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Gauss Sums on the Iwahori-Hecke Algebras of Type A

Dedicated to Professor Ken-ichi SHINODA

Yasushi GOMI

Sophia University

Abstract. In this paper, we determine $\tilde{\tau}_q(\chi_q^\lambda)$, the Gauss sums on the Iwahori-Hecke algebras of type A for irreducible characters χ_q^λ, which are q-analogues of those on the symmetric groups. We also explicitly determine the values of the corresponding trace function $\psi_q^{(n)} = \sum_{\lambda \vdash n} \tilde{\tau}_q(\chi_q^\lambda)\chi_q^\lambda$.

1. Introduction

Based on the classical Gauss sum

$$\sum_{x=0}^{p-1} e^{2x^2\pi\sqrt{-1}/p} = \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e^{2\pi x\sqrt{-1}/p} ,$$

where p is an odd prime and $\left(\frac{x}{p}\right)$ is the Legendre symbol, Gomi, Maeda and Shinoda [3] have defined Gauss sums on arbitrary finite groups as follows. Let G be a finite group, and fix a modular representation $\rho : G \to \mathrm{GL}_n(\mathbb{F}_q)$ over $\mathbb{F}_q$, a finite field with q elements. Fix a nontrivial additive character $e : \mathbb{F}_q \to \mathbb{C}$. For a class function $\chi : G \to \mathbb{C}$, the Gauss sum has been defined by

$$\tau(\chi) = \sum_{x \in G} \chi(x) e(\mathrm{Tr}\rho(x)) .$$

In case $G = \mathbb{F}_p^\times$, if we take ρ and e naturally and χ as the Legendre symbol, then $\tau(\chi)$ coincides with the classical Gauss sum. As we mentioned in [3], it is useful to consider $|G|^{-1}\tau(\chi)$ rather than $\tau(\chi)$, so we put $\tilde{\tau}(\chi) = |G|^{-1}\tau(\chi)$ and the Gauss sum for χ shall mean $\tilde{\tau}(\chi)$ in this paper.

Before [3], there are several previous works for Gauss sums on finite linear algebraic groups with natural representations $\rho : G \hookrightarrow \mathrm{GL}_n(\mathbb{F}_q)$. Kondo [7] determined the values of Gauss sums on general linear groups for any irreducible characters. Kim and Lee [6] determined the values for $\mathrm{O}^+(2n, q)$ and for linear characters. In a series of papers after [6],

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they also studied for some other classical groups for linear characters. Saito and Shinoda [9, 10] studied Gauss sums on finite reductive groups for the Deligne-Lusztig generalized characters and determined the values for $\mathrm{Sp}(4, q)$ and $G_2(q)$.

In [3], we started to study Gauss sums on arbitrary finite groups and explicitly determined the values for Weyl groups and for the complex reflection groups $G(m, r, n)$ for any irreducible characters. Based on the results in [3], we expect that there exist the q-analogues of the Gauss sums on Weyl groups, which means the Gauss sums on the Iwahori-Hecke algebras.

In this paper, we consider the case of type A, that is the case of the symmetric groups, and determine the Gauss sums on the Iwahori-Hecke algebras of type A.

This paper is organized as follows. In §2, we shall briefly review the Gauss sums on the symmetric groups. In §3, based on the Gauss sums on the symmetric groups, we determine $\tilde{\tau}_q\big(\chi_q^\lambda\big)$, the Gauss sums on the Iwahori-Hecke algebras of type A. In §4, we explicitly determine the values $\psi_q^{(n)}(T_w)$, where $\psi_q^{(n)}$ is the corresponding trace function defined by $\psi_q^{(n)} = \sum_{\lambda\vdash n} \tilde{\tau}_q\big(\chi_q^\lambda\big)\chi_q^\lambda$. We will see some relation between the Gauss sums and the Markov traces on the Iwahori-Hecke algebras of type A.

2. Preliminaries

For a prime p, let $\mathbb{F}_p$ be a finite field with p elements, and $\rho : \mathcal{S}_n \to \mathrm{GL}_n(\mathbb{F}_p)$ be the permutation representation of $\mathcal{S}_n$, the symmetric group of degree n. Throughout this paper we fix a nontrivial additive character $e : \mathbb{F}_p \to \mathbb{C}$ and a primitive p-th root of unity $\zeta = e(1) \in \mathbb{C}$. Using this e, we define a class function $\psi^{(n)}$ on $\mathcal{S}_n$ as follows:

$$\psi^{(n)}(\sigma) = e(\mathrm{Tr}\,\rho(\sigma)) \quad \text{for } \sigma \in \mathcal{S}_n\,,$$

where Tr denotes the trace of a matrix. We note that

$$\psi^{(n)}(\sigma) = \zeta^{\mathrm{fix}(\sigma)} \quad \text{for } \sigma \in \mathcal{S}_n\,,$$

where $\mathrm{fix}(\sigma)$ is the number of fixed points by σ. Let $\lambda \vdash n$ be a partition of n and χ^λ be the irreducible character of $\mathcal{S}_n$ corresponding to λ. Let

$$\tilde{\tau}_n\big(\chi^\lambda\big) = \big\langle \psi^{(n)}, \chi^\lambda \big\rangle \quad \text{for } \lambda \vdash n\,,$$

where the right hand side of the above is the usual scalar product of $\mathbb{C}$-valued class functions on $\mathcal{S}_n$. Then we have

$$\psi^{(n)} = \sum_{\lambda\vdash n} \tilde{\tau}_n\big(\chi^\lambda\big)\cdot\chi^\lambda\,.$$

