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A Study of Submanifolds of the Complex Grassmannian Manifold with Parallel Second Fundamental Form

Isami KOGA and Yasuyuki NAGATOMO

Kyushu University and Meiji University

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Abstract. We prove an extension of a theorem of A. Ros on *a characterization of seven compact Kaehler submanifolds by holomorphic pinching* [5] to certain submanifolds of the complex Grassmannian manifolds.

1. Introduction

Let $\mathbf{C}P^n(1)$ be the n-dimensional complex projective space with the constant holomorphic sectional curvature 1 and M^m an m-dimensional compact Kähler submanifold immersed in $\mathbf{C}P^n(1)$. In [5] Ros has proved that the holomorphic sectional curvature of M is greater than or equal to $\frac{1}{2}$ if and only if M has the parallel second fundamental form. Our goal in the present paper is to extend this result to submanifolds immersed in the complex Grassmannian manifold.

Let $Gr_p(\mathbf{C}^n)$ be the complex Grassmannian manifold of complex p-planes in $\mathbf{C}^n$. Since the tautological bundle $S \to Gr_p(\mathbf{C}^n)$ is a subbundle of a trivial bundle $Gr_p(\mathbf{C}^n) \times \mathbf{C}^n \to Gr_p(\mathbf{C}^n)$, we obtain the quotient bundle $Q \to Gr_p(\mathbf{C}^n)$. This is called the *universal quotient bundle*. We notice the fact that the holomorphic tangent bundle $T_{1,0}M$ over $Gr_p(\mathbf{C}^n)$ can be identified with the tensor product of holomorphic vector bundles S^* and Q, where $S^* \to Gr_p(\mathbf{C}^n)$ is the dual bundle of $S \to Gr_p(\mathbf{C}^n)$. If $\mathbf{C}^n$ has a Hermitian inner product, S, Q have Hermitian metrics and Hermitian connections and so $Gr_p(\mathbf{C}^n)$ has a Hermitian metric induced by the identification of $T_{1,0}Gr$ and $S^* \otimes Q$, which is called the *standard metric* on $Gr_p(\mathbf{C}^n)$. In the present paper, we prove the following theorem:

THEOREM 1. *Let $Gr_p(\mathbf{C}^n)$ be the complex Grassmannian manifold of complex p-planes in $\mathbf{C}^n$ with the standard metric h_{Gr} induced from a Hermitian inner product on $\mathbf{C}^n$ and f a holomorphic isometric immersion of a compact Kähler manifold (M, h_M) with a Hermitian metric h_M into $Gr_p(\mathbf{C}^n)$. We denote by $Q \to Gr_p(\mathbf{C}^n)$ the universal quotient bundle over $Gr_p(\mathbf{C}^n)$ of rank $q (:= n - p)$. We assume that the pull-back bundle of $Q \to Gr_p(\mathbf{C}^n)$*

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is projectively flat. Then the holomorphic sectional curvature of M is greater than or equal to $\frac{1}{q}$ if and only if f has parallel second fundamental form.

We regard $Gr_{n-1}(\mathbf{C}^n)$ as the complex projective space. When we consider a holomorphic map $f : M \to Gr_{n-1}(\mathbf{C}^n)$ of a compact complex manifold into the complex projective space, then the pull-back bundle of $Q \to Gr_{n-1}(\mathbf{C}^n)$ is projectively flat since the rank of Q is 1. Thus a holomorphic map of a compact complex manifold into the complex Grassmannian manifold which satisfies the condition that the pull-back bundle of the universal quotient bundle is projectively flat is a kind of generalization of a holomorphic map into the complex projective space. In the case that $p < n - 1$, see the latter part of Section 2.

It is why Theorem 1 is an extension of a theorem of Ros in [5]. In the case that $p = n-1$, the sufficient condition in our theorem is that the holomorphic sectional curvature is greater than or equal to 1, which is distinct from $\frac{1}{2}$ in a theorem of Ros. This is because we take a metric of Fubini-Study type with constant holomorphic sectional curvature 2.

REMARK 1. We can suppose that $p \geq q$ without loss of generality. In fact we can show that there is no immersion satisfying projectively flatness in the case that $p < q$. (See Remark 4.)

2. Preliminaries

Let $Gr_p(\mathbf{C}^n)$ be the complex Grassmannian manifold of complex p-planes in $\mathbf{C}^n$ with a standard metric h_{Gr} induced from a Hermitian inner product on $\mathbf{C}^n$. We denote by $S \to Gr_p(\mathbf{C}^n)$ the tautological vector bundle over $Gr_p(\mathbf{C}^n)$. Since $S \to Gr_p(\mathbf{C}^n)$ is a subbundle of a trivial vector bundle $\underline{\mathbf{C}^n} = Gr_p(\mathbf{C}^n) \times \mathbf{C}^n \to Gr_p(\mathbf{C}^n)$, we obtain a holomorphic vector bundle $Q \to Gr_p(\mathbf{C}^n)$ as a quotient bundle. This is called the universal quotient bundle over $Gr_p(\mathbf{C}^n)$. For simplicity, it is called the quotient bundle. Consequently we have a short exact sequence of vector bundles:

$$0 \to S \to \underline{\mathbf{C}^n} \to Q \to 0\,.$$

Taking the orthogonal complement of S in $\underline{\mathbf{C}^n}$ with respect to the Hermitian inner product on $\mathbf{C}^n$, we obtain a complex subbundle $S^\perp \to Gr_p(\mathbf{C}^n)$ of $\underline{\mathbf{C}^n}$. As C^∞ complex vector bundle, Q is naturally isomorphic to $S^\perp$. Consequently, the vector bundle $S \to Gr_p(\mathbf{C}^n)$ (resp. $Q \to Gr_p(\mathbf{C}^n)$) is equipped with a Hermitian metric h_S (resp.h_Q) and so a Hermitian connection ∇^S (resp.∇^Q). The holomorphic tangent bundle $T_{1,0}Gr_p(\mathbf{C}^n)$ over $Gr_p(\mathbf{C}^n)$ is identified with $S^* \otimes Q \to Gr_p(\mathbf{C}^n)$ and the Hermitian metric on the holomorphic tangent bundle is induced from the tensor product $h_{S^*} \otimes h_Q$ of h_{S^*} and h_Q.

Let $w_1, \dots, w_n$ be a unitary basis of $\mathbf{C}^n$. We denote by $\mathbf{C}^p$ the subspace of $\mathbf{C}^n$ spanned by $w_1, \dots, w_p$ and by $\mathbf{C}^q$ the orthogonal complement of $\mathbf{C}^p$, where $q = n - p$. The orthogonal projection to $\mathbf{C}^p$, $\mathbf{C}^q$ is denoted by π_p, π_q respectively. Let G be the special unitary group $SU(n)$ and P the subgroup $S(U(p) \times U(q))$ of $SU(n)$ according to the decomposition. Then $Gr_p(\mathbf{C}^n) \cong G/P$. The vector bundles S, Q are identified with $G \times_P \mathbf{C}^p$,

$G \times_P \mathbf{C}^q$ respectively. We denote by $\Gamma(S)$, $\Gamma(Q)$ spaces of sections of S, Q respectively. Let $\pi_Q : \mathbf{C}^n \to \Gamma(Q)$ be a linear map defined by

$$\pi_Q(w)([g]) := [g, \pi_q(g^{-1}w)] \in G \times_P \mathbf{C}^q\,, \quad w \in \mathbf{C}^n, g \in G\,.$$

The bundle injection $i_Q : Q \to \underline{\mathbf{C}^n}$ can be expressed as the following:

$$i_Q([g, v]) = ([g], gv)\,, \quad v \in \mathbf{C}^q\,, \quad g \in G\,, \quad [g] \in Gr_p(\mathbf{C}^n) \cong G/P\,.$$

Let t be a section of $Q \to Gr_p(\mathbf{C}^n)$. Since $i_Q(t)$ can be regarded as a $\mathbf{C}^n$-valued function $t : Gr_p(\mathbf{C}^n) \to \mathbf{C}^n$, $\pi_Q d(i_Q(t))$ defines a connection on Q. This is nothing but ∇^Q.

Similarly, we can write a bundle injection $i_S : S \to \underline{\mathbf{C}^n}$ and a linear map $\pi_S : \mathbf{C}^n \to \Gamma(S)$:

$$\begin{aligned} i_S([g, u]) &= ([g], gu)\,, && u \in \mathbf{C}^p\,, \quad g \in G\,, \quad [g] \in G/P\,, \\ \pi_S(w)([g]) &: = [g, \pi_p(g^{-1}w)]\,, && w \in \mathbf{C}^n\,, \quad g \in G\,. \end{aligned}$$

The connection $\pi_S d(i_S(s))$ on S is nothing but ∇^S.

We introduce the second fundamental form H in the sense of Kobayashi [1], which is a (1,0)-form with values in $\mathrm{Hom}(S, Q) \cong S^* \otimes Q$:

$$di_S(s) = \nabla^S s + H(s)\,, \quad H(s) = \pi_Q d(i_S(s))\,, \quad s \in \Gamma(S)\,. \tag{1}$$

Similarly, we introduce the second fundamental form K, which is a (0,1)-form with values in $\mathrm{Hom}(Q, S) \cong Q^* \otimes S$:

$$di_Q(t) = K(t) + \nabla^Q t\,, \quad K(t) = \pi_S d(i_Q(t))\,, \quad t \in \Gamma(Q)\,. \tag{2}$$

LEMMA 1 ([1]). *The second fundamental forms H and K satisfy*

$$h_Q(H_U s, t) = -h_S(s, K_{\overline{U}} t)\,, \quad s \in S_x\,, \ \ t \in Q_x\,, \quad U \in T_{1,0_x} Gr_p(\mathbf{C}^n)\,,$$

for any $x \in Gr_p(\mathbf{C}^n)$.

For a proof, See [1].

LEMMA 2. *For a vector $w \in \mathbf{C}^n$, set $s = \pi_S(w)$ and $t = \pi_Q(w)$. Then*

$$\nabla^S_{\overline{U}} s = -K_{\overline{U}}(t)\,, \quad \nabla^Q_U t = -H_U(s)\,, \quad (U \in T_{1,0} Gr_p(\mathbf{C}^n))\,.$$

PROOF. Since $i_S(s) + i_Q(t) = ([g], w)$, we have

$$0 = \pi_S\left(di_S(s) + di_Q(t)\right) = \nabla^S(s) + K(t)\,.$$

Thus $\nabla^S s = -K(t)$. Similarly $\nabla^Q t = -H(s)$. □

Since H is a $(1,0)$-form with values in $S^* \otimes Q$, then H can be regarded as a section of $T_{1,0} Gr_p(\mathbf{C}^n)^* \otimes T_{1,0} Gr_p(\mathbf{C}^n)$.

