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Isomorphism Classes of Modules over Iwasawa Algebra with $\lambda = 4$

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Abstract. We classify the isomorphism classes of finitely generated torsion $\mathcal{O}_E[[T]]$-modules which are free over $\mathcal{O}_E$ of rank 4, where $\mathcal{O}_E$ is the ring of the integers of a local field E. We apply this classification to the Iwasawa module associated to the cyclotomic $\mathbf{Z}_p$-extension of an imaginary quadratic field.

1. Introduction

Let p be a fixed prime number. Let E be a finite extension over the field $\mathbf{Q}_p$ of p-adic numbers and $\mathcal{O}_E$ the ring of integers of E. Let π be a prime element of $\mathcal{O}_E$. We put $\Lambda_E = \mathcal{O}_E[[T]]$ the ring of power series in one variable over $\mathcal{O}_E$. For a distinguished polynomial $f(T) \in \mathcal{O}_E[T]$, we consider finitely generated torsion Λ_E-modules whose characteristic ideals are $(f(T))$, and define the set $\mathcal{M}^E_{f(T)}$ by

$$\mathcal{M}^E_{f(T)} = \left\{ [M] \;\middle|\; \begin{array}{l} M \text{ is a finitely generated torsion } \Lambda_E\text{-module,} \\ \mathrm{char}(M) = (f(T)) \text{ and } M \text{ is free over } \mathcal{O}_E \end{array} \right\}, \tag{1}$$

where $[M]$ denotes the isomorphism class of M as a Λ_E-module. Sumida proved that $\mathcal{M}^E_{f(T)}$ is a finite set if and only if $f(T)$ is separable [12]. The case of $\deg(f(T)) \leq 3$ was treated in [2], [6], [7], [8], [12], and [13]. Sumida and Koike classified $\mathcal{M}^E_{f(T)}$ in the case of $\deg(f(T)) \leq 2$ ([6], Theorem 2.1 and [12], Proposition 10). Kurihara also classified $\mathcal{M}^E_{f(T)}$ in the case of $\deg(f(T)) = 2$, using higher Fitting ideals ([7], Corollary 9.3).

In the previous paper [8], the author classified Λ_E-modules in the case of $\lambda = 3$ and $\mu = 0$ (namely, Λ_E-modules which are free over $\mathcal{O}_E$ of rank 3) and gave numerical examples, applying Theorem 3.5 in [8] to imaginary quadratic fields. In that case, the distinguished

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polynomial $f(T)$ is of the form

$$f(T)=(T-\alpha)(T-\beta)(T-\gamma)\,,$$

where α,β,γ are distinct elements of the maximal ideal of $\mathcal{O}_E$. Using a famous structure theorem of Λ_E-modules (cf. [14], Chapter 13), we regard such a Λ_E-module N as a Λ_E-submodule of $\Lambda_E/(T-\alpha)\oplus\Lambda_E/(T-\beta)\oplus\Lambda_E/(T-\gamma)$. We note that $\Lambda_E/(T-\alpha)\oplus\Lambda_E/(T-\beta)\oplus\Lambda_E/(T-\gamma)$ is an integral closure of $\Lambda_E/(T-\alpha)(T-\beta)(T-\gamma)$. For each isomorphism class $\mathfrak{C}\in\mathcal{M}^E_{f(T)}$, we can take a submodule

$$N(m,n,x):=\langle(1,1,1),(0,\pi^m,x),(0,0,\pi^n)\rangle_{\mathcal{O}_E}$$

of $\Lambda_E/(T-\alpha)\oplus\Lambda_E/(T-\beta)\oplus\Lambda_E/(T-\gamma)$ with $[N(m,n,x)]=\mathfrak{C}$. Here m and n are non-negative integers and x is an element of $\mathcal{O}_E$ and $\langle*\rangle_{\mathcal{O}_E}$ denotes the $\mathcal{O}_E$-submodule generated by $*$. The non-negative integers m and n are determined only by $[N(m,n,x)]$ ([8], Corollary 4.2). Theorem 3.5 in [8] explicitly gives a necessary and sufficient condition for two Λ_E-modules $N(m,n,x)$ and $N(m,n,x')$ to be isomorphic.

