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Nested Square Roots and Poincaré Functions

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Abstract. We are concerned with finitely nested square roots which are roots of iterations of a real quadratic polynomial $x^2 - c$ with $c \geq 2$, and the limits of such nested square roots. We investigate how they are related to a Poincaré function $f(x)$ satisfying the functional equation $f(sx) = f(x)^2 - c$, where $s = 1 + \sqrt{1+4c}$. Our main theorems can be viewed as a natural generalization of the work of Wiernsberger and Lebesgue for the case $c = 2$. The key ingredients of the proof are some analytic properties of $F(x)$, which have been intensively studied by the second author using infinite compositions.

1. Introduction

Let c be a real number with $c \geq 2$ and $\varepsilon_1, \varepsilon_2, \ldots$ an infinite sequence consisting of ± 1. In this paper we are concerned with nested square roots of the form

$$R_c(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m) = \varepsilon_1\sqrt{c + \varepsilon_2\sqrt{c + \varepsilon_3\sqrt{c + \cdots + \varepsilon_m\sqrt{c}}}} \tag{1}$$

and infinite nested square roots

$$R_c(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) := \lim_{m\to\infty} R_c(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m)\,. \tag{2}$$

The existence of the limit (2) is proved in §7. In the case of $c = 2$, it is known that the nested root (1) can be expressed by the sine function:

$$R_2(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m) = 2\sin\frac{\pi}{2}\left(\frac{\varepsilon_1}{2} + \frac{\varepsilon_1\varepsilon_2}{2^2} + \cdots + \frac{\varepsilon_1\varepsilon_2\cdots\varepsilon_m}{2^m}\right). \tag{3}$$

This formula may be rewritten as

$$R_2(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m) = 2\cos\pi\left(\frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_m}{2^m} + \frac{1}{2^{m+1}}\right), \tag{4}$$

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where

$$a_i = \frac{1-\varepsilon_1\cdots\varepsilon_i}{2} = \begin{cases} 0 & (\text{if } \varepsilon_1\cdots\varepsilon_i = 1)\,, \\ 1 & (\text{if } \varepsilon_1\cdots\varepsilon_i = -1)\,. \end{cases}$$

Taking $\lim_{m\to\infty}$ of (4), we obtain a simple formula for the infinite nested square root:

$$R_2(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) = 2\cos\alpha\pi\,, \tag{5}$$

where α is a real number defined by the 2-adic expansion

$$\alpha = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots\,.$$

These formulas were proved by Wiernsberger [10] in 1905, and about thirty years later Lebesgue [7] (see also [8]) independently found the same formulas.

The purpose of this paper is to give a generalization of the formulas (4) and (5) to the case $c \geq 2$. To accomplish the task, we need a suitable function which will take the place of $\cos x$. In the proof of the formulas (4) and (5), the duplication formula

$$2\cos 2x = (2\cos x)^2 - 2$$

was crucial. It is therefore natural to seek for a function $f(x)$ satisfying the functional equation

$$f(sx) = f(x)^2 - c\,, \tag{6}$$

where s is a constant depending only on c. Such functional equations were studied by Poincaré, who showed that there exists an entire function $f(x)$ satisfying (6). In [4], [5] and [6] the second author of the present paper studied intensively analytic properties of such functions using a technique of infinite compositions.

In §2 and §3 we define an infinite composition $F(x)$ of a family of certain quadratic functions and study its analytic properties. We refer the reader to [1], [4], [5] and [6] for more details. In §4 we study the function $f(x) := s(F(x) + 1/2)$, which is the main object of the present paper. In particular, the zero sets of $f(x)$ and $f'(x)$ are crucial in studying nested square roots of the form (1) and its limit (2). In §5 we study the zero set of $F(x)$. Most results in §4 and §5 were proved by the second author in his master thesis [4]. Our main results (Theorem 6.7 and Theorem 7.3) give explicit descriptions of finite or infinite nested square roots in terms of special values of $f(x)$. As an application of Theorem 6.7, we compute the zeros of $f(x)$ and $F(x)$ (Theorem 6.10).

Another aspect of nested square roots in the case of $c = 2$ is a famous formula due to Viéta:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots\,. \tag{7}$$

In the final section we prove a formula on an infinite product involving nested square roots (Theorem 8.2), which may be regarded as a generalization of (7).

2. Infinite compositions of quadratic functions

For any two $\mathbb{C}$-valued functions $u(x)$, $v(x)$ on $\mathbb{C}$, we write

$$u(x) \circ v(x) = u(v(x)).$$

In this notation, for any complex number α, we can write $u(\alpha) = u(x) \circ \alpha$. More generally, if $\{u_n(x)\}_{n=1}^{\infty}$ is a sequence of $\mathbb{C}$-valued functions on $\mathbb{C}$, we write

$$u_1(x) \circ u_2(x) \circ \cdots \circ u_N(x) = u_1(u_2(\cdots u_N(x) \cdots)).$$

We also adopt the following notation used in [4], [5] and [6]:

$$\mathop{\mathcal{R}}_{n=1}^{N} u_n(x) = u_1(x) \circ u_2(x) \circ \cdots \circ u_N(x),$$
$$\mathop{\mathcal{R}}_{n=1}^{\infty} u_n(x) = \lim_{N \to \infty} \mathop{\mathcal{R}}_{n=1}^{N} u_n(x).$$

In the following we will study the infinite composition of quadratic functions

$$F(x, s) := \mathop{\mathcal{R}}_{n=1}^{\infty} \left(x + \frac{x^2}{s^n} \right), \tag{8}$$

where $s \in \mathbb{C}$ is a constant such that $|s| > 1$. By the definition, the function $F(x, s)$ is the limit of

$$F_N(x, s) := \mathop{\mathcal{R}}_{n=1}^{N} \left(x + \frac{x^2}{s^n} \right).$$

If no confusion arises, we simply write $F(x) = F(x, s)$ and $F_N(x) = F_N(x, s)$. The existence of the limit is proved in [6, Proposition 1.2] (see also [4] and [5]). It is clear from the definition that $F(0) = 0$ and $F'(0) = 1$. When $s = 2, 4, -2$, the function $F(x, s)$ is an elementary function (see [1] and [6]). More precisely we have

$$F(x, 2) = \frac{1}{2}\left(e^{2x} - 1\right),$$
$$F(x, 4) = \frac{1}{2}\left(\cos\sqrt{-4x} - 1\right),$$
$$F(x, -2) = \sin\left(\frac{2x}{\sqrt{3}} + \frac{\pi}{6}\right) - \frac{1}{2}$$

for any $x \in \mathbb{C}$. These are shown by the following:

PROPOSITION 2.1. *If* $|s| > 1$, *then the function* $F(x)$ *defined by* (8) *satisfies the functional equation*

$$F(sx) = s(F(x)^2 + F(x)) . \tag{9}$$

Conversely, if a complex valued function $H(x)$ *differentiable at* $x = 0$ *satisfies the functional equation* (9) *together with* $H(0) = 0$, $H'(0) = 1$, *then* $H(x) = F(x)$.

PROOF. Let $N > 1$ be an integer. Note that $F_N(sx)/s$ and $x + x^2/s^{n-1}$ are "conjugate" to $F_N(x)$ and $x + x^2/s^n$ respectively in the following sense:

$$\frac{F_N(sx)}{s} = \frac{x}{s} \circ F_N(x) \circ (sx) ,$$

$$x + \frac{x^2}{s^{n-1}} = \frac{x}{s} \circ \left(x + \frac{x^2}{s^n}\right) \circ (sx) .$$

Therefore

$$\begin{aligned}
\frac{F_N(sx)}{s} &= \frac{x}{s} \circ F_N(x) \circ (sx) \\
&= \frac{x}{s} \circ \left(\mathop{\mathcal{R}}_{n=1}^{N} \left(x + \frac{x^2}{s^n}\right)\right) \circ (sx) \\
&= \mathop{\mathcal{R}}_{n=1}^{N} \left(\frac{x}{s} \circ \left(x + \frac{x^2}{s^n}\right) \circ (sx)\right) \\
&= \mathop{\mathcal{R}}_{n=1}^{N} \left(x + \frac{x^2}{s^{n-1}}\right) \\
&= (x + x^2) \circ F_{N-1}(x) \\
&= F_{N-1}(x) + F_{N-1}(x)^2 .
\end{aligned}$$

Taking the limit $N \to \infty$, we obtain the functional equation

$$\frac{F(sx)}{s} = F(x) + F(x)^2 .$$

This proves the first part of the proposition.