We call $\tilde{\tau}_n\big(\chi^\lambda\big)$ the Gauss sum on $\mathcal{S}_n$ associated with the irreducible character χ^λ.

Now we review the isomorphism $R \simeq \Lambda$, where R is the algebra of class functions of symmetric groups and Λ is the algebra of symmetric functions. For the details, see [8]. We denote by R_n the vector space over $\mathbb{C}$ of class functions of $\mathcal{S}_n$. The irreducible characters $\{\chi^\lambda \mid \lambda \vdash n\}$ of $\mathcal{S}_n$ form a basis of R_n. Let $R = \bigoplus_{n\geqslant 0} R_n$ with $R_0 = \langle\chi^{(0)}\rangle = \mathbb{C}$. Then R has a ring structure, defined as follows:

$$u \cdot v = \mathrm{ind}_{\mathcal{S}_m\times\mathcal{S}_n}^{\mathcal{S}_{m+n}}(u \times v) \quad \text{for } u \in R_m, v \in R_n\,.$$

With this multiplication, R is a commutative, associative, graded $\mathbb{C}$-algebra with identity element $\chi^{(0)} = 1$. Let Λ^n be the homogeneous symmetric functions over $\mathbb{C}$ of degree n. Then $\Lambda = \bigoplus_{n\geqslant 0} \Lambda^n$ naturally has a graded $\mathbb{C}$-algebra structure. For each $n \geq 0$ the nth complete symmetric function h_n is the sum of all monomials of total degree n with $h_0 = 1$. Then we have

$$\Lambda = \mathbb{Z}[h_1, h_2, \dots]$$

and the h_n are algebraically independent over $\mathbb{C}$. For each partition $\lambda \vdash n$, we denote by s_λ the Schur function corresponding to λ with $s_{(0)} = 1$ for $\lambda = (0) \vdash 0$. Then the s_λ $(\lambda \vdash n)$ form a basis of Λ^n. We define a linear mapping $\mathrm{ch} : R \to \Lambda$ by $\mathrm{ch}(\chi^\lambda) = s_\lambda$ for each partition λ, which is called the characteristic map. In case $\lambda = (n)$, $\chi^{(n)}$ is the trivial character of $\mathcal{S}_n$ and the Schur function $s_{(n)}$ coincides with the nth complete function h_n. The next theorem is a basic fact.

THEOREM 2.1 ([8, I, (7.3)]). *The characteristic map* ch *is an isomorphism from R to Λ.*

We define an endomorphism $\phi : \Lambda \to \Lambda$ by

$$\phi(h_n) = \sum_{i=0}^{n} h_i\,.$$

Let λ and μ be partitions such that the Young diagram of λ contains that of μ. If the skew diagram λ/μ contains at most one box in each column, then λ/μ is called a horizontal strip which we abbreviate to h.s.

LEMMA 2.2. *For each Schur function s_λ, we have*

$$\phi(s_\lambda) = \sum_{\substack{\mu \\ \lambda/\mu \text{ is h.s.}}} s_\mu\,,$$

where the summation is over all partitions μ such that λ/μ is a horizontal strip.

For a proof, see [3, Th. 2.10].

In [3], the values of the Gauss sums on the symmetric groups are determined explicitly.

THEOREM 2.3 ([3]). *We define an algebra homomorphism $\Phi : \Lambda \to \mathbb{C}$ by*

$$\Phi(h_n) = \frac{(\zeta - 1)^n}{n!} \quad \textit{for } n = 0, 1, 2, \dots\,.$$

(1) *We define a linear mapping* $\tilde{\tau} : R = \bigoplus_{n\geqslant 0} R_n \to \mathbb{C}$ *by*

$$\tilde{\tau}\Big(\sum_{n\geq 0} f_n\Big) = \sum_{n\geq 0} \tilde{\tau}_n(f_n)\,.$$

Then we have

$$\tilde{\tau} = \Phi \circ \phi \circ \mathrm{ch}\,, \tag{2.1}$$

which shows that $\tilde{\tau}$ *is an algebra homomorphism.*

(2) *We define a generating function*

$$W(t) = \sum_{n\geq 0} \tilde{\tau}_n\big(\chi^{(n)}\big)t^n\,.$$

Then we have

$$W(t) = \frac{\exp((\zeta-1)t)}{1-t} = \frac{\exp(\zeta t)}{(1-t)\exp t}\,.$$

(3) *For the trivial character* $\chi^{(n)}$ *of* $\mathcal{S}_n$, *we have*

$$\tilde{\tau}_n\big(\chi^{(n)}\big) = \sum_{k=0}^{n} \frac{(\zeta-1)^k}{k!}\,.$$