PROPOSITION 1 ([3]). *The second fundamental form H can be regarded as the identity transformation of $T_{1,0}Gr_p(\mathbf{C}^n)$.*

The unitary basis $w_1, \dots, w_n$ of $\mathbf{C}^n$ provides us with the corresponding sections

$$s_A = \pi_S(w_A) \in \Gamma(S)\,, \quad t_A = \pi_Q(w_A) \in \Gamma(Q)\,, \quad A = 1, \dots, n\,.$$

PROPOSITION 2 ([3]). *For arbitrary $(1,0)$-vectors U and V on $Gr_p(\mathbf{C}^n)$, we have*

$$h_{Gr}(U, V) = \sum_{A=1}^{n} h_S(K_{\overline{V}} t_A, K_{\overline{U}} t_A) = \sum_{A=1}^{n} h_Q(H_U s_A, H_V s_A)\,.$$

Proposition 1 and Proposition 2 were proved by the second author in [3].

REMARK 2. Let U, V be (1,0)-vectors on $Gr_p(\mathbf{C}^n)$ at $x \in Gr_p(\mathbf{C}^n)$. From Lemma 1 and Proposition 2, we have

$$h_{Gr}(U, V) = -\mathrm{trace}_Q H_U K_{\overline{V}} = -\overline{\mathrm{trace}_S K_{\overline{V}} H_U}\,, \tag{3}$$

where $\mathrm{trace}_Q H_U K_{\overline{V}}$ is the trace of the endomorphism $H_U K_{\overline{V}}$ of Q_x and $\mathrm{trace}_S K_{\overline{V}} H_U$ is the trace of the endomorphism $K_{\overline{V}} H_U$ of S_x.

Since any vectors in S_x (resp. Q_x) can be expressed by a linear combination of $s_1(x), \dots, s_n(x)$ (resp. $t_1(x), \dots, t_n(x)$), it follows from Lemma 2 that the curvature R^S of ∇^S and R^Q of ∇^Q are expressed by the following:

$$R^S(U, \overline{V}) s_A = \nabla^S_U (\nabla^S s_A)(\overline{V}) - \nabla^S_{\overline{V}} (\nabla^S s_A)(U) = K_{\overline{V}} H_U s_A\,, \tag{4}$$

$$R^Q(U, \overline{V}) t_A = \nabla^Q_U (\nabla^Q t_A)(\overline{V}) - \nabla^Q_{\overline{V}} (\nabla^Q t_A)(U) = -H_U K_{\overline{V}} t_A\,. \tag{5}$$

It follows from $h_{Gr} = h_{s^*} \otimes h_Q$ that the curvature R^{Gr} of $Gr_p(\mathbf{C}^n)$ can be expressed as $R^{S^*} \otimes \mathrm{Id}_Q + \mathrm{Id}_{S^*} \otimes R^Q$. Thus we can compute R^{Gr} as follows:

$$R^{Gr}(U, \overline{V}) Z = -H_Z K_{\overline{V}} H_U - H_U K_{\overline{V}} H_Z\,, \tag{6}$$

for $(1,0)$-vectors U, V, Z.

REMARK 3. Let us compute the holomorphic sectional curvature of $Gr_{n-1}(\mathbf{C}^n)$. Since the quotient bundle over $Gr_{n-1}(\mathbf{C}^n)$ is of rank 1, then it follows from the equations (3) and (6) that

$$R^{Gr}(U, \overline{V}) Z = -H_Z K_{\overline{V}} H_U - H_U K_{\overline{V}} H_Z. = h_{Gr}(Z, V) U + h_{Gr}(U, V) Z\,,$$

where U, V is (1,0)-vectors. Thus for any unit (1,0)-vector U we obtain

$$\mathrm{Hol}^{Gr}(U) = h_{Gr}(R^{Gr}(U, \overline{U}) U, U) = h_{Gr}(2U, U) = 2\,,$$

where $\mathrm{Hol}^{Gr}(U)$ is the holomorphic sectional curvature along U of $Gr_{n-1}(\mathbf{C}^n)$.

From now on, we introduce a relation between holomorphic vector bundles over a compact complex manifold and holomorphic maps into the complex Grassmannian manifold. For a detail, see [3].

Let M be a compact complex manifold and $V \to M$ a holomorphic vector bundle with Hermitian metric and Hermitian connection ∇^V. We denote by $(W, (\cdot,\cdot)_W)$ the space of holomorphic sections of $V \to M$ with L_2-Hermitian inner product. Assume that the bundle homomorphism, which is called an *evaluation map*,

$$ev : M \times W \longrightarrow V : (x, t) \longmapsto t(x)$$

is surjective. In this case $V \to M$ is called *globally generated* by W. Then the linear map $ev_x : W \to V_x : t \mapsto t(x)$ is surjective for each $x \in M$. Then we obtain complex vector subspace $\mathrm{Ker}\, ev_x$ of W for each $x \in M$. We denoted by p the dimension of $\mathrm{Ker}\, ev_x$, which is not depend on $x \in M$. Therefore we obtain a holomorphic map

$$f_0 : M \longrightarrow Gr_p(W) : x \longmapsto \mathrm{Ker}\, ev_x .$$

This is called the *standard map* induced by $V \to M$.