In order to prove the second part of the proposition, let $H(x)$ be a complex valued function defined on $\mathbb{C}$ that is differentiable at $x = 0$ and satisfies the functional equation

$$H(sx) = s(x + x^2) \circ H(x) \quad (x \in \mathbb{C})$$

with the initial condition $H(0) = 0$, $H'(0) = 1$. Then we have

$$\begin{aligned}
H(x) &= \left(x + \frac{x^2}{s}\right) \circ sH\left(\frac{x}{s}\right) \\
&= \left(x + \frac{x^2}{s}\right) \circ s^2(x + x^2) \circ \frac{x}{s^2} \circ s^2 x \circ H\left(\frac{x}{s^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(x + \frac{x^2}{s}\right) \circ \left(x + \frac{x^2}{s^2}\right) \circ s^2 H\left(\frac{x}{s^2}\right) \\
&= \cdots \\
&= \left(\mathop{\mathcal{R}}_{k=1}^{n} \left(x + \frac{x^2}{s^k}\right)\right) \circ s^n H\left(\frac{x}{s^n}\right).
\end{aligned}
$$

Since $H(0) = 0$, $H'(0) = 1$ and $|s| > 1$,

$$\lim_{n\to\infty} s^n H\left(\frac{x}{s^n}\right) = x \lim_{n\to\infty} \frac{H\left(x/s^n\right)}{x/s^n} = xH'(0) = x \quad (\text{if } x \neq 0).$$

If $x = 0$, then the equality is trivial. Therefore

$$\lim_{n\to\infty} s^n H\left(\frac{x}{s^n}\right) = x$$

for any x. Moreover the sequence $\left\{\mathcal{R}_{k=1}^{n} \left(x + \frac{x^2}{s^k}\right)\right\}$ is equicontinuous on every compact subset of $\mathbb{C}$. (For a proof of the equicontinuity of the sequence, see [6].) Hence

$$
\begin{aligned}
H(x) &= \lim_{n\to\infty} \left(\left(\mathop{\mathcal{R}}_{k=1}^{n} \left(x + \frac{x^2}{s^k}\right)\right) \circ s^n H\left(\frac{x}{s^n}\right)\right) \\
&= \mathop{\mathcal{R}}_{k=1}^{\infty} \left(x + \frac{x^2}{s^k}\right) \circ \lim_{n\to\infty} s^n H\left(\frac{x}{s^n}\right) \\
&= \mathop{\mathcal{R}}_{k=1}^{\infty} \left(x + \frac{x^2}{s^k}\right) \circ x = \mathop{\mathcal{R}}_{k=1}^{\infty} \left(x + \frac{x^2}{s^k}\right) = F(x).
\end{aligned}
$$

Therefore

$$H(x) = F(x)$$

for any $x \in \mathbb{C}$.

$\square$

REMARK 2.2. For a given function $h(x)$, the functional equation of the form

$$P(sx) = h(P(x)) \tag{10}$$

has been studied by several mathematicians. Suppose that $|s| \neq 0, 1$. Koenigs [3] proved that if $h(x)$ is analytic at the origin and $h(0) = 0$, $h'(0) = s$, then the functional equation (10) has a unique solution $P(x)$ which is analytic at $x = 0$ and $P(0) = 0$, $P'(0) = 1$. This kind of function is called a Poincaré function. For example, $F(x)$ defined by (8) is a Poincaré function since it satisfies the functional equation (10) with $h(x) = s(x + x^2)$. For more details, see [2] or [9].

3. $F(x)$ as a real-valued function

From now on, s stands for a real number such that $s > 2$. Thus the function $F(x)$ defined in the previous section is a real valued function on $\mathbb{R}$.

THEOREM 3.1. *If $s > 2$, then the following statements hold.*

(i) $F(\mathbb{R}) = [-\frac{s}{4}, \infty)$.

(ii) *Let $\omega \in \mathbb{R}$ be the maximal value such that $F(\omega) = -s/4$. Then $F'(x) > 0$ for any $x > \omega$ and $F'(\omega) = 0$.*

Before giving the proof of Theorem 3.1, we prove two lemmas.

LEMMA 3.2. *$F'(x) \geq 1$ for any $x \in [0, \infty)$.*

PROOF. It is easy to see that the Taylor expansion of $F_n(x)$ at $x = 0$ is of the form

$$F_n(x) = x + \sum_{r=2}^{\infty} c_{n,r} x^r ,$$

where the coefficients $c_{n,r}$ are non-negative real numbers. Therefore $F_n'(x) \geq 1$ for any $x \geq 0$. $\square$

In order to state the next lemma, we need some notation. For each positive integer n, consider a real valued function

$$\varphi_n(x) = \frac{-1 + \sqrt{1 + 4s^{-n}x}}{2s^{-n}}$$

defined on the interval $[-s^n/4, \infty)$. Note that

$$\varphi_n\Big(\Big[-\frac{s^n}{4}, \infty\Big)\Big) = \Big[-\frac{s^n}{2}, \infty\Big) \subset \Big[-\frac{s^{n+1}}{4}, \infty\Big)$$

for any $n \geq 1$. Thus we can define composite functions

$$G_n(x) := \varphi_n(x) \circ \varphi_{n-1}(x) \circ \cdots \circ \varphi_1(x)$$

on the interval $[-s/4, \infty)$. For convenience, we put

$$G_0(x) = x .$$

Note that $\varphi_n(x) > 0$ for any $x > 0$, $\varphi_n(0) = 0$, and $\varphi_n(x) < 0$ for any $x < 0$, hence $G_n(x) > 0$ for any $x > 0$, $G_n(0) = 0$, and $G_n(x) < 0$ for any $x \in [-s/4, 0)$. Moreover, since

$$\left(x + \frac{x^2}{s^n}\right) \circ \varphi_n(x) = x ,$$

we have

$$F_n(x) \circ G_n(x) = x \tag{11}$$

for any $x \in [-s/4, \infty)$.

LEMMA 3.3. *Let the notation be as above and suppose $s > 2$. Then*:

(i) *The sequence $G_n(x)$ converges uniformly on every compact subset of* $[-s/4, \infty)$, *and define a function*

$$G(x) = \lim_{n \to \infty} G_n(x)$$

on $[-s/4, \infty)$ which is real analytic on $(-s/4, \infty)$.

(ii) *For any $x \in [-s/4, \infty)$, it holds that $F(x) \circ G(x) = x$.*

(iii) *The function $G(x)$ is strictly increasing on $[-s/4, \infty)$.*

(iv) *If we set $\omega_0 = G(-s/4)$, then the function $F(x)$ is strictly increasing on $[\omega_0, \infty)$.*

PROOF. (i) It follows from the definition of $G_n(x)$ that

$$\left(x + \frac{x^2}{s^n}\right) \circ G_n(x) = G_{n-1}(x) \quad (n \geq 1)\,,$$

that is,

$$G_n(x) \left(1 + \frac{G_n(x)}{s^n}\right) = G_{n-1}(x) \quad (n \geq 1)\,.$$

Therefore

$$G_n(x) = \frac{x}{\prod_{r=1}^{n} \left(1 + s^{-r} G_r(x)\right)} \tag{12}$$

for any $n \geq 1$. Here note that from the definition of $G_r(x)$ we have $G_r(x) \geq -\frac{s^r}{2}$, so $1 + s^{-r} G_r(x) \geq \frac{1}{2}$. Hence the denominators of the right hand side of (12) never vanish for any $r \in \mathbb{N}$.

Now, by the definition of $G_r(x)$, we have

$$\begin{aligned} G_r(x) &= \frac{-1 + \sqrt{1 + 4s^{-r}x}}{2s^{-r}} \circ G_{r-1}(x) \\ &= \frac{2x}{1 + \sqrt{1 + 4s^{-r}x}} \circ G_{r-1}(x) \\ &= \frac{2G_{r-1}(x)}{1 + \sqrt{1 + 4s^{-r}G_{r-1}(x)}}\,. \end{aligned}$$

Since $G_0(x) = x$, it follows that

$$G_n(x) \leq |x| \prod_{r=1}^{n} \frac{2}{1 + \sqrt{1 + 4s^{-r} G_{r-1}(x)}} \leq 2^n |x| \,.$$

Therefore

$$s^{-n} |G_n(x)| \leq \left(\frac{2}{s}\right)^n |x| \,.$$

This implies that if $s > 2$, then the infinite series

$$\sum_{n=1}^{\infty} s^{-n} |G_n(x)|$$

is convergent, hence the infinite product

$$\prod_{n=1}^{\infty} (1 + s^{-n} G_n(x))$$

is also convergent. Therefore, the limit $\lim_{n\to\infty} G_n(x)$ exists by (12), which proves (i).

(ii) The second assertion follows from the relation (11) and the equicontinuity of the sequence $\{F_n(x)\}$ on every compact subset of $[-s/4, \infty)$.

(iii) First we prove that the inequality

$$G_n(x) - G_n(y) \geq x - y \tag{13}$$

holds for any $x, y \in [-s/4, 0]$ with $x > y$ by induction on n.

In the case of $n = 0$, (13) is trivial. Suppose $n > 0$ and the inequality

$$G_{n-1}(x) - G_{n-1}(y) \geq x - y \tag{14}$$

holds for any $x, y \in [-s/4, 0]$ with $x > y$. Since $G_{n-1}(x) \leq 0$ for any $x \in [-s/4, 0]$, we have

$$\sqrt{1 + 4s^{-n} G_{n-1}(x)} \leq 1 \,.$$

Therefore

$$G_n(x) - G_n(y) = \frac{2\,(G_{n-1}(x) - G_{n-1}(y))}{\sqrt{1 + 4s^{-n} G_{n-1}(x)} + \sqrt{1 + 4s^{-n} G_{n-1}(y)}} \geq x - y \,.$$

Thus (13) holds for any $n \geq 0$.

Now, taking the limit $n \to \infty$ of (13) yields the inequality

$$G(x) - G(y) \geq x - y \,.$$

In particular, $G(x)$ is strictly increasing on $[-s/4, 0]$.