In this paper, we consider the case of $\deg(f(T))=4$. More precisely, we treat the case in which

$$f(T)=(T-\alpha)(T-\beta)(T-\gamma)(T-\delta)\,,$$

where α,β,γ, and δ are distinct elements of the maximal ideal of $\mathcal{O}_E$. By the same reason for the case of $\deg(f(T))=3$, for each isomorphism class $\mathfrak{C}\in\mathcal{M}^E_{f(T)}$, we can take a submodule

$$M(\ell,m,n;x,y,z):=\langle(1,1,1,1),(0,\pi^\ell,x,y),(0,0,\pi^m,z),(0,0,0,\pi^n)\rangle_{\mathcal{O}_E}$$

of $\Lambda_E/(T-\alpha)\oplus\Lambda_E/(T-\beta)\oplus\Lambda_E/(T-\gamma)\oplus\Lambda_E/(T-\delta)$ with $[M(\ell,m,n;x,y,z)]=\mathfrak{C}$, where ℓ,m,n are non-negative integers and x,y,z are elements of $\mathcal{O}_E$. We can prove that ℓ,m,n are determined by $\mathfrak{C}$ (see Proposition 1). In Section 2, we define the notion of "admissibility" (see Definition 1). Let $(\ell,m,n;x,y,z)$ be a 6-tuple with $\ell,m,n\in\mathbf{Z}_{\geq 0}$ and $x,y,z\in\mathcal{O}_E$ satisfying the conditions (a), (b), ..., (f) in Lemma 1 in Section 2. We prove that there is an admissible 6-tuple $(\ell,m,n;x,y,z)$ such that $[M]=[M(\ell,m,n;x,y,z)]$ for each $[M]\in\mathcal{M}_{f(T)}$ (see Proposition 4 (2)). By the definition of admissibility of $(\ell,m,n;x,y,z)$, we have $[M(\ell,m,n;x,y,z)]\in\mathcal{M}^E_{f(T)}$ if $(\ell,m,n;x,y,z)$ is admissible (see Proposition 4 (1)).

The following is our main theorem, which gives a necessary and sufficient condition for two Λ_E-modules $M(\ell,m,n;x,y,z)$ and $M(\ell,m,n;x',y',z')$ to be isomorphic:

THEOREM 1. *Let* $(\ell,m,n;x,y,z)$ *and* $(\ell,m,n;x',y',z')$ *be two admissible* 6*-tuples. Suppose that* $\mathrm{ord}_E(x)=\mathrm{ord}_E(x')$ *and* $\mathrm{ord}_E(z)=\mathrm{ord}_E(z')$, *where* ord_E *is the normalized additive valuation on E such that* $\mathrm{ord}_E(\pi)=1$. *Suppose also that* $\mathrm{ord}_E(1-x)=\mathrm{ord}_E(1-x')$ *if* $\ell=0$. *Then the following statements are equivalent*:

(i) *We have* $M(\ell,m,n;x,y,z)\cong M(\ell,m,n;x',y',z')$ *as* Λ_E*-modules.*

(ii) (I), (II), ..., *and* (XII) *hold for* $(\ell, m, n; x, y, z)$ *and* $(\ell, m, n; x', y', z')$.
Here the statements (I), (II), ..., *and* (XII) *are described in Section* 3.

We note that our assumptions $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, and $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$ are necessary conditions for the two modules to be isomorphic (see Proposition 5, Lemma 2).

In Theorem 1, the number of the quantities we have to check is at most 12, because for given two 6-tuples $(\ell, m, n; x, y, z)$, $(\ell, m, n; x', y', z')$, we have only to apply one statement among (I), (II), ..., (XII).

We note that the classification in the case of $\lambda = 4$ is essentially different from that of $\lambda = 3$. In fact, we have to investigate three elements $x, y, z \in \mathcal{O}_E$ to study $M(\ell, m, n; x, y, z)$ in the case of $\lambda = 4$, though in that of $\lambda = 3$, we need only one element $x \in \mathcal{O}_E$ to study $N(m, n, x)$.

