda}$. Thus we may assume that there is some $C_\lambda \in \mathscr{F}_0 \cap \{C_\lambda\}_{\lambda \in \Lambda}$. Then there are $L \subseteq \{1, \dots, n\}$ and a finite downset $F \subseteq X - \{w_1, \dots, w_n\}$ such that $C \cap C_\lambda = \bigcup_{i \in L} \downarrow w_i \cup F \in \mathscr{F}_0$. Replacing C by $C \cap C_\lambda$, we may assume that $\{C_\lambda\}_{\lambda \in \Lambda} \subseteq \mathscr{F}_0$. Since each intersection $\downarrow x \cap \downarrow x'$ for any distinct elements $x \neq x' \in X$ is finite, by the forms of elements of $\mathscr{F}_0$, there are $L \subseteq \{1, \dots, n\}$ and a finite downset $F \subseteq X - \{w_1, \dots, w_n\}$ such that $\bigcap_{\lambda \in \Lambda} C_\lambda = \bigcup_{i \in L} \downarrow w_i \cup F \in \mathscr{F}_0$. Thus $\mathscr{F}_X$ is closed under arbitrary intersections.

For $L \subseteq \{1, \dots, n\}$, denote $U_L = X - \bigcup_{i \in L} \downarrow w_i$. Then there is an open basis $\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$, where $\mathscr{B}_0 = \{V \cap U_L : V \text{ is a cofinite upset in } X, L \subseteq \{1, \dots, n\}\}$, $\mathscr{B}_1 = \{U \subseteq Y : \text{finite}\}$. Notice that $\mathscr{B}_0 = \{X - C \mid C \in \mathscr{F}_0\}$ and $\mathscr{B}_1 = \{X - C \mid C \in \mathscr{F}_1\}$. Hence $\mathscr{B}$ is the set of all open subsets. We will show that $\mathscr{B}$ consists of compact subsets. It suffices to show the following claim:

Claim 2.2. *For $L \subseteq \{1, \dots, n\}$ and a cofinite upset $V \subseteq X$, the open subset $U = V \setminus \bigcup_{i \in L} \downarrow w_i$ is compact.*

Indeed, let $L_i = \{1, \dots, n\} - \{i\}$. Since $U_L \subseteq Y \cup \bigcup_{i \notin L} \downarrow w_i$, $Y \subseteq U_{L_i}$, and $\downarrow w_i \setminus W \subseteq U_{L_i}$, these imply that $U_L \setminus W \subseteq \bigcup_{i \notin L} U_{L_i}$. Since $U_L \supseteq \bigcup_{i \notin L} U_{L_i}$ and W is finite, we have that $U_L \setminus W$ is cofinite in $\bigcup_{i \notin L} U_{L_i}$. Let U as in Claim 2.2. Since $U' := U \setminus W \subseteq U_L \setminus W$ is open and cofinite in $\bigcup_{i \notin L} U_{L_i}$, the finiteness of W implies that $U' \cap U_{L_i}$ is cofinite in U_{L_i} for any $i \notin L$. Since all nonempty open subset in U_{L_i} is cofinite in U_{L_i}, we obtain that $U' \cap U_{L_i}$ is compact for any $i \notin L$.

Hence $U' = \bigcup_{i \notin L}(U' \cap U_{L_i})$ is compact. Since W is finite, $U = U' \cup (U \cap W)$ is compact.

In particular, Claim 2.2 implies that X is compact. Therefore the following claim completes this proof.

Claim 2.3. *X is sober.*

Indeed, let F be a closed subset. Then F is either a cosaturation $F = \bigcup_{i=1}^{l} \downarrow x_i$ of a finite subset or $F = X - S$ where $S \subseteq Y$ is a upset. It suffices to show that F is reducible or has a generic point. Therefore we may assume that $F = X - S$. If $S \neq Y$, then there is an element $x \in Y \backslash S \subset \mathrm{Max}\, X$ such that $\downarrow x \subseteq F$ and $F - x = X - (\{x\} \sqcup S)$ are closed. Thus F is reducible or $\downarrow x = F$. Otherwise $S = Y$. Then $F = \bigcup_{i=1}^{n} \downarrow w_i$. If $n = 1$, then F has a generic point w_1. Otherwise F is reducible. □