It remains to show that $G(x)$ is strictly increasing on $(0, \infty)$. Since $F(G(x)) = x$ and $F'(x) \neq 0$ on $(0, \infty)$ by Lemma 3.2, it follows from the implicit function theorem that $G(x)$ is differentiable and the formula

$$F'(G(x))G'(x) = 1 \tag{15}$$

holds on $(0, \infty)$. Since $F'(x) > 0$ for any $x > 0$ by Lemma 3.2 again and $G(x) > 0$ for any $x > 0$, the formula (15) shows that $G'(x) > 0$, hence $G(x)$ is strictly increasing on $(0, \infty)$.

(iv) This is an immediate consequence of (ii) and (iii). □

We can now prove Theorem 3.1.

PROOF OF THEOREM 3.1. (i) Since $y^2 + y \geq -1/4$ for any $y \in \mathbb{R}$, the functional equation (9) shows that $F(x) \geq -s/4$ for any $x \in \mathbb{R}$. If $x \geq 0$, then $x + x^2/s^n \geq 0$ for any $n \geq 1$, hence $F(x) \geq 0$ for any $x \geq 0$. Moreover, if we set $\omega_0 = G(-s/4) < 0$, then

$$F(\omega_0) = F\left(G\left(-\frac{s}{4}\right)\right) = -\frac{s}{4}$$

by Lemma 3.3 (ii). Hence $F(x)$ actually attains the minimal value $-s/4$ at $x = \omega_0$. Therefore $F(\mathbb{R}) = [-s/4, \infty)$.

(ii) Let ω_0 be as in (i) and $\omega \in \mathbb{R}$ the maximal value such that $F(\omega) = -s/4$. As we have seen in (i), $F(x) \geq 0$ if $x \geq 0$, so ω is negative. Since $F(\omega_0) = -s/4$, this shows that ω_0 is the maximal real number attaining the minimal value of $F(x)$, hence $\omega = \omega_0$. It follows that $F'(\omega) = 0$ since $F(x)$ attains the minimal value.

It remains to show that $F'(x) > 0$ for any $x > \omega$. To see this, let ω_1 be the maximal real zero of $F'(x)$. If $\omega_1 > \omega$, then $\omega_1/s > \omega/s$, so $F(\omega_1/s) + 1/2 > 0$ since $F(x)$ is strictly increasing on $[\omega, \infty)$. But

$$F'\left(\frac{\omega_1}{s}\right)\left(1 + 2F\left(\frac{\omega_1}{s}\right)\right) = F'(\omega_1) = 0\,,$$

hence $F'(\omega_1/s) = 0$, which contradicts the maximality of ω_1. Therefore ω must be the maximal real zero of $F'(x)$. In other words, $F'(x) > 0$ for any $x > \omega$. This completes the proof.

4. The zeros of $f(x)$ and $f'(x)$

Throughout this section we assume that $s \geq 4$. Let

$$c = \frac{s^2}{4} - \frac{s}{2}\,.$$

Obviously, we have $c \geq 2$, and $c = 2$ if and only if $s = 4$. Let $F(x, s)$ be the function defined by (8) and put

$$f(x, s) = s\left(F(x, s) + \frac{1}{2}\right). \tag{16}$$

Then the following proposition shows that $f(x) := f(x, s)$ is the desired function mentioned in the introduction.

PROPOSITION 4.1. *The function $f(x)$ and its derivative $f'(x)$ satisfy the following functional equations*:

$$f(sx) = f(x)^2 - c\,, \tag{17}$$

$$sf'(sx) = 2f(x)f'(x)\,. \tag{18}$$

PROOF. It follows from (9) that

$$\begin{aligned} f(sx) &= s\left(F(sx) + \frac{1}{2}\right) \\ &= s^2(F(x)^2 + F(x)) + \frac{s}{2} \\ &= \left\{s\left((F(x) + \frac{1}{2}\right)\right\}^2 - \frac{s^2}{4} + \frac{s}{2} \\ &= f(x)^2 - c\,. \end{aligned}$$

Thus (17) holds. Differentiating the functional equation (17) yields (18). □

PROPOSITION 4.2. *Let ω be as in Theorem* 3.1. *Then $f(x) \geq -c$ for any $x \in \mathbb{R}$ and $f(\omega) = -c$. Moreover, $f'(x) > 0$ for any $x > \omega$.*

PROOF. Since $F(x) \geq -s/4$ for any $x \in \mathbb{R}$, we have $f(x) \geq -c$. Moreover, since $F(x)$ attains the minimal value $-s/4$ at $x = \omega$, $f(x)$ also attains the minimal value at $x = \omega$ and

$$f(\omega) = s\left(F(\omega) + \frac{1}{2}\right) = s\left(-\frac{s}{4} + \frac{1}{2}\right) = -\frac{s^2}{4} + \frac{s}{2} = -c\,.$$

The last statement follows from Theorem 3.1 (ii). □

As we will see later, the zeros of $f(x)$ and $f'(x)$ will play an important role in this paper. First note that $f(x)$ has at least one negative real zero. Indeed, since $f(0) = s(F(0)+1/2) = s/2 > 0$ and $f(\omega) = -c < 0$, it follows that $f(x)$ has a real zero in the interval $(\omega, 0)$.

PROPOSITION 4.3. *If ρ is a zero of $f(x)$, then the following statements hold.*

(i) *$f(s\rho) = -c$. In particular, $f(s\rho) < 0$.*

(ii) *$f(s^i\rho) \geq c$ for any $i \geq 2$, and the equality $f(s^i\rho) = c$ holds if and only if $c = 2$. In particular, $f(s^i\rho) > 0$ for any $i \geq 2$.*

(iii) *$f'(s^i\rho) = 0$ for any $i \geq 1$.*

PROOF. (i) The functional equation (17) shows that

$$f(s\rho) = f(\rho)^2 - c = -c\,,$$

which proves (i).

(ii) Suppose $|f(s^i\rho)| \geq c$ for some $i \geq 1$. Then

$$f(s^{i+1}\rho) = f(s^i\rho)^2 - c \geq c^2 - c = c(c-1)\,.$$

Since $c \geq 2$, we have $c(c-1) \geq c$, hence $f(s^{i+1}\rho) \geq c$. Clearly the equality holds if and only if $c = 2$. Since $|f(s\rho)| = c$, this implies that $f(s^i\rho) \geq c$ for any $i \geq 2$.

(iii) From the functional equation (18), we have

$$s^i f'(s^i x) = 2^i f(s^{i-1}x)\cdots f(sx)f(x)f'(x)$$

for any $i \geq 1$. Therefore, $f'(s^i\rho) = 0$, which proves (iii). □

COROLLARY 4.4. *The function $f(x)$ has infinitely many negative real zeros, and the same holds for $f'(x)$.*

PROOF. If ρ is a negative real zero of $f(x)$, then $f(s\rho) < 0$ and $f(s^2\rho) > 0$ by Proposition 4.3. Hence there exists at least one zero ρ' of $f(x)$ such that $s^2\rho < \rho' < s\rho$. In particular, $\rho' < \rho$. Therefore $f(x)$ has infinitely many real negative zeros. The second statement of the corollary is then clear from this, or directly follows from Proposition 4.3 (iii). □

PROPOSITION 4.5. *Suppose $s \geq 4$. Then*:

(i) *Every zero of $f(x)$ is a negative real number.*

(ii) *Every zero of $f'(x)$ is of the form $s^i\rho$, where ρ is a zero of $f(x)$ and i is a positive integer.*

(iii) *$f(x)$ and $f'(x)$ have no common zero.*

(iv) *Every zero of $f(x)f'(x)$ is simple.*

PROOF. (i) It is proved in [1, Theorem 1.1, (ii)] that if $s \geq 4$ then $F^{-1}([-s/4, 0]) \subset (-\infty, 0]$. Since $s \geq 4$, we have $-1/2 \in [-s/4, 0]$, and so $F^{-1}(-1/2) \subset (-\infty, 0]$. Since $f^{-1}(0) = F^{-1}(-1/2)$ and $f(0) = s/2 \neq 0$, it follows that $f^{-1}(0) \subset (-\infty, 0)$, which proves (i).

(ii) Let X denote the set of zeros of $f(x)$ and Y the set of zeros of $f'(x)$. Then (18) shows that $Y = sX \cup sY$. Since $0 \notin Y$ and Y has no accumulation points, this implies that

$$Y = \bigcup_{i=1}^{\infty} s^i X\,,$$

which proves (ii).

(iii) Proposition 4.3 shows that $X \cap s^i X = \emptyset$ for any $i \geq 1$, hence $X \cap Y = \emptyset$ by (ii). This proves (iii).

(iv) It follows from (iii) that every zero of $f(x)$ is simple. Since $f(x)$ and $f'(x)$ have no common zeros, we have only to show that $f'(x)$ has no zero of order ≥ 2.