In the end of Section 3, we also give an algorithm to determine the isomorphism classes of modules (see Remark 1).

Chase Franks [2] also studies the Λ_E-isomorphism classes. He gave an algorithm to determine whether two Λ_E-modules are isomorphic or not for any separable polynomial $f(T)$ of degree $\lambda \geq 0$. He determined all the elements of $\mathcal{M}^E_{f(T)}$ for a separable distinguished polynomial $f(T)$ with $\deg(f(T)) = 4$ satisfying some conditions ([2], Section 5.3). This algorithm is proceeded by checking whether some matrices he defined belong to $GL_\lambda(\mathcal{O}_E)$, where $\lambda = \deg(f(T))$. In the case of $\lambda = 4$, his method is similar to our method in this paper, but there are some differences, which we will explain here. Let E be a splitting field of $f(T)$ and $\mathcal{O}_E$ the ring of integers of E. He got equations ([2], Section 2.1, Section 5) which are essentially equivalent to our congruence equations in Proposition 3. He did not solve his equations explicitly. He considered a map

$$\varphi_{1,2} : \left(\mathcal{O}_E^\times\right)^4 \longrightarrow \mathrm{GL}_4(E)$$

for Λ_E-modules $M_1 = M(\ell_1, m_1, n_1; x_1, y_1, z_1)$ and $M_2 = M(\ell_2, m_2, n_2; x_2, y_2, z_2)$. This map is defined by $\varphi_{1,2}(u_1, u_2, u_3, u_4) = G_2^{-1}\mathrm{diag}(u_1, u_2, u_3, u_4)G_1$, where $\mathrm{diag}(u_1, u_2, u_3, u_4)$ is the diagonal matrix with $u_1, u_2, u_3, u_4 \in \mathcal{O}_E^\times$ along its diagonal and

$$G_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \pi^{\ell_i} & 0 & 0 \\ 1 & x_i & \pi^{m_i} & 0 \\ 1 & y_i & z_i & \pi^{n_i} \end{pmatrix} \quad \text{for } i = 1, 2 .$$

He proved that $M_1 \cong M_2$ as Λ_E-modules if and only if $\mathrm{im}(\varphi_{1,2}) \cap \mathrm{GL}_4(\mathcal{O}_E) \neq \emptyset$ ([2], Section 2, Theorem 2.1.2), which corresponds to finding units $a_1, a_2, a_3, a_4 \in \mathcal{O}_E^\times$ in our Proposition 2. In order to check this property $\mathrm{im}(\varphi_{1,2}) \cap \mathrm{GL}_4(\mathcal{O}_E) \neq \emptyset$, he took some finite set $S \subset (\mathcal{O}_E^\times)^4$ and reduced this property to $\varphi_{1,2}(S) \cap \mathrm{GL}_4(\mathcal{O}_E) \neq \emptyset$ ([2], Section 5.2, Theorem 5.2.1). Consequently, he gave an algorithm ([2], Section 5.3) which is proceeded by checking the property above for all elements in S. It is a merit of his algorithm to work for arbitrary λ and separable polynomial $f(T)$. Our algorithm is different from his and more explicit. The key to get our main theorem is to solve our congruence equations in Proposition 3 completely and to give a necessary and sufficient condition whether the roots of our congruence equations exist in $\mathcal{O}_E$ or not. The merit of our method is as follows. First, we reduce the problem to checking the p-adic valuations of the quantities obtained from modules $M(\ell, m, n; x, y, z)$, $M(\ell, m, n; x', y', z')$ (see (I), (II), ..., and (XII) in Section 3). Furthermore, the number of the quantities we have to check is at most 12 for each statement. On the other hand, the number of S does not have a good upper bound (at least $\sharp S \leq p^{\ell+m+n}$ in the case of $E = \mathbf{Q}_p$). Thus, we get a complete algorithm, see Table 1 in Remark 1.