Conversely, let M be a compact Kähler manifold and $f : M \to Gr_p(\mathbf{C}^n)$ a holomorphic isometric immersion. It follows from a Borel-Weil Theorem that $\mathbf{C}^n$ can be regarded as the space of holomorphic sections of $Q \to Gr$. By restricting each sections of $Q \to Gr$ to M, we obtain a linear map from $\mathbf{C}^n$ to the space of holomorphic sections of $f^*Q \to M$. Then we obtain an evaluation map

$$ev_{\mathbf{C}} : M \times \mathbf{C}^n \longrightarrow f^*Q : (x, t) \longmapsto t(x) , \quad \text{for } x \in M, t \in \mathbf{C}^n .$$

The bundle isomorphism $ev_{\mathbf{C}}$ is surjective and we have $\mathrm{Ker}\, ev_{\mathbf{C}_x} = S_{f(x)} = f(x)$. Therefore by using $ev_{\mathbf{C}}$, f is expressed that $f(x) = \mathrm{Ker}\, ev_{\mathbf{C}_x}$.

Here we assume that $f^*Q \to M$ is projectively flat. It follows from the holonomy theorem and $(*)$ in Section 3 that there exists a holomorphic line bundle $L \to M$ such that $f^*Q \to M$ is decomposed to orthogonal direct sum of q-copies of $L \to M$, where $q = n - p$. We denote by $\tilde{L} \to M$ the orthogonal direct sum bundle of q-copies of $L \to M$ and also denote by W and $\tilde{W}$ the space of holomorphic sections of $L \to M$ and $\tilde{L} \to M$ respectively. We fix an L_2-Hermitian inner product $(\cdot,\cdot)_W$ and $(\cdot,\cdot)_{\tilde{W}}$ of W and $\tilde{W}$ respectively. Then $\tilde{W}$ is regarded as the orthogonal q-direct sum of W. Let $f_0 : M \to Gr_{N-1}(W)$ be the standard map induced by $L \to M$, where N is the dimension of W. When we denote by $\tilde{f} : M \to Gr_{q(N-1)}(\tilde{W})$ the standard map induced by $\tilde{L} \to M$, $\tilde{f}$ can be expressed as

$$\tilde{f}(x) = f_0(x) \oplus \cdots \oplus f_0(x) \subset W \oplus \cdots \oplus W . \quad \text{for } x \in M .$$

Since $f^*Q \to M$ is isomorphic to $\tilde{L} \to M$ with metrics and connections, we have a linear map $\iota : \mathbf{C}^n \to \tilde{W}$. We assume that ι is injective. Then it follows from Theorem 5.5 in [3] that there exists a semi-positive Hermitian endomorphism T of $\tilde{W}$ such that $f : M \to$

$Gr_p(\mathbf{C}^n)$ can be expressed as

$$f(x) = (\iota^* T \iota)^{-1}(\tilde{f}(x) \cap \iota(\mathbf{C}^n)),$$

where $\iota^* : \tilde{W} \to \mathbf{C}^n$ is the adjoint linear map of ι.

Consequently, if $f : M \to Gr_p(\mathbf{C}^n)$ is holomorphic isometric immersion with the condition that $f^*Q \to M$ is projectively flat, then f can be expressed by using a holomorphic map into the complex projective space and a semi-positive Hermitian endomorphism.

3. Proof of Theorem 1

Let M be a compact Kähler manifold and $f : M \to Gr_p(\mathbf{C}^n)$ a holomorphic isometric immersion, where $Gr_p(\mathbf{C}^n)$ has the metric h_{Gr} induced by the Hermitian inner product of $\mathbf{C}^n$. We denote by ∇^M and ∇^{Gr} the Hermitian connections of M and $Gr_p(\mathbf{C}^n)$ respectively. We have a short exact sequence of holomorphic vector bundles:

$$0 \to T_{1,0}M \to T_{1,0}Gr|_M \to N \to 0,$$

where $T_{1,0}Gr|_M$ is a holomorphic vector bundle induced by f from the holomorphic tangent bundle over $Gr_p(\mathbf{C}^n)$ and N is a quotient bundle. In the same manner as in the previous section, we obtain second fundamental forms σ and A of TM and N:

$$\nabla^{Gr}_U V = \nabla^M_U V + \sigma(U, V), \qquad U \in T_{\mathbf{C}}M, \quad V \in \Gamma(T_{1,0}M), \tag{7}$$

$$\nabla^{Gr}_U \xi = -A_\xi U + \nabla^N_U \xi, \qquad U \in T_{\mathbf{C}}M, \quad \xi \in \Gamma(N). \tag{8}$$

For each point $x \in M$, $\sigma : T_{1,0_x}M \times T_{1,0_x}M \to N_x$ is a symmetric bilinear mapping. This is called the second fundamental form of f. The second fundamental form $A : N_x \times T_{0,1_x}M \to T_{1,0_x}M$ is a bilinear mapping. This is called the shape operator of f. We follow a convention of submanifold theory to define the shape operator.