Suppose $f'(x)$ and $f''(x)$ have a common zero, and let α be the maximum of such zeros. By (18), we have

$$0 = sf'(\alpha) = 2f\left(\frac{\alpha}{s}\right)f'\left(\frac{\alpha}{s}\right),$$

and exactly one of $f(\alpha/s)$ and $f'(\alpha/s)$ is zero by (iii). Differentiating (18), we get

$$s^2 f''(sx) = 2\{f'(x)^2 + f(x)f''(x)\}.$$

It follows that

$$0 = s^2 f''(\alpha) = 2\left\{f'\left(\frac{\alpha}{s}\right)^2 + f\left(\frac{\alpha}{s}\right)f''\left(\frac{\alpha}{s}\right)\right\}. \tag{19}$$

If $f(\alpha/s) = 0$, then (19) implies that $f'(\alpha/s) = 0$, which is impossible since $f(x)$ and $f'(x)$ have no common zero. Hence $f(\alpha/s) \neq 0$ and $f'(\alpha/s) = 0$. It then follows from (19) again that $f''(\alpha/s) = 0$, which contradicts the choice of α since $\alpha < \alpha/s$. Therefore $f'(x)$ and $f''(x)$ have no common zero, and so $f'(x)$ has only simple zeros. This proves (iv). □

Recall that both $f(x)$ and $f'(x)$ have infinitely many zeros by Corollary 4.4, all of which are negative real numbers by Proposition 4.5. Numbering the zeros of $f(x)$ and $f(x)f'(x)$ in descending order, respectively, we write

$$(0 >)\ \rho(1) > \rho(2) > \rho(3) > \cdots,$$

and

$$(0 >)\ \tau(1) > \tau(2) > \tau(3) > \cdots.$$

For convenience, we set $\tau(0) = \tau(-1) = \infty$.

Recall that we have defined ω to be the maximal zero of $F'(x)$. Then ω is also the maximal zero of $f'(x)$.

PROPOSITION 4.6. *Notation being as above, we have* $\tau(1) = \rho(1)$ *and* $\tau(2) = \omega = s\rho(1)$.

PROOF. Note that $f'(x) > 0$ for any $x > \omega$ and $f(\omega) = -c < 0$ by Proposition 4.2. This implies that $\tau(1) = \rho(1)$ and $\rho(1)$ is the unique zero of $f(x)$ in the interval $(\omega, 0)$. Moreover, since $f'(\omega) = 0$, it follows that $\tau(2) = \omega$.

To see that $\omega = s\rho(1)$, note that $\omega < \omega/s < 0$, hence $f'\left(\frac{\omega}{s}\right) \neq 0$. But

$$2f\left(\frac{\omega}{s}\right)f'\left(\frac{\omega}{s}\right) = sf'(\omega) = 0\,,$$

hence $f(\omega/s) = 0$. This implies that $\omega/s = \rho(1)$, so $\omega = s\rho(1)$. □

Proposition 4.6 can be generalized as follows.

THEOREM 4.7. *Let n be a positive integer. Then*

$$\tau(2n-1) = \rho(n)\,, \qquad \tau(2n) = s\tau(n)\,. \tag{20}$$

In particular, $f(\tau(n)) = 0$ if n is odd and $f'(\tau(n)) = 0$ if n is even.

Although this theorem is proved in [4], we give a slightly simplified proof here for the sake of the reader. In the proof of the theorem we need the following notation: For a real number x, define

$$\mathrm{sgn}(x) = \begin{cases} 1 & (\text{if } x > 0)\,, \\ 0 & (\text{if } x = 0)\,, \\ -1 & (\text{if } x < 0)\,. \end{cases}$$

PROOF OF THEOREM 4.7. First, note that the statement for $n = 1$ is nothing but Proposition 4.6. For each $n \geq 1$, consider the open interval $I_n = (s\tau(n), s\tau(n-1))$. In particular, $\rho(1) \in I_1 = (s\tau(1), \infty)$.

Let $k \geq 1$ be an integer and assume that (20) holds for any n with $1 \leq n \leq k$. In order to show that the assertion of the theorem for $n = k+1$ is true, we first prove that $f(x)$ has a unique zero in I_n for any integer n with $1 \leq n \leq 2k$. For such an integer n, take arbitrary $x \in I_n$. Then $\tau(n) < x/s < \tau(n-1)$, hence neither $f(x/s)$ nor $f'(x/s)$ vanishes. Since

$$\mathrm{sgn}(f'(x)) = \mathrm{sgn}\left(f\left(\frac{x}{s}\right)f'\left(\frac{x}{s}\right)\right)$$

by (18), $\mathrm{sgn}(f'(x))$ is constant on I_n. Hence $f(x)$ is either monotonously increasing or monotonously decreasing on the interval I_n. Moreover, if n is even, say $n = 2l$ with $1 \leq l \leq k$, then

$$s\tau(n) = s\tau(2l) = s^2\tau(l)$$

by the inductive hypothesis. Hence $f(s\tau(n)) > 0$ by Proposition 4.3 (ii). If n is odd, say $n = 2l-1$ with $1 \leq l \leq k$, then

$$s\tau(n) = s\tau(2l-1) = s\rho(l)$$

by the inductive hypothesis. Then $f(s\tau(n)) < 0$ by Proposition 4.3 (i). Therefore, $f(x)$ has a unique zero in I_n for any integer n with $1 \leq n \leq 2k$. In particular, there exists a unique zero in $I_{k+1} = (s\tau(k+1), s\tau(k)) = (s\tau(k+1), \tau(2k))$, namely, there exists a unique positive integer u such that

$$s\tau(k+1) < \rho(u) < \tau(2k)\ (< \tau(2k-1) = \rho(k))\,. \tag{21}$$

On the other hand, in the notation of the proof of Proposition 4.5 (ii), we have

$$Y = s(X \cup Y) = \{s\tau(n) \mid n = 1, 2, 3, \ldots\}\,. \tag{22}$$

From (21), (22), we find that $\rho(u) = \tau(2k+1)$ and $s\tau(k+1) = \tau(2k+2)$. In particular, $\rho(u)$ is the maximal zero of $f(x)$ less than $\rho(k)$, so $u = k+1$. Therefore $\tau(2k+1) = \rho(k+1)$ and $\tau(2k+2) = s\tau(k+1)$. This proves that the theorem holds for $n = k+1$. Thus the theorem holds for any positive integer n. □

For each positive integer n, define a nonnegative integer $v(n)$ and a positive integer $n^{\#}$ by the rule

$$n = 2^{v(n)}(2n^{\#} - 1)\,. \tag{23}$$

Obviously, both $v(n)$ and $n^{\#}$ are uniquely determined by n. The following corollary is an immediate consequence of Theorem 4.7.

COROLLARY 4.8. *Notation being as above, we have*

$$\tau(n) = s^{v(n)}\rho(n^{\#})\,.$$

THEOREM 4.9. *Let n be a non-negative integer. Then*

$$f(x) > 0 \quad \text{if } \tau(4n+1) < x < \tau(4n-1)\,,$$

$$f(x) < 0 \quad \text{if } \tau(4n+3) < x < \tau(4n+1)\,,$$

and

$$f'(x) > 0 \quad \text{if } \tau(4n+2) < x < \tau(4n)\,,$$

$$f'(x) < 0 \quad \text{if } \tau(4n+4) < x < \tau(4n+2)\,.$$

PROOF. Theorem 4.7 shows that the set of zeros of $f(x)$ is $\{\tau(2m-1) \mid m \in \mathbb{N}\}$, and so $\mathrm{sgn}(f(x))$ is constant on the open interval $(\tau(2m+1), \tau(2m-1))$ for any $m \in \mathbb{N}$. Therefore $\mathrm{sgn}(f(x)) = \mathrm{sgn}(f(\tau(2m)))$ for any $x \in (\tau(2m+1), \tau(2m-1))$. Moreover, combining Theorem 4.7 with Proposition 4.3, we see that

$$\mathrm{sgn}(f(\tau(2m))) = (-1)^m\,.$$

This proves the first statement of the theorem.

For the second statement, recall that we have seen in the proof of Theorem 4.7 that $\mathrm{sgn}(f'(x))$ is constant on the interval $(\tau(2m+2), \tau(2m))$ for any $m \in \mathbb{N}$. Since

$$\mathrm{sgn}(f'(\tau(2m+1))) = \mathrm{sgn}(f'(\rho(m+1))) = (-1)^m$$

for any $m \geq 0$, we have $\mathrm{sgn}(f'(x)) > 0$ if and only if $x \in (\tau(4n+2), \tau(4n))$ for some $n \geq 0$. This completes the proof. □

Now, for an integer n, define $a_0(n),\ a_1(n) \in \{0, 1\}$ by the rule

$$n \equiv a_0(n) + 2a_1(n) \pmod 4 .$$

COROLLARY 4.10. *If $x \in (\tau(n+1), \tau(n))$, then*

$$\mathrm{sgn}(f'(x)) = (-1)^{a_1(n)} , \tag{24}$$

$$\mathrm{sgn}(f(x)) = (-1)^{a_0(n)+a_1(n)} . \tag{25}$$

PROOF. Note that the following equivalence holds:

$$a_1(n) \equiv 0 \pmod 2 \iff n \equiv 0, 1 \pmod 4 ,$$

$$a_0(n) + a_1(n) \equiv 0 \pmod 2 \iff n \equiv 0, 3 \pmod 4 .$$

Therefore, the corollary immediately follows from Theorem 4.9. □

COROLLARY 4.11. *The function $f(x)$ takes extreme values at $x = \tau(2n)$ for any $n \in \mathbb{N}$. If n is odd, then*

$$f(\tau(2n)) = -c ,$$

which is independent of n. On the other hand, if n is even, then

$$f(\tau(2n)) = \left(\mathop{\mathcal{R}}_{j=1}^{v(2n)} \left(x^2 - c\right) \right) \circ 0 ,$$

all of which are positive.