The outline of this paper is as follows. In Section 2, we prepare some notation and introduce the notion of admissibility of a 6-tuple $(\ell, m, n; x, y, z)$. In Section 3, we describe the statements (I), (II), ..., and (XII). In Section 4, we prove our main theorem. Applying Theorem 1, we determine in Corollary 1 the number of the isomorphism classes of $\mathcal{M}^E_{f(T)}$ in the case of $E = \mathbf{Q}_p$ and $\mathrm{ord}_p(\alpha - \beta) = \mathrm{ord}_p(\beta - \gamma) = \mathrm{ord}_p(\gamma - \delta) = \mathrm{ord}_p(\delta - \alpha) = \mathrm{ord}_p(\beta - \delta) = \mathrm{ord}_p(\alpha - \gamma)=1$, where we write ord_p for $\mathrm{ord}_{\mathbf{Q}_p}$. In Section 5, we determine the isomorphism classes of Iwasawa modules associated to the cyclotomic $\mathbf{Z}_3$-extension of imaginary quadratic fields for $\mathbf{Q}(\sqrt{-12453})$ and $\mathbf{Q}(\sqrt{-78730})$.

2. Preliminaries

As in Introduction, let p be a prime number. Let E be a finite extension over the field $\mathbf{Q}_p$ of p-adic numbers. Let $\mathcal{O}_E$, π, and ord_E be the ring of integers in E, a prime element, and the normalized additive valuation on E such that $\mathrm{ord}_E(\pi) = 1$, respectively. We put $\Lambda_E := \mathcal{O}_E[[T]]$ the ring of power series over $\mathcal{O}_E$.

Let M be a finitely generated torsion Λ_E-module. By the structure theorem of Λ_E-modules, there is a Λ_E-homomorphism

$$\varphi : M \longrightarrow \left(\bigoplus_i \Lambda_E / \left(\pi^{m_i}\right)\right) \oplus \left(\bigoplus_j \Lambda_E / \left(f_j(T)^{n_j}\right)\right)$$

with finite kernel and finite cokernel, where m_i, n_j are non-negative integers and $f_j(T) \in \mathcal{O}_E[T]$ is a distinguished irreducible polynomial. We put

$$\mathrm{char}(M) = \left(\prod_i \pi^{m_i} \prod_j f_j(T)^{n_j}\right)$$

which is an ideal in Λ_E. We denote the Λ_E-isomorphism class of M by $[M]_E$ or $[M]$.

For a distinguished polynomial $f(T) \in \mathcal{O}_E[T]$, we consider finitely generated torsion

Λ_E-modules whose characteristic ideals are $(f(T))$, and define the set $\mathcal{M}^E_{f(T)}$ by

$$\mathcal{M}^E_{f(T)} = \left\{ [M]_E \,\middle|\, \begin{array}{l} M \text{ is a finitely generated torsion } \Lambda_E\text{-module}, \\ \mathrm{char}(M) = (f(T)) \text{ and } M \text{ is free over } \mathcal{O}_E \end{array} \right\}. \tag{2}$$

Now we consider

$$f(T) = (T-\alpha)(T-\beta)(T-\gamma)(T-\delta), \tag{3}$$

where α, β, γ, and δ are distinct elements of $\pi\mathcal{O}_E$. We classify all the elements of $\mathcal{M}^E_{f(T)}$ in the next section.

Let $[M]_E \in \mathcal{M}^E_{f(T)}$. As in Introduction, we may regard the Λ_E-module M as a Λ-submodule of $\Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) \oplus \Lambda_E/(T-\delta)$. Namely, since M has no non-trivial finite Λ_E-submodule, there exists an injective Λ_E-homomorphism

$$\varphi : M \hookrightarrow \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) \oplus \Lambda_E/(T-\delta) =: \mathcal{E}$$

with finite cokernel. We write $\mathcal{E}$ for the right hand side.