Throughout this section, the symbol ∇ means the suitable connection of covariant tensor fields induced by ∇^M, ∇^{Gr} and ∇^N. Second fundamental forms σ and A satisfy the following formulas.

FORMULAS 1. For any $U, V, Z, W \in T_{1,0_x}M$, we have

- $\sigma(\overline{U}, V) = 0, \qquad A_\xi U = 0,$
- $h_{Gr}(\sigma(U, V), \xi) = h_{Gr}\left(V, A_\xi \overline{U}\right),$
- $h_{Gr}\left(R^M(U, \overline{V})Z, W\right) = h_{Gr}\left(R^{Gr}(U, \overline{V})Z, W\right) - h_{Gr}(\sigma(U, Z), \sigma(V, W)),$
- $h_{Gr}\left(R^N(U, \overline{V})\xi, \eta\right) = h_{Gr}\left(R^{Gr}(U, \overline{V})\xi, \eta\right) + h_{Gr}\left(A_\xi \overline{V}, A_\eta \overline{U}\right),$
- $(\nabla_V \sigma)(U, Z) = (\nabla_U \sigma)(V, Z),$
- $\left(\nabla_{\overline{V}} \sigma\right)(U, Z) = -\left(R^{Gr}(U, \overline{V})Z\right)^\perp.$

Note that the quotient bundle N is isomorphic to the orthogonal complement bundle $T_{1,0}^\perp M$ as a C^∞ complex vector bundle. The third, fourth and fifth formulas are called the equation of Gauss, the equation of Ricci and the equation of Codazzi respectively. From the equation of Codazzi,

$$\nabla\sigma : T_{1,0_x}M \otimes T_{1,0_x}M \otimes T_{1,0_x}M \longrightarrow N_x$$

is a symmetric tensor for any $x \in M$.

We assume that $f^*Q \to M$ is projectively flat. The vector bundle $f^*Q \to M$ is projectively flat if and only if

$$R^{f^*Q}(U, \overline{V}) = \alpha(U, \overline{V})\mathrm{Id}_{Q_{f(x)}}, \quad \text{for } U, V \in T_{1,0_x}M,$$

where α is a complex 2-form on M. Since R^{f^*Q} is a (1,1)-form, so is α. It follows from the equation (3) that

$$h_M(U, V) = \mathrm{trace}R^Q(U, \overline{V}) = q \cdot \alpha(U, \overline{V}).$$

Therefore, $f^*Q \to M$ is projectively flat if and only if

$$R^{f^*Q}(U, \overline{V}) = \frac{1}{q}h_M(U, V)\mathrm{Id}_{Q_{f(x)}}, \quad \text{for } U, V \in T_{1,0_x}M. \tag{$*$}$$

REMARK 4. It follows from the equation (5) that

$$R^{f^*Q}(U, \overline{V}) = -H_U K_{\overline{V}} : Q_x \longrightarrow S_x \longrightarrow Q_x.$$

Therefore, if an immersion f satisfies the equation $(*)$, the rank of S is greater than or equal to that of Q.

We denote by Hol the holomorphic sectional curvature of a Kähler manifold. By the equation of Gauss, if U is a unit (1,0)-vector on M, then

$$\begin{aligned}\mathrm{Hol}^M(U) = h_M(R^M(U, \overline{U})U, U) &= h_{Gr}(R^{Gr}(U, \overline{U})U, U) - \|\sigma(U, U)\|^2 \\ &= \mathrm{Hol}^{Gr}(U) - \|\sigma(U, U)\|^2.\end{aligned} \tag{9}$$

LEMMA 3. *Under the assumption of Theorem* 1, *for any unit* $(1, 0)$*-vector* U *on* M *we have*

$$\mathrm{Hol}^{Gr}(U) = \frac{2}{q}.$$

PROOF. Let U be a unit (1,0)-vector at $x \in M$. By the equation $(*)$, we have

$$-H_U K_{\overline{U}} = R^{f^*Q}(U, \overline{U}) = \frac{1}{q}\mathrm{Id}_{Q_x}. \tag{10}$$

It follows from equations (6) and (10) that

$$\begin{aligned}\mathrm{Hol}^{Gr}(U) &= h_{Gr}(R^{Gr}(U, \overline{U})U, U) = -2h_{S^*\otimes Q}(H_U K_{\overline{U}} H_U, H_U) \\ &= \frac{2}{q}h_{S^*\otimes Q}(H_U, H_U) = \frac{2}{q}.\end{aligned}$$

□

LEMMA 4. *Under the assumption of Theorem* 1, *for any* $(0,1)$*-vector* $\overline{V}$ *on* M *we have*

$$\nabla_{\overline{V}}\sigma = 0\,.$$

PROOF. It follows from equation (6) and $(*)$ that

$$\begin{aligned} R^{Gr}(U,\overline{V})Z &= -H_Z K_{\overline{V}} H_U - H_U K_{\overline{V}} H_Z \\ &= \frac{1}{q} h_{Gr}(Z,V)U + \frac{1}{q} h_{Gr}(U,V)Z\,, \end{aligned} \tag{11}$$

where U, V, Z are $(1,0)$-vectors on M. By the equation of Codazzi, we have

$$\nabla_{\overline{V}}\sigma(U,Z) = -(R^{Gr}(U,\overline{V})Z)^{\perp} = 0\,.$$

□

In [5] A. Ros has proved the following Lemma.