(4) *For each irreducible character* χ^λ *of* $\mathcal{S}_n$, *we have*

$$\tilde{\tau}_n\big(\chi^\lambda\big) = \sum_{\substack{\mu \\ \lambda/\mu \text{ is h.s.}}} \frac{(\zeta-1)^{|\mu|}}{h(\mu)}\,,$$

where $h(\mu) = \prod_{x\in\mu} h(x)$ *is the product of hook lengths of* μ *and the summation is over all partitions* μ *such that* λ/μ *is a horizontal strip.*

3. q-analogue of $\tilde{\tau}$

In this section, we construct Gauss sums on the Iwahori-Hecke algebras of type A. In case of the symmetric groups, first we consider the class function $\psi^{(n)}(\sigma) = \zeta^{\mathrm{fix}(\sigma)}$, then the Gauss sums $\tilde{\tau}_n\big(\chi^\lambda\big)$ are defined as structure constants of $\psi^{(n)}$, that is,

$$\psi^{(n)} = \sum_{\lambda\vdash n} \tilde{\tau}_n\big(\chi^\lambda\big)\chi^\lambda\,.$$

The $\tilde{\tau}_n$ induce the algebra homomorphism $\tilde{\tau} : R \to \mathbb{C}$ which is decomposed to $\tilde{\tau} = \Phi\circ\phi\circ\mathrm{ch}$ (see Theorem 2.3(1)). In case of the Iwahori-Hecke algebra of type A, first we consider Φ_q, the q-analogue of Φ, and then we define $\tilde{\tau}_q$, the q-analogues of $\tilde{\tau}$, by $\tilde{\tau}_q = \Phi_q\circ\phi\circ\mathrm{ch}$, which is parallel to the equation (2.1). After that, we define $\psi_q^{(n)}$, the q-analogue of $\psi^{(n)}$, which is a trace function of the Iwahori-Hecke algebra of type A_{n-1} in the next section.

Let $S = \{s_i = (i, i+1) \in \mathcal{S}_n \mid i = 1, 2, \ldots n-1\}$. Then $(\mathcal{S}_n, S)$ has a structure of a Coxeter system and have a usual length function l with respect to S. Let q be a parameter in $\mathbb{C}$ which is not a root of unity. The algebra $\mathcal{H}_n$, the Iwahori-Hecke algebra of type A_{n-1} is a $\mathbb{C}$-algebra with basis $\{T_w \mid w \in \mathcal{S}_n\}$, and whose multiplication is determined by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ qT_{sw} + (q-1)T_w & \text{if } l(sw) = l(w) - 1, \end{cases}$$

for all $s \in S, w \in \mathcal{S}_n$. Let $R(\mathcal{H}_n)$ be the set of trace functions on $\mathcal{H}_n$ and let $R_q = \oplus_{n \geq 0} R(\mathcal{H}_n)$ with $R(\mathcal{H}_0) = \langle \chi_q^{(0)} \rangle = \mathbb{C}$. Since the algebra $\mathcal{H}_n$ is semisimple, the irreducible characters $\{\chi_q^\lambda \mid \lambda \vdash n\}$ form a basis of $R(\mathcal{H}_n)$. It is a basic fact that the algebras R and R_q are isomorphic by the correspondence $\chi^\lambda \mapsto \chi_q^\lambda$ for all partitions λ (see [2, §9.1]). So we have an isomorphism from R_q to Λ, which we also denote by ch, so that $\mathrm{ch}\big(\chi_q^\lambda\big) = s_\lambda$ for each partition λ. We define an algebra homomorphism $\Phi_q : \Lambda \to \mathbb{C}$ by

$$\Phi_q(h_n) = \frac{(\zeta - 1)(\zeta - q) \cdots (\zeta - q^{n-1})}{[n]_q!},$$

where

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q \quad \text{for } n = 1, 2, \ldots .$$

LEMMA 3.1. *For each Schur function s_λ, we have*

$$\Phi_q(s_\lambda) = \frac{q^{n(\lambda)}}{h[\lambda]_q} \prod_{x \in \lambda} \big(\zeta - q^{c(x)}\big),$$

where the product is over all boxes x in the Young diagram of λ, and

$$h[\lambda]_q = \prod_{x \in \lambda} [h(x)]_q, \ \textit{the product of } q\textit{-hook lengths of } \lambda,$$
$$c(x) = j - i, \ \textit{the content of } x = (i, j) \in \lambda,$$
$$n(\lambda) = \sum_{i=1}^{r} (i-1)\lambda_i \quad \textit{for } \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \textit{ with } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r.$$

The proof is an easy application of [8, I,3,Ex.3].