Now we fix notation to express such submodules in $\mathcal{E}$. First, by using the canonical isomorphism $\Lambda_E/(T-\alpha) \cong \mathcal{O}_E$ $(f(T) \longmapsto f(\alpha))$, we define an isomorphism

$$\iota : \mathcal{E} = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) \oplus \Lambda_E/(T-\delta) \longrightarrow \mathcal{O}_E^{\oplus 4}$$

by $(f_1(T), f_2(T), f_3(T), f_4(T)) \longmapsto (f_1(\alpha), f_2(\beta), f_3(\gamma), f_4(\delta))$. We identify $\mathcal{E}$ with $\mathcal{O}_E^{\oplus 4}$ via ι. Thus an element in $\mathcal{E}$ is expressed as $(a_1, a_2, a_3, a_4) \in \mathcal{O}_E^{\oplus 4}$. Since the rank of M is equal to 4, we can write M of the form

$$M = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4), (d_1, d_2, d_3, d_4) \rangle_{\mathcal{O}_E} \subset \mathcal{E},$$

where $\langle * \rangle_{\mathcal{O}_E}$ is the $\mathcal{O}_E$-submodule generated by $*$. Further, using this notation, we can express the action of T by

$$T(a_1, a_2, a_3, a_4) = (\alpha a_1, \beta a_2, \gamma a_3, \delta a_4).$$

Let M be an $\mathcal{O}_E$-submodule of $\mathcal{E}$ with $\mathrm{rank}(M) = 4$.

$$M = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4), (d_1, d_2, d_3, d_4) \rangle_{\mathcal{O}_E} \subset \mathcal{E}.$$

By the same method as [8], we have

$$M = \langle (\pi^s, a, b, c), (0, \pi^t, d, e), (0, 0, \pi^u, f), (0, 0, 0, \pi^v) \rangle_{\mathcal{O}_E}$$

for some non-negative integers s, t, u, v and $a, b, c, d, e, f \in \mathcal{O}_E$. Further, by Lemma 1 in [13], we may assume that a Λ_E-module M is of the form

$$M = \langle (1, 1, 1, 1), (0, \pi^\ell, x, y), (0, 0, \pi^m, z), (0, 0, 0, \pi^n) \rangle_{\mathcal{O}_E} \subset \mathcal{E}$$

for some non-negative integers ℓ, m, n and $x, y, z \in \mathcal{O}_E$. We define an $\mathcal{O}_E$-module M by

$$M(\ell, m, n; x, y, z) := \langle (1,1,1,1), (0, \pi^{\ell}, x, y), (0,0,\pi^m, z), (0,0,0,\pi^n)\rangle_{\mathcal{O}_E} \subset \mathcal{E},$$

where ℓ, m and n are non-negative integers. We can prove the next lemma by the same method as Lemma 3.1 in [8].

LEMMA 1. *The following two statements are equivalent*:
(i) *The $\mathcal{O}_E$-module $M(\ell, m, n; x, y, z)$ is a Λ_E-submodule.*
(ii) *The integers ℓ, m, n and $x, y, z \in \mathcal{O}_E$ satisfy*

$$\begin{cases} \text{(a)} \quad \ell \leq \mathrm{ord}_E(\beta-\alpha), \\ \text{(b)} \quad m \leq \mathrm{ord}_E\{(\gamma-\alpha)-(\beta-\alpha)\pi^{-\ell}x\}, \\ \text{(c)} \quad n \leq \mathrm{ord}_E\left[(\delta-\alpha)-(\beta-\alpha)\pi^{-\ell}y-\{(\gamma-\alpha)-(\beta-\alpha)\pi^{-\ell}x\}\pi^{-m}z\right], \\ \text{(d)} \quad m \leq \mathrm{ord}_E(\gamma-\beta)+\mathrm{ord}_E(x), \\ \text{(e)} \quad n \leq \mathrm{ord}_E\{(\delta-\beta)y-(\gamma-\beta)x\pi^{-m}z\}, \\ \text{(f)} \quad n \leq \mathrm{ord}_E(\delta-\gamma)+\mathrm{ord}_E(z). \end{cases}$$

PROPOSITION 1. *Let $[M]_E, [M']_E \in \mathcal{M}^E_{f(T)}$, $M = M(\ell, m, n; x, y, z)$, and $M' = M(\ell', m', n'; x', y', z')$. If $[M]_E = [M']_E$, then we have $\ell = \ell'$, $m = m'$ and $n = n'$.*