REMARK 4.12. Corollary 4.11 shows that $f(x)$ takes local maximums at $x = \tau(2n)$ for even integers $n > 0$ and they depend only on $v(2n)$. For positive integers ν, let $M_\nu = f(\tau(2^{\nu+1}))$. If $s = 4$, then $M_\nu = 2$ for any $\nu \geq 1$. On the contrary, if $s > 4$, then M_ν becomes arbitrarily large as $\nu \to \infty$ (see Figure 1.). For example, one can easily see that

$$M_\nu \geq c(c-1)^{2^\nu - 1} . \tag{26}$$

Indeed, this holds for $\nu = 1$ since $M_1 = c(c-1)$. If $M_\nu \geq c(c-1)^{2^\nu - 1}$, then

$$\begin{aligned} M_{\nu+1} &= (x^2 - c) \circ M_\nu \\ &= M_\nu^2 - c \\ &\geq c^2(c-1)^{2(2^\nu - 1)} - c \\ &= c\{c(c-1)^{2^{\nu+1}-2} - 1\} \\ &= c\{(c-1)(c-1)^{2^{\nu+1}-2} + (c-1)^{2^{\nu+1}-2} - 1\} \end{aligned}$$

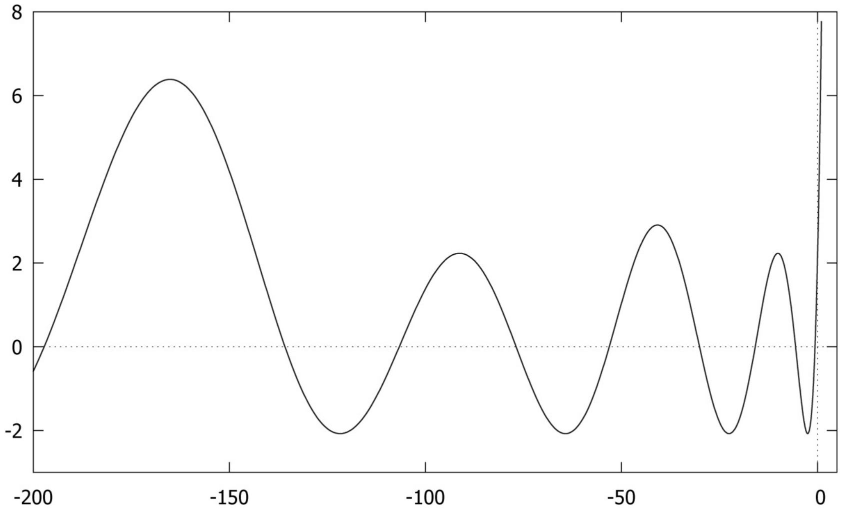

FIGURE 1. The graph of $f(x)$ for $s = 4.05$

$$> c(c-1)^{2^{\nu+1}-1}.$$

The last inequality holds since $c - 1 > 1$. This proves that the inequality (26) holds for any $\nu \geq 1$.

5. The zeros of $F(x)$

In this section we assume that $s > 4$. From the definition of $f(x)$ we deduce that

$$F(x) = \frac{f(x)}{s} - \frac{1}{2}.$$

Since $f'(\tau(2n)) = 0$ for any integer $n > 0$ by Theorem 4.7, we have

$$F'(\tau(2n)) = 0.$$

To study the distribution of the zeros of $F(x)$ we start with the following lemma.

LEMMA 5.1. *For any integer $n \geq 0$, there exists a unique zero of $F(x)$ in every open interval $(\tau(2n+2), \tau(2n))$. Here, we set $\tau(0) = \infty$ for convenience.*

PROOF. First suppose n is odd, say $n = 2k - 1$. Then Proposition 4.3 shows that

$$F(\tau(2n)) = \frac{f(\tau(2n))}{s} - \frac{1}{2} = \frac{-c}{s} - \frac{1}{2} = -\frac{s}{4} < 0.$$

On the other hand, we have $2n+2 = 4k$, so $\tau(2n+2) = s^2\tau(k)$. Hence Proposition 4.3 again shows that

$$F(\tau(2n+2)) = \frac{f(\tau(2n+2))}{s} - \frac{1}{2} > \frac{c}{s} - \frac{1}{2} = \frac{s}{4} - 1 > 0\,.$$

Moreover, since $F'(x) < 0$ for any $x \in (\tau(2n+2), \tau(2n))$ by Theorem 4.9, this shows that there is a unique zero of $F(x)$ in the interval $(\tau(2n+2), \tau(2n))$. The proof of the case n even is quite similar. □

For each $n \geq 0$, we denote by $\mu(n)$ the unique zero of $F(x)$ in the interval $(\tau(2n+2), \tau(2n))$. Thus,

$$0 = \mu(0) > \mu(1) > \mu(2) > \cdots\,.$$

PROPOSITION 5.2. *For any $n \in \mathbb{N}$, we have*

$$\tau(4n+1) < \mu(2n) < \tau(4n) < \mu(2n-1) < \tau(4n-1)\,.$$

PROOF. Theorem 4.9 shows that $F'(x) < 0$ for any $x \in (\tau(4n), \tau(4n-1))$. Since

$$F(\tau(4n-1)) = F(\rho(2n)) = -\frac{1}{2} < 0\,, \qquad F(\tau(4n)) > 0$$

we find that

$$\tau(4n) < \mu(2n-1) < \tau(4n-1)\,,$$

which proves the half of the proposition. The proof of the remaining part of the proposition is quite similar. □

REMARK 5.3. If $s = 4$, then $c = 2$ and

$$f(x) = f(x,4) = 2\cos\sqrt{-4x} = \begin{cases} 2\cos(2\sqrt{-x}) & (x \leq 0)\,, \\ 2\cosh(2\sqrt{x}) & (x > 0)\,, \end{cases}$$

$$F(x) = F(x,4) = \frac{\cos\sqrt{-4x}}{2} - \frac{1}{2} = \begin{cases} -\sin^2(\sqrt{-x}) & (x \leq 0)\,, \\ \sinh^2(\sqrt{x}) & (x > 0)\,. \end{cases}$$

It follows that $\tau(n) = -\pi^2 n^2/4^2$ for any positive integer n, so $F(\tau(4n)) = F'(\tau(4n)) = 0$. Therefore, the case $s = 4$ can be regarded as a degenerate case where "$\mu(2n) = \mu(2n-1)$". This is the reason why we have excluded the case $s = 4$.

Now, the functional equation of $F(x)$ shows that $s\mu(n)$ is a zero of $F(x)$ for any $n \geq 0$. Thus, given $n \geq 0$, we have $s\mu(n) = \mu(n')$ for some $n' \geq 0$. The following theorem gives an explicit relationship between n and n'. To state it, for any integer $n > 0$, we define an odd integer n^* by

$$n = 2^{v(n)} n^*\,, \tag{27}$$

where $v(n) \geq 0$ is the integer defined in (23). Thus, $n^* = 2n^\# - 1$ in the notation of (23).

THEOREM 5.4. *Let $s > 4$. Then the following hold for any $n \geq 1$.*

$$\mu(2n) = s^{v(n)}\mu(2n^*) \tag{28}$$

$$\mu(2n-1) = s^{v(n)}\mu(2n^*-1) \tag{29}$$

PROOF. It suffices to prove that

$$s\mu(2n) = \mu(4n), \qquad s\mu(2n-1) = \mu(4n-1) \tag{30}$$

for any $n \geq 1$. To prove the first equation of (30), note that the inequalities

$$\tau(4n+1) < \mu(2n) < \tau(4n)$$

hold by Proposition 5.2. Hence

$$s\tau(4n+1) < s\mu(2n) < s\tau(4n).$$

Since $s\tau(4n+1) = \tau(8n+2)$ and $s\tau(4n) = \tau(8n)$, it follows that

$$\tau(8n+2) < s\mu(2n) < s\tau(8n). \tag{31}$$

Since $\mu(4n)$ is the unique zero of $F(x)$ in the interval $(\tau(8n+2), \tau(8n))$, it follows from (31) that $s\mu(2n) = \mu(4n)$.