Now we define an algebra homomorphism $\tilde{\tau}_q : R_q \to \mathbb{C}$ by

$$\tilde{\tau}_q = \Phi_q \circ \phi \circ \mathrm{ch},$$

which is parallel to the equation (2.1). For each irreducible character χ_q^λ of $\mathcal{H}_n$, we call $\tilde{\tau}_q\big(\chi_q^\lambda\big)$ a Gauss sum on $\mathcal{H}_n$.

PROPOSITION 3.2. *For each χ_q^λ, we have*

$$\tilde{\tau}_q\big(\chi_q^\lambda\big) = \sum_{\substack{\mu \\ \lambda/\mu \text{ is h.s.}}} \frac{q^{n(\mu)}}{h[\mu]_q} \prod_{x\in\mu}\big(\zeta - q^{c(x)}\big)\,,$$

where the summation is over all partitions μ such that λ/μ is a horizontal strip.

PROOF. Using Lemma 3.1, direct calculation proves the proposition as follows:

$$\begin{aligned}\tilde{\tau}_q\big(\chi_q^\lambda\big) &= \Phi_q \circ \phi(s_\lambda) \\ &= \sum_{\substack{\mu \\ \lambda/\mu \text{ is h.s.}}} \Phi_q(s_\mu) \\ &= \sum_{\substack{\mu \\ \lambda/\mu \text{ is h.s.}}} \frac{q^{n(\mu)}}{h[\mu]_q} \prod_{x\in\mu}\big(\zeta - q^{c(x)}\big).\end{aligned}$$

□

For the index character $\chi_q^{(n)}$ of $\mathcal{H}_n$ which corresponds to the trivial character $\chi^{(n)}$ of $\mathcal{S}_n$, the value $\tilde{\tau}_q\big(\chi_q^{(n)}\big)$ can be expressed simply. In order to do that, we prepare some notation. For $\sigma \in \mathcal{S}_n$, the descent set of σ is defined to be

$$\mathrm{DES}(\sigma) = \{\, i \in \{1, 2, \dots, n-1\} \mid \sigma(i) > \sigma(i+1)\,\}\ ,$$

and the major index of σ is defined by

$$\mathrm{maj}(\sigma) = \sum_{i\in\mathrm{DES}} i\ .$$

We define two generation functions

$$W_q(t) = \sum_{n\ge 0} \tilde{\tau}_q\big(\chi_q^{(n)}\big)\, t^n = \sum_{n\ge 0} \Phi_q \circ \phi(h_n)\, t^n\ ,$$

and

$$V_q(t) = \sum_{n\ge 0} \Phi_q(h_n)\, t^n\ .$$

LEMMA 3.3. *We have*

$$W_q(t) = \frac{\exp_q(\zeta t)}{(1-t)\exp_q(t)}\ ,$$

where

$$\exp_q(t) = \sum_{n\ge 0} \frac{t^n}{[n]_q!}\ .$$

PROOF. Since $\phi(h_n) = \sum_{i=0}^{n} h_i$, we have

$$\begin{aligned} W_q(t) &= \sum_{n\geq 0}\sum_{i=0}^{n} \Phi_q(h_i)\, t^n \\ &= \sum_{i\geq 0}\sum_{n\geq i} \Phi_q(h_i)\, t^i t^{n-i} \\ &= \frac{1}{1-t} V_q(t). \end{aligned}$$

Using the formula in [1, Th.2.1], we have

$$V_q(t) = \frac{\exp_q(\zeta t)}{\exp_q(t)},$$

and the proof is completed. □

THEOREM 3.4. *For the index character $\chi_q^{(n)}$ of $\mathcal{H}_n$, we have*

$$\tilde{\tau}_q\big(\chi_q^{(n)}\big) = \frac{1}{[n]_q!}\sum_{\sigma\in\mathcal{S}_n} q^{\mathrm{maj}(\sigma)}\zeta^{\mathrm{fix}(\sigma)}.$$

PROOF. We define a generating function

$$W'_q(t) = \sum_{n\geq 0}\sum_{\sigma\in\mathcal{S}_n} q^{\mathrm{maj}(\sigma)}\zeta^{\mathrm{fix}(\sigma)}\frac{t^n}{[n]_q!}.$$

In order to prove the theorem, we shall show $W'_q(t) = W_q(t)$. By [11, Th.1.2] or [4], we have

$$W'_q(t) = \frac{(1-q)\exp_q(\zeta t)}{\exp_q(qt) - q\exp_q(t)}. \tag{3.1}$$

Since we have

$$\exp_q(t) = \prod_{n\geq 0}\big(1 - q^n(1-q)t\big)^{-1}$$

by [1, Cor.2.2], the denominator of the right hand side of the equation (3.1) is

$$\begin{aligned} \exp_q(qt) - q\exp_q(t) &= \prod_{n\geq 0}\big(1 - q^{n+1}(1-q)t\big)^{-1} - q\exp_q(t) \\ &= (1-(1-q)t)\exp_q(t) - q\exp_q(t) \\ &= (1-q)(1-t)\exp_q(t). \end{aligned}$$