PROOF. For any Λ-module M and $\xi \in \Lambda_E$, we define a map $\Pi_\xi = \Pi^M_\xi : M \longrightarrow M$ by $\Pi_\xi(y) = \xi y$. Then we have

$$\sharp\left(\mathrm{Ker}\left(\Pi^M_{(T-\alpha)}\right)/\mathrm{Im}\left(\Pi^M_{(T-\beta)}\right)\right) = q^{\{\mathrm{ord}_E(\delta-\alpha)+\mathrm{ord}_E(\delta-\beta)+\mathrm{ord}_E(\delta-\gamma)-n\}},$$

$$\sharp\left(\mathrm{Ker}\left(\Pi^M_{(T-\gamma)}\right)/\mathrm{Im}\left(\Pi^M_{(T-\alpha)(T-\beta)(T-\delta)}\right)\right) = q^{\{\mathrm{ord}_E(\gamma-\alpha)+\mathrm{ord}_E(\gamma-\beta)+\mathrm{ord}_E(\gamma-\delta)-m\}}.$$

We put $N = \mathrm{Im}\left(\Pi^M_{(T-\gamma)(T-\delta)}\right)$. Then we have

$$\sharp\left(\mathrm{Ker}\left(\Pi^N_{(T-\beta)}\right)/\mathrm{Im}\left(\Pi^N_{(T-\alpha)}\right)\right) = q^{\{\mathrm{ord}_E(\beta-\alpha)-\ell\}}.$$

Since $M \cong M'$, we have $\mathrm{Ker}\left(\Pi^M_{(T-\gamma)}\right) \cong \mathrm{Ker}\left(\Pi^{M'}_{(T-\gamma)}\right)$ and $\mathrm{Im}\left(\Pi^M_{(T-\alpha)(T-\beta)(T-\delta)}\right) \cong \mathrm{Im}\left(\Pi^{M'}_{(T-\alpha)(T-\beta)(T-\delta)}\right)$. This implies $m = m'$. We get $\ell = \ell'$ and $n = n'$ by the same method. $\square$

For $M = M(\ell, m, n; x, y, z)$, we put $e_1 = (1,1,1,1)$, $e_2 = (0, \pi^\ell, x, y)$, $e_3 = (0,0,\pi^m, z)$, $e_4 = (0,0,0,\pi^n)$. For $M' = M(\ell, m, n; x', y', z')$, we also put $e_1{}' = (1,1,1,1)$, $e_2{}' =$

$(0, \pi^{\ell}, x', y')$, $e_3{}' = (0, 0, \pi^m, z')$, $e_4{}' = (0, 0, 0, \pi^n)$ and

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \pi^{\ell} & 0 & 0 \\ 1 & x & \pi^m & 0 \\ 1 & y & z & \pi^n \end{pmatrix}, \quad G' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \pi^{\ell} & 0 & 0 \\ 1 & x' & \pi^m & 0 \\ 1 & y' & z' & \pi^n \end{pmatrix}.$$

The matrix G is the transition matrix from the basis e_1, e_2, e_3, e_4 to the basis $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$. The matrix G' is the transition matrix from the basis $e_1{}', e_2{}', e_3{}', e_4{}'$ to the basis $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$. Let $g : M \longrightarrow M'$ be a Λ_E-isomorphism. Since we have $g(Tx) = Tg(x)$ for $x \in M$ and $T(1,0,0,0) = (\alpha,0,0,0), T(0,1,0,0) = (0,\beta,0,0), T(0,0,1,0) = (0,0,\gamma,0)$, $T(0,0,0,1) = (0,0,0,\delta)$, we can prove the next proposition by the same method as Proposition 4.3 in [8].

PROPOSITION 2. *Let $M = M(\ell, m, n;\ x, y, z)$ and $M' = M(\ell, m, n; x', y', z')$ be Λ_E-modules satisfying $[M]_E, [M']_E \in \mathcal{M}^E_{f(T)}$. Assume that $g : M \longrightarrow M'$ is a Λ_E-isomorphism. We take $\langle e_1, e_2, e_3, e_4\rangle$ and $\langle e_1{}', e_2{}', e_3{}', e_4{}'\rangle$ as a basis of M and that of M', respectively. Let A be the matrix corresponding to g with respect to the basis $\langle e_1, e_2, e_3, e_4\rangle$ and the basis $\langle e_1{}', e_2{}', e_3{}', e_4{}'\rangle$. Then we have*

$$G'AG^{-1} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

for some $a_1, a_2, a_3, a_4 \in \mathcal{O}_E^{\times}$.