On the other hand Proposition 5.2 shows that

$$\tau(4n) < \mu(2n-1) < \tau(4n-1),$$

ands so

$$s\tau(4n) < s\mu(2n-1) < s\tau(4n-1).$$

Since $s\tau(4n) = \tau(8n)$ and $s\tau(4n-1) = \tau(8n-2)$, it follows that

$$\tau(8n) < s\mu(2n-1) < s\tau(8n-2). \tag{32}$$

Note that $\mu(4n-1)$ is the unique zero of $F(x)$ in the interval $(\tau(8n), \tau(8n-2))$ by Proposition 5.2. Therefore we see that $s\mu(2n-1) = \mu(4n-1)$ by (32). $\square$

6. Finite nested square roots

From now on, we assume that $s \geq 4$. Let m be a positive integer. Given a finite sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) \in \{\pm 1\}^m$, consider a real valued function

$$R_c(\varepsilon_1, \ldots, \varepsilon_m; x) = \varepsilon_1\sqrt{c + \varepsilon_2\sqrt{c + \varepsilon_3\sqrt{c + \cdots + \varepsilon_m\sqrt{c+x}}}}$$

defined for $x \geq -c$. This can be written as

$$R_c(\varepsilon_1, \ldots, \varepsilon_m; x) = \mathop{\mathcal{R}}_{k=1}^{m} \varepsilon_k\sqrt{c+x}.$$

PROPOSITION 6.1. *Let $\alpha \in \mathbb{R}$. If we set $\varepsilon_k = \operatorname{sgn}(f(\alpha/s^k))$ for $k = 1, 2, \dots, m$, then*

$$f\left(\frac{\alpha}{s^m}\right) = R_c(\varepsilon_m, \varepsilon_{m-1}, \dots, \varepsilon_1; f(\alpha)).$$

PROOF. The functional equation $f(sx) = f(x)^2 - c$ shows that

$$f(x) = \operatorname{sgn}(f(x))\sqrt{c + f(sx)}.$$

Hence

$$f\left(\frac{\alpha}{s^m}\right) = \varepsilon_m \sqrt{c + f\left(\frac{\alpha}{s^{m-1}}\right)}.$$

Repeating this process yields the proposition. □

For any integer $m \geq 0$, define $a_k(m) \in \{0, 1\}$ ($k = 0, 1, \dots$) by the 2-adic expansion of m:

$$m = a_0(m) + 2a_1(m) + 2^2 a_2(m) + \cdots.$$

If $\tau(m+1) < x < \tau(m)$, then $\operatorname{sgn}(f(x)) = (-1)^{a_0(m)+a_1(m)}$ by Corollary 4.10. The following theorem determines $\operatorname{sgn}(f(x/s^k))$ for $k \geq 1$.

THEOREM 6.2. *If $\tau(2m+2) < x < \tau(2m)$, then*

$$\operatorname{sgn}\left(f\left(\frac{x}{s^k}\right)\right) = (-1)^{a_{k-1}(m)+a_k(m)}$$

for any integer $k \geq 1$.

PROOF. Put $B_k(m) = 2^k a_k(m) + 2^{k+1} a_{k+1}(m) + \cdots$. Then

$$2^{k-1} a_{k-1}(m) + B_k(m) \leq m < m + 1 \leq 2^{k-1}(1 + a_{k-1}(m)) + B_k(m),$$

hence

$$2^k a_{k-1}(m) + 2B_k(m) \leq 2m < 2m + 2 \leq 2^k(1 + a_{k-1}(m)) + 2B_k(m).$$

Therefore

$$\begin{aligned} \tau(2^k(1 + a_{k-1}(m)) + 2B_k(m)) &\leq \tau(2m+2) \\ &< \tau(2m) \leq \tau(2^k a_{k-1}(m) + 2B_k(m)), \end{aligned}$$

which implies that

$$\tau(1 + a_{k-1}(m) + 2B_k(m)/2^k) < \frac{x}{s^k} < \tau(a_{k-1}(m) + 2B_k(m)/2^k).$$

Thus, if we put $n = a_{k-1}(m) + 2a_k(m) + \cdots$, then $\tau(1+n) < x/s^k < \tau(n)$. Hence

$$\operatorname{sgn}\left(f\left(\frac{x}{s^k}\right)\right) = (-1)^{a_0(n)+a_1(n)}$$

by Corollary 4.10. But $a_0(n) + a_1(n) = a_{k-1}(m) + a_k(m)$, and so the theorem holds. □

REMARK 6.3. For any real number x, let $[x]$ denote the largest integer not greater than x. Then

$$a_{k-1}(m) + a_k(m) \equiv \left[\frac{m}{2^k} + \frac{1}{2}\right] \pmod 2 .$$

COROLLARY 6.4. *Let k be an integer with $k \geq 1$. Then*

$$\operatorname{sgn}\left(f\left(\frac{\rho(m+1)}{s^k}\right)\right) = (-1)^{a_k(m)+a_{k-1}(m)} .$$

PROOF. Since $\rho(m+1) = \tau(2m+1)$, it follows that

$$\tau(2m+2) < \rho(m+1) < \tau(2m) .$$

Hence, applying Theorem 6.2 with $x = \rho(m+1)$, we obtain the corollary. □

THEOREM 6.5. *If $\tau(2m+2) < \alpha < \tau(2m)$, then*

$$f\left(\frac{\alpha}{s^N}\right) = \left(\mathop{\mathcal{R}}_{n=1}^{N} (-1)^{a_{N+1-n}(m)+a_{N-n}(m)}\sqrt{c+x}\right) \circ f(\alpha)$$

for any $N \geq 1$.

PROOF. For simplicity we put $a_k = a_k(m)$. Then Proposition 6.1 gives

$$f\left(\frac{\alpha}{s^N}\right) = \left(\mathop{\mathcal{R}}_{n=1}^{N} \operatorname{sgn}\left(f\left(\frac{\alpha}{s^{N+1-n}}\right)\right)\sqrt{c+x}\right) \circ f(\alpha) . \tag{33}$$

From Theorem 6.2, we deduce that

$$\operatorname{sgn}\left(f\left(\frac{\alpha}{s^{N+1-n}}\right)\right) = (-1)^{a_{N+1-n}+a_{N-n}} . \tag{34}$$

Then the theorem follows from (33) and (34). □

Taking $\alpha = \rho(m+1), \alpha = \mu(m)$ in Theorem 6.5, we obtain the following corollary.

COROLLARY 6.6. *For any integer $N \geq 1$, we have*

$$f\left(\frac{\rho(m+1)}{s^N}\right) = \left(\mathop{\mathcal{R}}_{n=1}^{N} (-1)^{a_{N+1-n}(m)+a_{N-n}(m)}\sqrt{c+x}\right) \circ 0 \quad (s \geq 4) .$$

$$f\left(\frac{\mu(m)}{s^N}\right) = \left(\mathop{\mathcal{R}}_{n=1}^{N} (-1)^{a_{N+1-n}(m)+a_{N-n}(m)}\sqrt{c+x}\right) \circ \frac{s}{2} \quad (s > 4) .$$

Now we can state one of our main theorems.

THEOREM 6.7. *Given $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$, let*

$$a_i = \frac{1 - \varepsilon_1 \cdots \varepsilon_i}{2} \in \{0, 1\},$$

for $i = 1, \dots, k$ and put $A_k = 2^{k-1}a_1 + 2^{k-2}a_2 + \cdots + 2a_{k-1} + a_k$. Then

$$R_c(\varepsilon_1, \dots, \varepsilon_k) = f\left(\frac{\rho(A_k + 1)}{s^k}\right) \quad (s \geq 4). \tag{35}$$

$$R_c\left(\varepsilon_1, \dots, \varepsilon_k; \frac{s}{2}\right) = f\left(\frac{\mu(A_k)}{s^k}\right) \quad (s > 4). \tag{36}$$

PROOF. Applying Corollary 6.6 with $m = A_k$ and $N = k$, we get

$$f\left(\frac{\rho(A_k + 1)}{s^k}\right) = \left(\mathop{\mathcal{R}}_{n=1}^{k} (-1)^{a_n + a_{n-1}} \sqrt{c + x}\right) \circ 0, \tag{37}$$

where we put $a_0 = 0$ for convenience. From the definition of a_n, we have

$$\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n = 1 - 2a_n = (-1)^{a_n}$$

for any $n \geq 1$, which implies that

$$\varepsilon_n = (-1)^{a_n + a_{n-1}}. \tag{38}$$

Therefore, from (37) and (38), we conclude that

$$f\left(\frac{\rho(A_k + 1)}{s^k}\right) = \left(\mathop{\mathcal{R}}_{n=1}^{k} \varepsilon_n \sqrt{c + x}\right) \circ 0 = R_c(\varepsilon_1, \dots, \varepsilon_k),$$

which proves (35). The same argument gives (36). □

REMARK 6.8. As we have seen in Remark 5.3, if $s = 4$, then $f(x) = 2\cos\sqrt{-4x}$ and $\rho(n) = -\pi^2(2n-1)^2/4^2$ for any positive integer n, hence

$$\frac{\rho(A_k + 1)}{s^k} = -\frac{\pi^2(2A_k + 1)^2}{4^{k+2}}.$$

Therefore

$$\begin{aligned} f\left(\frac{\rho(A_k + 1)}{s^k}\right) &= 2\cos\frac{\pi(2A_k + 1)}{2^{k+1}} \\ &= 2\cos\pi\left(\frac{a_1}{2} + \cdots + \frac{a_k}{2^k} + \frac{1}{2^{k+1}}\right). \end{aligned}$$

From this and Theorem 6.7 we obtain

$$2\cos\pi\left(\frac{a_1}{2} + \cdots + \frac{a_k}{2^k} + \frac{1}{2^{k+1}}\right)$$

$$= (-1)^{a_1}\sqrt{2+(-1)^{a_1+a_2}\sqrt{2+(-1)^{a_2+a_3}\sqrt{2+\cdots+(-1)^{a_{k-1}+a_k}\sqrt{2}}}}\,,$$

which is the formula (4) in the introduction.