By Lemma 3.3, we obtain the result. □

4. q-analogue of $\psi^{(n)}$

We define $\psi_q^{(n)}$, a trace function of $\mathcal{H}_n$ by

$$\psi_q^{(n)} = \sum_{\lambda \vdash n} \tilde{\tau}_q\big(\chi_q^\lambda\big)\, \chi_q^\lambda\,,$$

which is a q-analogue of $\psi^{(n)}$. In this section, we determine the values $\psi_q^{(n)}(T_w)$ for $w \in \mathcal{S}_n$. First we briefly review the theory of trace functions on $\mathcal{H}_n$ (see [2, §8.2]). Let Cl be the set of conjugacy classes of $\mathcal{S}_n$, and let

$$C_{\min} = \{\, w \in C \mid w \text{ has minimal length in } C \,\} \quad \text{for } C \in \mathrm{Cl}\,.$$

Then we have the following.

THEOREM 4.1 ([2, (8.2.3), (8.2.6)]). *Let $\psi : \mathcal{H}_n \to \mathbb{C}$ be a trace function on $\mathcal{H}_n$.*

(1) $\psi(T_w) = \psi(T_{w'})$ *for* $w, w' \in C_{min}$.

(2) *For each $w \in \mathcal{S}_n$, there exist $f_{w,C} \in \mathbb{C}$ which are uniquely determined by the equations*

$$T_w \equiv \sum_{C \in \mathrm{Cl}} f_{w,C}\, T_{w_C} \mod [\mathcal{H}_n, \mathcal{H}_n]\,,$$

where $[\mathcal{H}_n, \mathcal{H}_n] \subset \mathcal{H}_n$ is the subspace spanned by all commutators $[h, h'] = hh' - h'h$ for $h, h' \in \mathcal{H}_n$.

(3) *We choose a representative $w_C \in C_{min}$ for each $C \in$ Cl. Then we have*

$$\psi(T_w) = \sum_{C \in \mathrm{Cl}} f_{w,C}\, \psi(T_{w_C})\,.$$

Since the values of $f_{w,C}$ can be obtained inductively on the length of w, in order to give the values $\psi_q^{(n)}(T_w)$ for all $w \in \mathcal{S}_n$, it is sufficient to give them only for $w = w_C \in C_{\min}$ for $C \in$ Cl.

LEMMA 4.2. *Let $x \in \mathcal{S}_m$ and $y \in \mathcal{S}_{n-m}$ and let $\bar{y}$ be the image of y by the injection $\mathcal{S}_{n-m} \hookrightarrow \mathcal{S}_n$ with $s_i \mapsto s_{i+m}$ for $i = 1, 2, \ldots, n-m-1$. Then we have*

$$\psi_q^{(n)}(T_x T_{\bar{y}}) = \psi_q^{(m)}(T_x)\psi_q^{(n-m)}(T_y)\,.$$

PROOF. Let $c^\nu_{\lambda,\mu}$ be the Littlewood-Richardson coefficients for partitions $\nu \vdash n$, $\lambda \vdash m$ and $\mu \vdash n-m$, which give the multiplication rule for Schur functions:

$$s_\lambda s_\mu = \sum_{\nu \vdash n} c^\nu_{\lambda,\mu} s_\nu\,.$$

By the isomorphism ch : $R_q \to \Lambda$, we have

$$\chi_q^\lambda \cdot \chi_q^\mu = \mathrm{ind}^{\mathcal{H}_n}_{\mathcal{H}_m \otimes \mathcal{H}_{n-m}} \chi_q^\lambda \otimes \chi_q^\mu = \sum_{\nu \vdash n} c^\nu_{\lambda,\mu} \chi_q^\nu\,,$$

and by the Frobenius reciprocity, we have

$$\chi_q^\nu|_{\mathcal{H}_m\otimes\mathcal{H}_{n-m}} = \sum_{\lambda\vdash m,\ \mu\vdash n-m} c^\nu_{\lambda,\mu}\ \chi_q^\lambda\otimes\chi_q^\mu .$$

Then we obtain

$$\begin{aligned}\psi_q^{(n)}(T_xT_{\bar{y}}) &= \sum_{\nu\vdash n}\tilde{\tau}_q\big(\chi_q^\nu\big)\chi_q^\nu(T_xT_{\bar{y}})\\ &= \sum_{\nu\vdash n}\tilde{\tau}_q\big(\chi_q^\nu\big)\sum_{\lambda\vdash m,\ \mu\vdash n-m} c^\nu_{\lambda,\mu}\chi_q^\lambda(T_x)\chi_q^\mu(T_y)\\ &= \sum_{\lambda\vdash m,\ \mu\vdash n-m}\left(\sum_{\nu\vdash n} c^\nu_{\lambda,\mu}\tilde{\tau}_q\big(\chi_q^\nu\big)\right)\chi_q^\lambda(T_x)\chi_q^\mu(T_y)\\ &= \sum_{\lambda\vdash m}\tilde{\tau}_q\big(\chi_q^\lambda\big)\chi_q^\lambda(T_x)\sum_{\mu\vdash n-m}\tilde{\tau}_q\big(\chi_q^\mu\big)\chi_q^\mu(T_y)\\ &= \psi_q^{(m)}(T_x)\psi_q^{(n-m)}(T_y),\end{aligned}$$