LEMMA 5 (A. Ros [5]). *Let* T *be a* k*-covariant tensor on a compact Riemannian manifold* M. *Then*

$$\int_{UM} (\nabla T)(X,\dots,X)dX = 0\,,$$

where UM *is the unit tangent bundle of* M *and* dX *is the canonical measure of* UM *induced by the Riemannian metric on* M.

For a proof, see [5].

We use the complexification of the above Lemma.

LEMMA 6. *Let* T *be a* (p,q)*-covariant tensor on an* m*-dimensional compact Kähler manifold* (M,h_M). *We consider* M *as an* $2m$*-dimensional real manifold with the almost complex structure* J. *We denote by* g_M *the Riemannian metric induced by* h_M. *Then we have the canonical measure* dX *of* UM. *We obtain the following equality*:

$$\int_{UM} (\nabla T)(\overline{U_X}, U_X,\dots,U_X,\overline{U_X},\dots,\overline{U_X})dX = 0\,,$$

where $U_X = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX)$ *and* $\overline{U_X} = \frac{1}{\sqrt{2}}(X + \sqrt{-1}JX)$ *and* X *is a real tangent vector on* M.

PROOF. We define real valued k-covariant tensors on Riemannian manifold (M, g_M) by

$$\begin{aligned} 2K(X_1,\dots,X_k) &= T(U_1,\dots,U_p,\overline{U_{p+1}},\dots,\overline{U_k}) + \overline{T(U_1,\dots,U_p,\overline{U_{p+1}},\dots,\overline{U_k})}, \\ 2L(X_1,\dots,X_k) &= \sqrt{-1}\{T(U_1,\dots,U_p,\overline{U_{p+1}},\dots,\overline{U_k}) \\ &\quad - \overline{T(U_1,\dots,U_p,\overline{U_{p+1}},\dots,\overline{U_k})}\}\,, \end{aligned}$$

where $k = p + q$, $U_i = U_{X_i}$ for $i = 1, \dots, k$. Then T, K and L satisfy the following equation:

$$T(U_1, \dots, U_p, \overline{U_{p+1}}, \dots, \overline{U_k}) = K(X_1, \dots, X_k) - \sqrt{-1}L(X_1, \dots, X_k)\,.$$

We get the covariant derivative of both sides of this equation:

$$\begin{aligned}(\nabla_{\overline{U}_X}T)(U_X, \dots, \overline{U}_X, \dots) &= \frac{1}{\sqrt{2}}(\nabla_{X+\sqrt{-1}JX}K)(X, \dots, X) \\ &\quad - \frac{\sqrt{-1}}{\sqrt{2}}(\nabla_{X+\sqrt{-1}JX}L)(X, \dots, X)\,.\end{aligned} \tag{12}$$

Since the covariant derivative is linear, then

$$(\nabla_{X+\sqrt{-1}JX}K)(X, \dots, X) = (\nabla_X K)(X, \dots, X) + \sqrt{-1}(\nabla_{JX}K)(X, \dots, X)\,. \tag{13}$$

Consequently it follows from Lemma 5 that we obtain

$$\begin{aligned}\int_{UM}(\nabla T)(\overline{U_X}, U_X, \dots, U_X, \overline{U_X}, \dots, \overline{U_X})dX &= \frac{\sqrt{-1}}{\sqrt{2}}\int_{UM}(\nabla_{JX}K)(X, \dots, X)dX \\ &\quad + \frac{1}{\sqrt{2}}\int_{UM}(\nabla_{JX}L)(X, \dots, X)dX\,.\end{aligned}$$

For the covariant tensor field K, we define a new covariant tensor fields $\tilde{K}$ by

$$\tilde{K}(X_1, \dots, X_k) = K(JX_1, \dots, JK_k)\,, \quad \text{for } X_1, \dots, X_k \in T_xM\ (x \in M)\,.$$

Since the almost complex structure J is parallel and preserves the inner product and orientation of each tangent space of M, it follows that

$$\begin{aligned}\int_{UM}(\nabla_{JX}K)(X, \dots, X)dX &= (-1)^k\int_{UM}(\nabla_{JX}K)(J(JX), \dots, J(JX))dX \\ &= (-1)^k\int_{UM}(\nabla_{JX}\tilde{K})(JX, \dots, JX)dX \\ &= (-1)^k\int_{UM}(\nabla_X\tilde{K})(X, \dots, X)dX \\ &= 0\,.\end{aligned}$$

The last equation follows from Lemma 5. Similarly we have

$$\int_{UM}(\nabla_{JX}L)(X, \dots, X)dX = 0\,.$$

Therefore we obtain the equality in Lemma 6. □

PROOF OF THEOREM 1. We define a (2,2)-covariant tensor T on M by

$$T(U, V, \overline{Z}, \overline{W}) = h_{Gr}\left(\sigma(U, V), \sigma(Z, W)\right)\,, \tag{14}$$

where U, V, Z, W are (1,0)-vectors on M. Using the equation of Ricci and the equation of Codazzi, we obtain

$$(\nabla^2 T)(\overline{U}, U, U, U, \overline{U}, \overline{U}) = h_M\big((\nabla^2\sigma)(\overline{U}, U, U, U), \sigma(U, U)\big) + \|(\nabla\sigma)(U, U, U)\|^2 .$$