COROLLARY 6.9. *Let a, a' be positive integers such that $a + a' = 2^m + 1$. Then*

$$f\left(\frac{\rho(a)}{s^m}\right) = -f\left(\frac{\rho(a')}{s^m}\right).$$

PROOF. Let a_i, a'_i $(i = 1, \dots, m)$ be the coefficients of the 2-adic expansion of $a - 1, a' - 1$ respectively, that is,

$$a - 1 = 2^{m-1}a_1 + 2^{m-2}a_2 + \cdots + 2a_{m-1} + a_m\,,$$

$$a' - 1 = 2^{m-1}a'_1 + 2^{m-2}a'_2 + \cdots + 2a'_{m-1} + a'_m\,.$$

Since $(a-1)+(a'-1) = 2^m - 1$, we have $a_i + a'_i = 1$ for any $i = 1, \dots, m$. Let $a_0 = a'_0 = 0$ and $\varepsilon_i = (-1)^{a_i+a_{i-1}}$, $\varepsilon'_i = (-1)^{a'_i+a'_{i-1}}$. Then

$$\varepsilon_i \cdot \varepsilon'_i = (-1)^{(a_i+a'_i)+(a_{i-1}+a'_{i-1})} = \begin{cases} 1 & (i = 2, \dots, m)\,, \\ -1 & (i = 1)\,. \end{cases}$$

Hence $R_c(\varepsilon_1, \dots, \varepsilon_m) = -R_c(\varepsilon'_1, \dots .\varepsilon'_m)$. The corollary is then an immediate consequence of Theorem 6.7. □

Solving (35) and (36) for $\rho(A_k + 1)$ and $\mu(A_k)$ respectively, we obtain the following theorem.

THEOREM 6.10. *Given a positive integer m, let*

$$m = 2^{k-1}a_1 + 2^{k-2}a_2 + \cdots + 2a_{k-1} + a_k \quad (k \geq 1,\ a_i \in \{0, 1\})$$

be the 2-adic expansion of m and define $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$ by

$$\varepsilon_i = (-1)^{a_i+a_{i-1}} \quad (i = 1, \dots, k)$$

with $a_0 = 0$. Then

$$\rho(m + 1) = s^k G\left(\frac{x}{s} - \frac{1}{2}\right) \circ R_c(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)\,. \tag{39}$$

Moreover, if $s > 4$, then

$$\mu(m) = s^k G\left(\frac{x}{s} - \frac{1}{2}\right) \circ R_c\left(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k; \frac{s}{2}\right). \tag{40}$$

PROOF. Observe that

$$\left(\frac{x}{s}-\frac{1}{2}\right)\circ sx\circ\left(x+\frac{1}{2}\right)=x$$

and

$$sx\circ\left(x+\frac{1}{2}\right)\circ F(x)=f(x)\,.$$

Since $G(x)\circ F(x)=x$ for any $x\geq s\rho(1)$, combining these formulas, we conclude that

$$G\left(\frac{x}{s}-\frac{1}{2}\right)\circ f(x)=x$$

for any $x\geq s\rho(1)$. Then (39) follows from (35) if we prove that

$$s\rho(1)\leq\frac{\rho(A_k+1)}{s^k}\,. \tag{41}$$

The inequality (41) can be proved as follows:

$$s^{k+1}\rho(1)=\tau(2^{k+1})<\tau(2^{k+1}-1)=\rho(2^k)\leq\rho(A_k+1)\,.$$

Consequently we get (41). This proves (39). The proof of (40) is quite similar. □

Using formulas (39) and (40), we can compute $\rho(m+1)$ and $\mu(m)$ for any $m\in\mathbb{N}$ if we know the value of $G(t)$ for $-1\leq t\leq 0$:

EXAMPLE 6.11. The 20-th zero $\rho(20)$ can be computed using the formula (39) as follows. Since $20-1=19=2^4+2+1$, we take $a_1=1, a_2=0, a_3=0, a_4=1, a_5=1$ and $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)=(-1,-1,1,-1,1)$. Thus

$$R_c(-1,-1,1,-1,1)=-\sqrt{c-\sqrt{c+\sqrt{c-\sqrt{c+\sqrt{c}}}}}\,,$$

and if we set $t=R_c(-1,-1,1,-1,1)/s-1/2$, then we get $\rho(20)=s^5G(t)$ from (39).

Now, let us study the behavior of the root $\rho(2^m)$ as $m\to\infty$. As for the roots of $F(x)$, we have $\mu(2^m)=\mu(2)s^{m-1}$ for any positive integer m by (28). Although $\rho(2^m)$ does not have such a simple formula, we can prove the following estimate of $\rho(2^m)$ when m is sufficiently large.

THEOREM 6.12. *Suppose $s>4$. Then*

$$\rho(2^m)=\mu(1)s^{m-1}+\frac{\rho(1)}{F'(\mu(1))}+O(s^{-m})$$

as $m\to\infty$

PROOF. Applying Corollary 6.9 with $a = 2^m$ and $a' = 1$, we have

$$f\left(\frac{\rho(2^m)}{s^m}\right) = -f\left(\frac{\rho(1)}{s^m}\right).$$

Since $F(x) = f(x)/s - 1/2$, it follows that

$$F\left(\frac{\rho(2^m)}{s^m}\right) = -1 - F\left(\frac{\rho(1)}{s^m}\right),$$

and so

$$\frac{\rho(2^m)}{s^m} = G\left(F\left(\frac{\rho(2^m)}{s^m}\right)\right) = G\left(-1 - F\left(\frac{\rho(1)}{s^m}\right)\right). \tag{42}$$

The function $G(x)$ is infinitely many times differentiable at any $x \in (-s/4, \infty)$ since $F(G(x)) = x$ and $F'(G(x)) > 0$ holds for $-s/4 < x$. Hence

$$G(-1-x) = G(-1) - G'(-1)x + O(x^2)$$

as $x \to 0$. From $F'(0) = 1$, we find that $F(x) = x + O(x^2)$, so from (42) we obtain an estimate

$$\frac{\rho(2^m)}{s^m} = G(-1) - G'(-1)\frac{\rho(1)}{s^m} + O(s^{-2m})$$

as $m \to \infty$. Since

$$sF\left(\frac{\mu(1)}{s}\right)\left\{F\left(\frac{\mu(1)}{s}\right) + 1\right\} = F(\mu(1)) = 0,$$

we have $F(\mu(1)/s) = -1$, and so $G(-1) = \mu(1)/s$. Moreover, using the formula $F'(sx) = (1 + 2F(x))F'(x)$, we see that

$$G'(-1) = \frac{1}{F'(G(-1))} = \frac{1}{F'(\mu(1)/s)} = -\frac{1}{F'(\mu(1))}.$$

Therefore

$$\frac{\rho(2^m)}{s^m} = \frac{\mu(1)}{s} + \frac{\rho(1)}{F'(\mu(1))}s^{-m} + O(s^{-2m}),$$

which completes the proof. □

7. Infinite nested square roots

In this section, we prove that if $c \geq 2$ then the infinite nested square roots

$$R(\varepsilon_1, \varepsilon_2, \ldots) := R_c(\varepsilon_1, \varepsilon_2, \ldots)$$

have a definite value for any $(\varepsilon_1, \varepsilon_2, \ldots) \in \{\pm 1\}^{\mathbb{N}}$ and express it as special values of the function $f(x)$.

We begin with a lemma.

LEMMA 7.1. *For any integers $m > 1$, we have*

$$\prod_{k=1}^{m} |R(\varepsilon_k, \dots, \varepsilon_m)| \geq \frac{2c}{s}.$$

PROOF. Since $c^2 - c \geq c$, we have

$$\begin{aligned}
&|R(\varepsilon_1, -\varepsilon_2, \varepsilon_3, \dots, \varepsilon_m) R(\varepsilon_1, \varepsilon_2, \varepsilon_3 \dots, \varepsilon_m)| \\
&\quad = \sqrt{c^2 - R(\varepsilon_2, \dots, \varepsilon_m)^2} \\
&\quad = \sqrt{c^2 - c - R(\varepsilon_3, \dots, \varepsilon_m)} \\
&\quad \geq \sqrt{c - R(\varepsilon_3, \dots, \varepsilon_m)} \\
&\quad = |R(\varepsilon_2, -\varepsilon_3, \varepsilon_4, \dots, \varepsilon_m)|.
\end{aligned}$$

Repeating this argument, we obtain

$$|R(\varepsilon_1, -\varepsilon_2, \dots, \varepsilon_m)| \prod_{k=1}^{m-1} |R(\varepsilon_k, \dots, \varepsilon_m)| \geq |R(\varepsilon_m)| = \sqrt{c}.$$

Since $|R(\varepsilon_1, -\varepsilon_2, \dots, \varepsilon_m)| \leq \frac{s}{2}$, the lemma holds. □

PROPOSITION 7.2. *For any $(\varepsilon_1, \varepsilon_2, \dots) \in \{\pm 1\}^{\mathbb{N}}$, the sequence $\{R(\varepsilon_1, \dots, \varepsilon_m)\}_{m=1}^{\infty}$ converges.*