which completes the proof. □

COROLLARY 4.3. *We have*

$$\psi_q^{(n)}(1) = \zeta^n \quad \textit{for } n = 1, 2, \dots .$$

PROOF. By Lemma 4.2, we have

$$\psi^{(n)}(1) = \left(\psi^{(1)}(1)\right)^n .$$

By the definition of $\psi_q^{(n)}$, we have

$$\psi_q^{(1)} = \tilde{\tau}_q\big(\chi_q^{(1)}\big)\chi_q^{(1)} ,$$

and by the definition of $\tilde{\tau}_q$, we have

$$\tilde{\tau}_q\big(\chi_q^{(1)}\big) = \Phi_q(h_1) + \Phi_q(h_0) = \zeta - 1 + 1 = \zeta .$$

Hence we obtain

$$\psi_q^{(n)}(1) = \left(\zeta\ \chi_q^{(1)}(1)\right)^n = \zeta^n \quad \text{for } n = 1, 2, \dots .$$

□

By Theorem 4.1 and Lemma 4.2, it is sufficient to give the value of $\psi_q^{(n)}(T_w)$ only for the Coxeter element $w = s_1s_2\cdots s_{n-1}$.

LEMMA 4.4. *We define a trace function $\varphi_q^{(n)}$ on $\mathcal{H}_n$ by*

$$\varphi_q^{(n)} = \sum_{\lambda\vdash n}\Phi_q(s_\lambda)\chi_q^\lambda .$$

Then we have

$$\psi_q^{(n)} = \sum_{k=0}^{n} \mathrm{ind}_{\mathcal{H}_k \otimes \mathcal{H}_{n-k}}^{\mathcal{H}_n} \varphi_q^{(k)} \otimes \chi_q^{(n-k)} ,$$

where $\chi_q^{(n-k)}$ is the index character of $\mathcal{H}_{n-k}$.

PROOF. The direct calculation shows that

$$\begin{aligned}
\psi_q^{(n)} &= \sum_{\lambda \vdash n} \Phi_q \circ \phi(s_\lambda)\, \chi_q^\lambda \\
&= \sum_{\lambda \vdash n} \sum_{\substack{\mu \\ \lambda/\mu \text{ is h.s.}}} \Phi_q(s_\mu)\, \chi_q^\lambda \\
&= \sum_{k=0}^{n} \sum_{\mu \vdash k} \Phi_q(s_\mu) \sum_{\substack{\lambda \vdash n \\ \lambda/\mu \text{ is h.s.}}} \chi_q^\lambda \\
&= \sum_{k=0}^{n} \sum_{\mu \vdash k} \Phi_q(s_\mu)\, \mathrm{ind}_{\mathcal{H}_k \otimes \mathcal{H}_{n-k}}^{\mathcal{H}_n} \chi_q^\mu \otimes \chi_q^{(n-k)} \quad \text{by Pieri's formula [8, I,(5.16)]} \\
&= \sum_{k=0}^{n} \mathrm{ind}_{\mathcal{H}_k \otimes \mathcal{H}_{n-k}}^{\mathcal{H}_n} \varphi_q^{(k)} \otimes \chi_q^{(n-k)},
\end{aligned}$$

which completes the proof. □

LEMMA 4.5. *Let $w_C = s_1 s_2 \cdots s_{n-1}$ be a Coxeter element of $\mathcal{S}_n$, then we have*

$$\varphi_q^{(n)}(T_{w_C}) = (1-q)^{n-1}(\zeta - 1) .$$

PROOF. By Lemma 3.1, we have

$$(\zeta - 1)^{-n} \varphi_q^{(n)} = \sum_{\lambda \vdash n} \frac{q^{n(\lambda)} \prod_{x \in \lambda} \left(\zeta - q^{c(x)}\right)}{(\zeta - 1)^n h[\lambda]_q} \chi_q^\lambda .$$

The argument of [5, §5] shows that $(\zeta - 1)^{-n} \varphi_q^{(n)}$ is the Markov trace on $\mathcal{H}_n$ with parameter $z = \dfrac{1-q}{\zeta - 1}$, and we have

$$\varphi_q^{(n)}(T_{w_C}) = (\zeta - 1)^n z^{n-1} = (1-q)^{n-1}(\zeta - 1) ,$$

which completes the proof. □

Here we review the formula of induced character of Iwahori-Hecke algebra (see [2, §9.1]). Let (W, S) be a Coxeter system and let $\mathcal{H}$ be the corresponding Iwahori-Hecke algebra. For $J \subset S$, let $W_J \subset W$ be the corresponding parabolic subgroup and let $\mathcal{H}_J = \langle T_w \mid w \in W_J \rangle \subset \mathcal{H}$ be the corresponding parabolic subalgebra. Let W^J be the right coset representatives of $W_J\backslash W$ such that $x \in W^J$ is the unique element of minimal length in $W_J x$. We denote by w_0 (resp. w_J) the longest element of W (resp. W_J), and define $d_J = w_J w_0$. Then d_J is the unique element of maximal element in W^J and any element $x \in W^J$ satisfies $l\big(x^{-1}d_J\big) = l(d_J) - l(x)$. For any $\mathcal{H}_J$-module V, the induced module is defined by