Using the Ricci identity, we obtain

$$(\nabla^2\sigma)(U, \overline{U}, U, U) - (\nabla^2\sigma)(\overline{U}, U, U, U) = R^N(U, \overline{U})\big(\sigma(U, U)\big) - 2\sigma\big(R^M(U, \overline{U})U, U\big) .$$

It follows from Lemma 4 that

$$(\nabla^2\sigma)(\overline{U}, U, U, U) = -R^N(U, \overline{U})\big(\sigma(U, U)\big) + 2\sigma\big(R^M(U, \overline{U})U, U\big) .$$

Therefore, we obtain

$$\begin{aligned}(\nabla^2 T)(\overline{U}, U, U, U, \overline{U}, \overline{U}) = &-h_{Gr}\big(R^N(U, \overline{U})(\sigma(U, U)), \sigma(U, U)\big)\\ &+ 2h_{Gr}\big(\sigma(R^M(U, \overline{U})U, U), \sigma(U, U)\big)\\ &+ \|(\nabla\sigma)(U, U, U)\|^2 .\end{aligned} \tag{15}$$

From the equation of Ricci and (6), we have

$$\begin{aligned}h_{Gr}\big(R^N(U, \overline{U})(\sigma(U, U)), \sigma(U, U)\big) &= h_{Gr}\big(R^{Gr}(U, \overline{U})(\sigma(U, U)), \sigma(U, U)\big)\\ &\quad + \|A_{\sigma(U,U)}\overline{U}\|^2.\\ &= h_{Gr}(-H_{\sigma(U,U)}K_{\overline{U}}H_U, H_{\sigma(U,U)})\\ &\quad + h_{Gr}(-H_U K_{\overline{U}}H_{\sigma(U,U)}, H_{\sigma(U,U)})\\ &\quad + \|A_{\sigma(U,U)}\overline{U}\|^2 .\end{aligned} \tag{16}$$

In the following calculation, we extend (1,0)-vectors to local holomorphic vector fields if necessary.

LEMMA 7. *For any* $(1,0)$*-vectors* U, V, Z *on* M, *we have*

$$-H_{\sigma(U,Z)}K_{\overline{V}} = \big(\nabla_Z R^{f^*Q}\big)(U, \overline{V}) .$$

PROOF. We have

$$\big(\nabla_Z R^{f^*Q}\big)(U, \overline{V}) = -\nabla_Z(H_U K_{\overline{V}}) + H_{\nabla_Z U}K_{\overline{V}} = -(\nabla_Z H)(U)K_{\overline{V}} .$$

Since we can easily show that $H_{\sigma(U,Z)} = (\nabla_U H)(Z)$, we obtain

$$-H_{\sigma(U,Z)}K_{\overline{V}} = (\nabla_U H)(Z)K_{\overline{V}} = \big(\nabla_Z R^{f^*Q}\big)(U, \overline{V}). \qquad \square$$

It follows from $(*)$ in Section 3 that

$$\begin{aligned}\big(\nabla_Z R^{f^*Q}\big)(U, \overline{V}) &= \nabla_Z^{f^*Q}\big(R^{f^*Q}(U, \overline{V})\big) - R^{f^*Q}(\nabla_Z^M U, \overline{V})\\ &= \frac{1}{q}\nabla_Z^M\big(h_M(U, V)\big)\mathrm{Id}_Q - \frac{1}{q}h_M(\nabla_Z^M U, V)\mathrm{Id}_Q = 0\end{aligned}$$

where U, V, Z are (1,0)-vectors on M. Then it follows from Lemma 7, the equations (10) and (16) that

$$\begin{aligned} h_{Gr}\big(R^N(U,\overline{U})(\sigma(U,U)),\sigma(U,U)\big) &= h_{Gr}(-H_U K_{\overline{U}} H_{\sigma(U,U)}, H_{\sigma(U,U)}) \\ &\quad + \|A_{\sigma(U,U)}\overline{U}\|^2 \\ &= \frac{1}{q}\|\sigma(U,U)\|^2 + \|A_{\sigma(U,U)}\overline{U}\|^2 . \end{aligned} \tag{17}$$

Using the equation of Gauss and the equation (11), we have

$$\begin{aligned} h_{Gr}\big(\sigma(R^M(U,\overline{U})U,U),\sigma(U,U)\big) &= h_{Gr}\big(R^M(U,\overline{U})U, A_{\sigma(U,U)}\overline{U}\big) \\ &= h_{Gr}\big(R^{Gr}(U,\overline{U})U, A_{\sigma(U,U)}\overline{U}\big) - \|A_{\sigma(U,U)}\overline{U}\|^2 \\ &= -2h_{Gr}(H_U K_{\overline{U}} H_U, H_{A_{\sigma(U,U)}\overline{U}}) - \|A_{\sigma(U,U)}\overline{U}\|^2 \\ &= \frac{2}{q}\|\sigma(U,U)\|^2 - \|A_{\sigma(U,U)}\overline{U}\|^2 . \end{aligned} \tag{18}$$