PROOF. It suffices to show that $\{R(\varepsilon_1, \dots, \varepsilon_m)\}_{m=1}^{\infty}$ is a Cauchy sequence. To see this, note that

$$|R(\varepsilon_1, \dots, \varepsilon_m) - R(\varepsilon_1, \dots, \varepsilon_n)| \leq \sum_{k=n}^{m-1} |R(\varepsilon_1, \dots, \varepsilon_{k+1}) - R(\varepsilon_1, \dots, \varepsilon_k)|$$

for any positive integers m, n with $m > n$. Here we have

$$\begin{aligned}
&|R(\varepsilon_1, \dots, \varepsilon_{k+1}) - R(\varepsilon_1, \dots, \varepsilon_k)| \\
&\quad = \frac{|R(\varepsilon_1, \dots, \varepsilon_{k+1})^2 - R(\varepsilon_1, \dots, \varepsilon_k)^2|}{|R(\varepsilon_1, \dots, \varepsilon_{k+1}) + R(\varepsilon_1, \dots, \varepsilon_k)|} \\
&\quad = \frac{|R(\varepsilon_2, \dots, \varepsilon_{k+1}) - R(\varepsilon_2, \dots, \varepsilon_k)|}{|R(\varepsilon_1, \dots, \varepsilon_{k+1}) + R(\varepsilon_1, \dots, \varepsilon_k)|} \\
&\quad\ \ \vdots \\
&\quad = \frac{\sqrt{c}}{\prod_{i=1}^{k} |R(\varepsilon_i, \dots, \varepsilon_{k+1}) + R(\varepsilon_i, \dots, \varepsilon_k)|}.
\end{aligned}$$

As for the denominator, using Lemma 7.1, we obtain

$$\begin{aligned}
&\prod_{i=1}^{k}|R(\varepsilon_i,\dots,\varepsilon_{k+1})+R(\varepsilon_i,\dots,\varepsilon_k)| \\
&\geq 2^k\prod_{i=1}^{k}|R(\varepsilon_i,\dots,\varepsilon_{k+1})R(\varepsilon_i,\dots,\varepsilon_k)|^{1/2} \\
&= 2^k|R(\varepsilon_{k+1})|^{-\frac{1}{2}}\left(\prod_{i=1}^{k+1}|R(\varepsilon_i,\dots,\varepsilon_{k+1})|\right)^{\frac{1}{2}}\left(\prod_{i=1}^{k}|R(\varepsilon_i,\dots,\varepsilon_k)|\right)^{\frac{1}{2}} \\
&\geq 2^k\cdot c^{-\frac{1}{4}}\cdot\left(\frac{2c}{s}\right)^{\frac{1}{2}}\cdot\left(\frac{2c}{s}\right)^{\frac{1}{2}} \\
&= \frac{2^{k+1}c^{\frac{3}{4}}}{s}\,.
\end{aligned}$$

Therefore

$$|R(\varepsilon_1,\dots,\varepsilon_m)-R(\varepsilon_1,\dots,\varepsilon_n)|\leq\sum_{k=n}^{m-1}\frac{s}{2^{k+1}c^{1/4}}<\frac{s}{2^nc^{1/4}}$$

for any $m>n$. This implies that $\{R(\varepsilon_1,\dots,\varepsilon_m)\}_{m=1}^{\infty}$ is a Cauchy sequence. $\square$

Now, recall that $f(x)$ is a monotonously increasing continuous function on $[\omega,\infty)$ and $f(\omega)=-c$. Note that $-c\leq -s/2$ and

$$R_c(\mathbf{e})\leq R_c(1,1,\dots)=\frac{s}{2}$$

for any $\mathbf{e}\in\{\pm1\}^{\mathbb{N}}$. Therefore for any $\mathbf{e}\in\{\pm1\}^{\mathbb{N}}$ there exists a unique real number $\lambda(\mathbf{e})\in[\omega,\infty)$ such that $f(\lambda(\mathbf{e}))=R(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_i,\dots)$.

THEOREM 7.3. *Given an infinite sequence* $\mathbf{e}=(\varepsilon_1,\varepsilon_2,\dots)\in\{\pm1\}^{\mathbb{N}}$, *define integers* A_m *as in Theorem* 6.7. *Then*

$$\lim_{m\to\infty}\frac{\rho(A_m+1)}{s^m}=\lambda(\mathbf{e})\,.$$

PROOF. Since $A_m+1\leq 2^m$, we have

$$\rho(A_m+1)\geq\rho(2^m)=\tau(2^{m+1}-1)>\tau(2^{m+1})=s^m\tau(2)=s^m\omega\,.$$

Hence $\omega<\rho(A_m+1)/s^m<0$. Then by Theorem 6.7 we have

$$\lim_{m\to\infty}f\left(\frac{\rho(A_m+1)}{s^m}\right)=\lim_{m\to\infty}R_c(\varepsilon_1,\dots,\varepsilon_m)=f(\lambda(\mathbf{e}))\,.$$

This implies that

$$\lim_{m\to\infty} \frac{\rho(A_m+1)}{s^m} = \lambda(\mathbf{e}),$$

which completes the proof. □

8. A generalization of Viéta's formula

In this section we give a generalization of Viéta's formula (7). Let us start with a proposition.

PROPOSITION 8.1. *Let s be a complex number with $|s|>1$. Then*

$$f(x)-f(y) = s(x-y)\prod_{n=1}^{\infty}\frac{1}{s}\left(f\left(\frac{x}{s^n}\right)+f\left(\frac{y}{s^n}\right)\right)$$

for any $x, y \in \mathbb{C}$.

PROOF. Using the functional equation

$$f(sx) = f(x)^2 - c,$$

we have

$$\begin{aligned} f(x)-f(y) &= f\left(\frac{x}{s}\right)^2 - f\left(\frac{y}{s}\right)^2 \\ &= s\left(f\left(\frac{x}{s}\right)-f\left(\frac{y}{s}\right)\right)\frac{1}{s}\left(f\left(\frac{x}{s}\right)+f\left(\frac{y}{s}\right)\right) \\ &= s\left(f\left(\frac{x}{s^2}\right)^2-f\left(\frac{y}{s^2}\right)^2\right)\frac{1}{s}\left(f\left(\frac{x}{s}\right)+f\left(\frac{y}{s}\right)\right) \\ &= s^2\left(f\left(\frac{x}{s^2}\right)-f\left(\frac{y}{s^2}\right)\right)\frac{1}{s}\left(f\left(\frac{x}{s}\right)+f\left(\frac{y}{s}\right)\right)\cdot \\ &\quad \frac{1}{s}\left(f\left(\frac{x}{s^2}\right)+f\left(\frac{y}{s^2}\right)\right) \\ &= \cdots \\ &= s^m\left(f\left(\frac{x}{s^m}\right)-f\left(\frac{y}{s^m}\right)\right)\prod_{n=1}^{m}\frac{1}{s}\left(f\left(\frac{x}{s^n}\right)+f\left(\frac{y}{s^n}\right)\right). \end{aligned}$$

Since the Taylor expansion of $f(x)$ at $x=0$ is

$$f(x) = \frac{s}{2} + sx + \cdots,$$

the limit

$$\lim_{m\to\infty} s^m\left(f\left(\frac{x}{s^m}\right)-f\left(\frac{y}{s^m}\right)\right)\prod_{n=1}^{m}\frac{1}{s}\left(f\left(\frac{x}{s^n}\right)+f\left(\frac{y}{s^n}\right)\right)$$

exists and equals

$$s(x-y)\prod_{n=1}^{\infty}\frac{1}{s}\left(f\left(\frac{x}{s^n}\right)+f\left(\frac{y}{s^n}\right)\right).$$

This proves the proposition. □

THEOREM 8.2. *Suppose $s \geq 4$. Then*

$$\frac{1}{2|\rho(k)|}=\prod_{n=1}^{\infty}\frac{1}{s}\left(f\left(\frac{\rho(k)}{s^n}\right)+\frac{s}{2}\right). \tag{43}$$

PROOF. Setting $x=\rho(k)$, $y=0$ in Proposition 8.1, we have

$$-\frac{s}{2}=s\rho(k)\prod_{n=1}^{\infty}\frac{1}{s}\left(f\left(\frac{\rho(k)}{s^n}\right)+\frac{s}{2}\right).$$

Since $\rho(k)<0$, we obtain the theorem. □

We should remark that the formula (43) can be viewed as a generalization of Viéta's formula. To see this, note that the quantity $f(\rho(k)/s^n)$ in the product of the right hand side of (43) is a nested square root by Corollary 6.6. For example, if $k=1$, then

$$\frac{1}{2|\rho(1)|}=\frac{1}{s}\left(\frac{s}{2}+\sqrt{c}\right)\frac{1}{s}\left(\frac{s}{2}+\sqrt{c+\sqrt{c}}\right)\cdots. \tag{44}$$

If $k=2$, then

$$\frac{1}{2|\rho(2)|}=\frac{1}{s}\left(\frac{s}{2}-\sqrt{c}\right)\frac{1}{s}\left(\frac{s}{2}+\sqrt{c-\sqrt{c}}\right)\cdots.$$

If we set $s=4$ (i.e. $c=2$) in (44), then $\rho(1)=-\pi^2/16$ by Remark 6.8. Hence (44) reduces to Viéta's formula

$$\frac{2}{\pi}=\frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots.$$

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