Let $A = (a_{ij})$, $1 \le i, j \le 4$. Using this proposition, we have $a_{ii} = a_i$ for $i = 1, 2, 3, 4$ and $a_{ij} = 0$ for $i < j$. Since we have $a_{ij} \in \mathcal{O}_E$ for $i > j$, we get the following proposition (cf. Proposition 4.5 and Lemma 4.6 in [8] and Lemma 2.1.2 in [2]). We note that we write a_1, a_2, a_3 for $\frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4}$, respectively, in the following proposition.

PROPOSITION 3. *Let $[M]_E, [M']_E \in \mathcal{M}^E_{f(T)}$, $M = M(\ell, m, n; x, y, z)$, and $M' = M(\ell, m, n, x', y', z')$. Then the following two statements are equivalent*:

(i) *We have $M \cong M'$ as Λ_E-modules.*

(ii) *There exist $a_1, a_2, a_3 \in \mathcal{O}_E^{\times}$ satisfying*

$$a_2 \equiv a_1 \mod \pi^{\ell}, \tag{4}$$

$$a_3 - a_1 - (a_2 - a_1)\pi^{-\ell}x' \equiv 0 \mod \pi^m, \tag{5}$$

$$1 - a_1 - (a_2 - a_1)\pi^{-\ell}y' - \{a_3 - a_1 - (a_2 - a_1)\pi^{-\ell}x'\}\pi^{-m}z' \equiv 0 \mod \pi^n, \tag{6}$$

$$a_3 x \equiv a_2 x' \mod \pi^m, \tag{7}$$

$$y - a_2 y' - (a_3 x - a_2 x')\pi^{-m} z' \equiv 0 \mod \pi^n , \tag{8}$$

$$z \equiv a_3 z' \mod \pi^n . \tag{9}$$

Let R be a set of complete representatives in $\mathcal{O}_E$ of the elements of the residue field $\mathcal{O}_E/(\pi)$. Namely, R is a subset of $\mathcal{O}_E$ and each class of $\mathcal{O}_E/(\pi)$ contains a unique element in R. We assume that R contains 0, 1 and fix this complete representatives R. For non-negative integers k, we set

$$S_k = \left\{ \sum_{i=0}^{k-1} a_i \pi^i \;\middle|\; a_i \in R \text{ for } i = 0, 1, \ldots, k-1 \right\} \quad \text{if } k > 0 ,$$
$$S_0 = \{0\} \quad \text{if } k = 0 .$$

DEFINITION 1. Let $(\ell, m, n; x, y, z)$ be a 6-tuple with $\ell, m, n \in \mathbf{Z}_{\geq 0}$ and $x, y, z \in \mathcal{O}_E$ satisfying the conditions (a), (b), ..., (f) in Lemma 1. We call a 6-tuple $(\ell, m, n; x, y, z)$ admissible if $x \in S_m$ and $y, z \in S_n$.

PROPOSITION 4. (1) *If a 6-tuple $(\ell, m, n; x, y, z)$ is admissible, then $M(\ell, m, n; x, y, z)$ becomes a Λ_E-module and $[M(\ell, m, n; x, y, z)] \in \mathcal{M}^E_{f(T)}$.*
(2) *Suppose that $[M] \in \mathcal{M}^E_{f(T)}$. Then there is an admissible 6-tuple $(\ell, m, n; x, y, z)$ such that $[M] = [M(\ell, m, n; x, y, z)]$.*

PROOF. (1) This follows from Lemma 1.