Combining the equations (17) and (18) with (15), we obtain

$$\begin{aligned} (\nabla^2 T)(\overline{U},U,U,U,\overline{U},\overline{U}) &= -\left(\frac{1}{q}\|\sigma(U,U)\|^2 + \|A_{\sigma(U,U)}\overline{U}\|^2\right) \\ &\quad + 2\left(\frac{2}{q}\|\sigma(U,U)\|^2 - \|A_{\sigma(U,U)}\overline{U}\|^2\right) + \|(\nabla\sigma)(U,U,U)\|^2 \\ &= \frac{3}{q}\big(\|\sigma(U,U)\|^2 - q\|A_{\sigma(U,U)}\overline{U}\|^2\big) + \|(\nabla\sigma)(U,U,U)\|^2 . \end{aligned} \tag{19}$$

By integrating both sides of the equation (19) ($U = U_X$), Lemma 6 yields

$$\begin{aligned} \frac{3}{q}\int_{UM} &\big(\|\sigma(U_X,U_X)\|^2 - q\|A_{\sigma(U_X,U_X)}\overline{U_X}\|^2\big)dX \\ &+ \int_{UM}\|(\nabla\sigma)(U_X,U_X,U_X)\|^2 dX = 0 . \end{aligned} \tag{20}$$

From now on we assume that the holomorphic sectional curvature of M is greater than or equal to $\frac{1}{q}$. Let us compute the first term of the left hand side of the equation (20). We define $\xi \in N$ as $\sigma(U,U) = \|\sigma(U,U)\|\xi$. Then we have

$$A_{\sigma(U,U)}\overline{U} = \|\sigma(U,U)\|A_\xi\overline{U} .$$

We denote by τ the involutive anti-holomorphic transformation of the complexification $T_{\mathbf{C}}M$ of TM having TM as the fixed point set. Let $B := A_\xi \circ \tau$. B is an anti-linear transformation and satisfies the following equation:

$$h_{Gr}(BU, V) = h_{Gr}(BV, U)\,, \quad \text{for } U, V \in T_{1,0_x}M\,, \ \ x \in M\,.$$

If we regard B as a real linear transformation on the real vector space with an inner product $\mathfrak{Re}(h_{Gr}(\cdot,\cdot))$, then B is a symmetric transformation. Let λ be the eigenvalue of B whose absolute value is maximum and e the corresponding unit eigenvector. By Cauchy-Schwarz inequality, we have

$$\lambda = h_{Gr}(Be, e) = h_{Gr}(A_\xi \overline{e}, e) = h_{Gr}(\xi, \sigma(e, e)) \leq \|\sigma(e, e)\|\,.$$

It follows from the equation (9), Lemma 3 and the hypothesis that

$$\|A_\xi \overline{U}\|^2 \leq \lambda^2 \leq \|\sigma(e, e)\|^2 \leq \frac{1}{q}\,.$$

It follows that

$$\begin{aligned} \|\sigma(U, U)\|^2 - q\|A_{\sigma(U,U)}\overline{U}\|^2 &= \|\sigma(U, U)\|^2(1 - q\|A_\xi \overline{U}\|^2) \\ &\geq \|\sigma(U, U)\|^2\left(1 - q \cdot \frac{1}{q}\right) = 0\,. \end{aligned}$$

Thus it follows from the equation (20) that

$$\|(\nabla\sigma)(U, U, U)\|^2 = 0\,.$$

Since $\nabla\sigma$ is a symmetric tensor, $\nabla\sigma$ vanishes.

Conversely, we assume that M has parallel second fundamental form. From the equation (9) and Lemmas 3 and 4, it is enough to prove that $\|\sigma(U, U)\|^2 \leq \frac{1}{q}$, where U is an arbitrary unit $(1, 0)$-vector on M. Let T be a $(2, 2)$-covariant tensor on M defined by the equation (14). Since the second fundamental form σ is parallel, T is also parallel and so $\nabla^2 T = 0$. It follows from the equation (19) that

$$\|\sigma(U, U)\|^2 - q\|A_{\sigma(U,U)}\overline{U}\|^2 = 0\,. \tag{21}$$

The Cauchy-Schwarz inequality and the equation (21) imply that

$$\begin{aligned} \|\sigma(U, U)\|^2 &= h_{Gr}\big(\sigma(U, U), \sigma(U, U)\big) = h_{Gr}\big(U, A_{\sigma(U,U)}\overline{U}\big) \\ &\leq \|A_{\sigma(U,U)}\overline{U}\| = \frac{1}{\sqrt{q}}\|\sigma(U, U)\|\,. \end{aligned}$$

Therefore, $\|\sigma(U, U)\|^2 \leq \frac{1}{q}$. □

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Present Addresses:
ISAMI KOGA
GRADUATE SCHOOL OF MATHEMATICS,
KYUSHU UNIVERSITY,
744 MOTOOKA, NISHI-KU, FUKUOKA 819–0366, JAPAN.
e-mail: i-koga@math.kyushu-u.ac.jp

YASUYUKI NAGATOMO
DEPARTMENT OF MATHEMATICS,
MEIJI UNIVERSITY,
HIGASHI-MITA, TAMA-KU, KAWASAKI-SHI, KANAGAWA 214–8571, JAPAN.
e-mail: yasunaga@meiji.ac.jp