$$\mathrm{ind}_J^S(V) = V \otimes_{\mathcal{H}_J} \mathcal{H}\,,$$

and if χ is the character of V, then we denote by $\mathrm{ind}_J^S(\chi)$ the character of $\mathrm{ind}_J^S(V)$. The direct sum decomposition $\mathcal{H} = \sum_{x\in W^J} \mathcal{H}_J T_x$ implies that

$$T_x T_w = \sum_{y\in W^J} h^w(x,y)T_y \quad \text{with unique } h^w(x,y) \in \mathcal{H}_J\,,$$

for any $w \in W$ and $x \in W^J$.

PROPOSITION 4.6 ([2, Prop.9.1.3]). *For any character χ of $\mathcal{H}_J$ and for any $w \in W$, we have*

$$\mathrm{ind}_J^S(\chi)(T_w) = \sum_{x\in W^J} \chi(h^w(x,x))\,.$$

We apply this proposition to the case of $W = \mathcal{S}_n$,

$$S = \{\, s_i = (i, i+1) \mid i = 1, 2, \dots n-1 \,\}\,,$$

$J = S - \{s_k\}$ and the character $\varphi_q^{(k)} \otimes \chi_q^{(n-k)}$ of the parabolic subalgebra $\mathcal{H}_J = \mathcal{H}_k \otimes \mathcal{H}_{n-k}$ for $k = 1, 2, \dots, n-1$.

LEMMA 4.7. *Let $w_C = s_1 s_2 \cdots s_{n-1}$ be a Coxeter element of $\mathcal{S}_n$ and $J = S - \{s_k\}$, $w_k = s_1 s_2 \cdots \hat{s_k} \cdots s_{n-1}$ for $k = 1, 2, \dots, n-1$. Then we have*

$$h^{w_C}(x,x) = \begin{cases} (q-1)T_{w_k} & \text{if } x = d_J\,, \\ 0 & \text{if } x \neq d_J\,, \end{cases}$$

for $x \in W^J$.

PROOF. Let $x \in W^J - \{d_J\}$, then there exist some $s_i \in S$ such that $l(xs_i) = l(x) + 1$ and $xs_i \in W^J$. If we assume that

$$x, xs_1, xs_1s_2, \dots, xs_1s_2\cdots s_{i-1} \in W_J x\,,$$

$$l(xs_i) = l(x) + 1 \ \text{ and } \ xs_i \in W^J\,,$$

for some i, then it is easy to see that $xw_C \in W_J y$ for some $y \in W^J$ with $l(y) > l(x)$, which implies that $T_x T_{w_C} \in \mathcal{H}_J y$. Hence we obtain $h^{w_C}(x,x) = 0$. In case

$$x, xs_1, xs_1s_2, \dots, xs_1s_2\cdots s_{i-1} \in W_J x\,,$$

$$l(xs_i) = l(x) - 1 \text{ and } xs_i \in W^J ,$$

for some i, the situation is more difficult than the previous case, but if you consider the action of W on $W_J \backslash W$ carefully, then it can be shown that $h^{w_C}(x, x) = 0$. Now we assume that $x = d_J$. We note that

$$d_J s_i = \begin{cases} s_{i+k} d_J & \text{for } i = 1, 2, \dots, n-k-1, \\ s_{i+k-n} d_J & \text{if } i = n-k+1, n-k+2, \dots, n-1, \end{cases}$$

and

$$d_J s_{n-k} \cdots s_{n-k+i-1} \in W^J$$

with $l(d_J s_{n-k} \cdots s_{n-k+i-1}) = l(d_J) - i$ for $i = 1, 2, \dots, k$. Then the direct calculation shows that

$$\begin{aligned} T_{d_J} T_{w_C} &= T_{s_{k+1}s_{k+2}\cdots s_{n-1}} T_{d_J} T_{s_{n-k}s_{n-k+1}\cdots s_{n-1}} \\ &= (q-1) T_{s_{k+1}s_{k+2}\cdots s_{n-1}} T_{d_J} T_{s_{n-k+1}s_{n-k+2}\cdots s_{n-1}} \\ &\quad + q T_{s_{k+1}s_{k+2}\cdots s_{n-1}} T_{d_J s_{n-k}} T_{s_{n-k+1}s_{n-k+2}\cdots s_{n-1}} \\ &= \cdots \\ &= (q-1) T_{w_k} T_{d_J} \\ &\quad + \sum_{i=1}^{k-1} q^i (q-1) T_{s_{i+1}s_{i+2}\cdots \hat{s}_k \cdots s_{n-1}} T_{d_J s_{n-k} \cdots s_{n-k+i-1}} \\ &\quad + q^n T_{s_{k+1}s_{k+2}\cdots s_{n-1}} T_{d_J s_{n-k} \cdots s_{n-1}} . \end{aligned}$$