Proof of Corollary 1.3. Since any ordered disjoint union of spectral posets is spectral, we may assume that X is connected. Suppose that there is a finite subset $\{w_1, \dots, w_n\} \subseteq X$ such that $\bigcup_{i=1}^{n} \downarrow w_i \supseteq \{x \in X : |\uparrow x| + |\downarrow x| = \infty\}$. Let $Z = \bigcup_{i=1}^{n} \uparrow\downarrow w_i$ and $Y = \mathrm{Min}\, X \backslash Z$. Notice that for any $y \in Y$, $|\uparrow y| + |\downarrow y| < \infty$. Since $\bigcup_{i=1}^{n} \downarrow w_i \supseteq \mathrm{Min}\, Z$, Theorem 1.1 implies that Z is a spectral poset. Since $\mathrm{Max}\, X \backslash Z$ has height 0 and so is a spectral poset, the order disjoint union $Z' := (\mathrm{Max}\, X \backslash Z) \sqcup Z$ is a spectral poset. Note that Y is a downset and $X = Z' \sqcup Y$. To apply Theorem 5.8 [LO76] to $X_1 = Y$ and $X_2 = Z'$, it is enough to show that, for any $x \in Z'$ and for any $y \in Y$, $\downarrow x \cap Y$ and $\uparrow y \cap Z'$ are finite. For $x \in Z$, the definition of Z implies that $\downarrow x \cap Y$ is finite. For $x \in Z' - Z$, $|\downarrow x \cap Y| \leq |\downarrow x| < \infty$. For any $y \in Y$, $|\uparrow y \cap Z'| \leq |\uparrow y| < \infty$. Hence Theorem 5.8 [LO76] implies that X is spectral. □

3. Examples

We describe some spectral posets.

Example 3.1. Let $X_0 = \{c_i\}_{i \in \mathbf{Z}_{>0}} \cup \{w\}$ be a set and $X_1 = \{b_i\}_{i \in \mathbf{Z}_{>0}} \cup \{a\}$ a set. Define a poset $X = X_0 \sqcup X_1$ with an order $\leq$ as follows: $c_i < a$, $w < b_i$ and $c_i < b_i$ for any i. Then Theorem 1.1 implies that X is spectral.

Example 3.2. Let X as in Example 3.1. Define a poset $Y = X \sqcup \{w_i\}_{i \in \mathbf{Z}_{>0}}$ with an extension order $\leq_Y$ of $\leq$ by $w, w_2 <_Y w_1$ and $w_{2i}, w_{2i+2} <_Y w_{2i+1}$ for any $i \in \mathbf{Z}_{>0}$. Then Corollary 1.3 implies that X is spectral.

The following example is a non-spectral poset which can not be embedded as a connected component in any spectral poset. Recall that the topology on a poset X which is generated by the closed base $\{\downarrow F \mid F \subseteq X \text{ is finite}\}$ is called the upper topology on X.

Example 3.3. Let $X_0 = \mathbf{Z}_{>1}$ and $X_1 = \operatorname{Spec} \mathbf{Z} - \{(0)\} = \{(2), (3), (5), \dots\}$. For $n \in \mathbf{Z}_{>1}$, define $X_{1n} := \{(p) \in X_1 \mid p \le n\}$. Define a poset $X_n = X_0 \sqcup X_{1n}$ with an order $\le$ as follows: $m < (p)$ if and only if $m/p \in \mathbf{Z}$. Then the dual statement of Corollary 1.3 implies that X_n is spectral. However the colimit $X = X_0 \sqcup X_1$ of X_n is not spectral and can not be embedded as a connected component in any spectral poset. Indeed, since $\bigcap_{(p) \in X_1} \downarrow(p) = \emptyset$, $\downarrow(p)$ is closed but not compact with respect to the upper topology. Thus X is not compact with respect to the upper topology. Since any order compatible spectral topology contains the upper topology, X can not be embedded as a connected component in any spectral poset.

The following example which is a non-spectral poset X with diameter 2 shows that the finiteness condition (i.e. $|\downarrow x \cap \downarrow y| < \infty$ for any elements $x \neq y \in X$) in Theorem 1.1 and Corollary 1.2 can not be dropped entirely.

Example 3.4. Let $X_0 = \{y_i \mid i \in \mathbf{Z}_{\ge 0}\}$ be a set and $X_1 = \{z_i \mid i \in \mathbf{Z}_{\ge 0}\}$ a set. Define a poset $X = X_0 \sqcup X_1$ with an order $\le$ as follows: $y_j \le z_i$ if and only if $i \le j \in \mathbf{Z}_{\ge 0}$. Then X is a non-spectral poset with diameter 2. Indeed, for any elements $z_i, z_j \in X$ with $i < j$, $\downarrow z_i \cap \downarrow z_j = \{y_k \mid k \in \mathbf{Z}_{\ge j}\}$ and thus $|\downarrow z_i \cap \downarrow z_j| = \infty$. Since $\uparrow\downarrow z_0 = X$, $\operatorname{diam} X = 2$. Since $\downarrow z_i$ are closed and $\bigcap_{i \ge 0} \downarrow z_i = \emptyset$, this implies that $\downarrow z_0$ is closed but not compact with respect to the upper topology. Thus X is not compact with respect to the upper topology. Since any order compatible spectral topology contains the upper topology, there is no spectral topology on X.

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