Hence we obtain that $h^{w_C}(d_J, d_J) = (q-1) T_{w_k}$ as required. □

THEOREM 4.8. *For the Coxeter element $w_C = s_1 s_2 \cdots s_{n-1} \in \mathcal{S}_n$, we have*

$$\psi_q^{(n)}(T_{w_C}) = \frac{q(q-1)\bigl(q^{n-2} - (1-q)^{n-2}\bigr)}{q - (1-q)} \zeta + \frac{q\bigl(q^{n-1} - (1-q)^{n-1}\bigr)}{q - (1-q)} .$$

PROOF. By Corollary 4.3, we have $\psi_q^{(1)}(1) = \zeta$ as required. Now we assume $n \geq 2$. By Lemma 4.4, we have

$$\psi_q^{(n)}(T_{w_C}) = \varphi_q^{(n)}(T_{w_C}) + \chi_q^{(n)}(T_{w_C}) + \sum_{k=1}^{n-1} \operatorname{ind}_{\mathcal{H}_k \otimes \mathcal{H}_{n-k}}^{\mathcal{H}_n} \varphi_q^{(k)} \otimes \chi_q^{(n-k)}(T_{w_C}) .$$

By Lemma 4.5, we have

$$\varphi_q^{(n)}(T_{w_C}) = (1-q)^{n-1}(\zeta - 1) .$$

Since $\chi_q^{(n)}$ is the index character of $\mathcal{H}_n$, we have

$$\chi_q^{(n)}(T_{w_C}) = q^{l(w_C)} = q^{n-1} .$$

By Proposition 4.6 and Lemma 4.7, we have

$$\begin{aligned}
\mathrm{ind}_{\mathcal{H}_k\otimes\mathcal{H}_{n-k}}^{\mathcal{H}_n}\varphi_q^{(k)}\otimes\chi_q^{(n-k)}(T_{w_C}) &= \sum_{x\in W^J}\varphi_q^{(k)}\otimes\chi_q^{(n-k)}\big(h^{w_C}(x,x)\big)\\
&= \varphi_q^{(k)}\otimes\chi_q^{(n-k)}\big((q-1)T_{w_k}\big)\\
&= (q-1)(1-q)^{k-1}(\zeta-1)q^{n-k-1}\\
&= -q^{n-k-1}(1-q)^k(\zeta-1).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\psi_q^{(n)}(T_{w_C}) &= (1-q)^{n-1}(\zeta-1)+q^{n-1}-\sum_{k=1}^{n-1}q^{n-k-1}(1-q)^k(\zeta-1)\\
&= \frac{q(q-1)\big(q^{n-2}-(1-q)^{n-2}\big)}{q-(1-q)}\zeta+\frac{q\big(q^{n-1}-(1-q)^{n-1}\big)}{q-(1-q)},
\end{aligned}$$

which completes the proof. □

Here we list the values $\psi_q^{(n)}(T_w)$ for $n = 1, 2, 3, 4$. For a partition μ, we abbreviate the element T_{w_μ} to T_μ, where w_μ is an element of minimal length in the conjugacy class corresponding to μ.

Case $n = 1$:

$$\psi_q^{(1)}\big(T_{(1)}\big)=\zeta\,.$$

Case $n = 2$:

$$\psi_q^{(2)}\big(T_{(1,1)}\big)=\zeta^2,\quad \psi_q^{(2)}\big(T_{(2)}\big)=q\,.$$

Case $n = 3$:

$$\psi_q^{(3)}\big(T_{(1,1,1)}\big)=\zeta^3,\quad \psi_q^{(3)}\big(T_{(2,1)}\big)=q\zeta,\quad \psi_q^{(3)}\big(T_{(3)}\big)=q(q-1)\zeta+q\,.$$

Case $n = 4$:

$$\psi_q^{(4)}\big(T_{(1,1,1,1)}\big)=\zeta^4,\quad \psi_q^{(4)}\big(T_{(2,1,1)}\big)=q\zeta^2,\quad \psi_q^{(4)}\big(T_{(2,2)}\big)=q^2\,,$$

$$\psi_q^{(4)}\big(T_{(3,1)}\big)=q(q-1)\zeta^2+q\zeta,\quad \psi_q^{(4)}\big(T_{(4)}\big)=q(q-1)\zeta+(q^2-q+1)q\,.$$

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Present Address:
DEPARTMENT OF INFORMATION AND COMMNUNICATION SCIENCES,
SOPHIA UNIVERSITY,
KIOICHO, CHIYODA-KU, TOKYO 102–8554, JAPAN.
e-mail: y-gomi@sophia.ac.jp