(2) We suppose that $[M] \in \mathcal{M}^E_{f(T)}$. Then, as we explained before Lemma 1, we can take $M(\ell, m, n; x', y', z')$ such that $[M] = [M(\ell, m, n; x', y', z')]$, where $\ell, m, n \geq 0$ and $x', y', z' \in \mathcal{O}_E$. We choose $x \in S_m$ and $y, z \in S_n$ satisfying $x' \equiv x \mod \pi^m$, $y' + (x - x')\pi^{-m} z' \equiv y \mod \pi^n$ and $z' \equiv z \mod \pi^n$. Then $(\ell, m, n; x, y, z)$ is admissible. Put $a_1 = a_2 = a_3 = 1$. Then equations (4), (5), (6), (7), (8), (9) hold. By Proposition 3, we have $[M] = [M(\ell, m, n; x', y', z')] = [M(\ell, m, n; x, y, z)]$. Thus we get (2). □

The next theorem is the main theorem of our previous paper, which will be used in the proof of our main theorem. We consider

$$g(T) = (T - \alpha)(T - \beta)(T - \gamma) ,$$

where α, β, γ are distinct elements of the maximal ideal of $\mathcal{O}_E$. As in Introduction, we write $N(m, n, x) = \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E} \subset \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta) \oplus \Lambda_E/(T - \gamma)$.

THEOREM 2 (Theorem 3.5 in [8]). *Let $[N(m, n, x)]$ and $[N(m, n, x')] \in \mathcal{M}^E_{g(T)}$. Suppose that* $\mathrm{ord}_E(x) < n$ *or* $x = 0$ *and that* $\mathrm{ord}_E(x') < n$ *or* $x' = 0$. *Then the following are equivalent*:

(i) *We have $N(m, n, x) \cong N(m, n, x')$ as Λ_E-modules.*

(ii) *Either* (I′), (II′), *or* (III′) *holds*, *where* (I′), (II′), *and* (III′) *are*

(I′) $m \neq 0,\ x' \neq 0$ and $\min\left\{\mathrm{ord}_E\left(\dfrac{\pi^n}{x'}\right), \mathrm{ord}_E(\pi^m - x')\right\} \leq \mathrm{ord}_E\left(\dfrac{x}{x'} - 1\right)$,

(II′) $x' = 0$,

(III′) $m = 0$ and $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$,

and $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ *holds.*

□

3. The statements (I) - (XII)

In this section, we describe the statements (I), (II), . . . , and (XII) in Theorem 1. For two 6-tuples $(\ell, m, n; x, y, z)$, $(\ell, m, n; x', y', z')$, we set the following quantities. If $x' \neq 0$, $z' \neq 0$, we put

$$\begin{aligned}
&A = \frac{\pi^n}{z'}\frac{x}{x'}y', \quad B = \frac{\pi^m}{x'}y' - z',\\
&C = -y + \frac{z}{z'}\frac{x}{x'}y', \quad D = x' - y',\\
&E = \pi^m - z', \qquad F = \pi^\ell - x' + (x' - y')\Big(1 - \frac{x}{x'}\Big),\\
&G = -\pi^m + (\pi^m - z')\Big(1 - \frac{x}{x'}\Big).
\end{aligned}$$

(I) If $x' \neq 0$, $z' \neq 0$, and $\mathrm{ord}_E(A) \leq \mathrm{ord}_E(B)$, then either the following (I-1), (I-2), or (I-3) hold.

(I-1) All of the following (I-1-a), (I-1-b), (I-1-c), and (I-1-d) are satisfied.

(I-1-a) $\min\left\{\mathrm{ord}_E\left(\dfrac{\pi^m}{x'}\right), \mathrm{ord}_E(F), \mathrm{ord}_E(G)\right\} = \mathrm{ord}_E\left(\dfrac{\pi^m}{x'}\right)$,

(I-1-b) $\mathrm{ord}_E(A) \leq \mathrm{ord}_E(C)$,

(I-1-c) $x = x'$,

(I-1-d) $\min\left\{\mathrm{ord}_E\left(D + \dfrac{x'}{z'}A^{-1}BF\pi^{n-m}\right), \mathrm{ord}_E\left(E + \dfrac{x'}{z'}A^{-1}BG\pi^{n-m}\right),\right.$
$\left.\mathrm{ord}_E\left(\dfrac{\pi^n}{y'}\right)\right\} \leq \mathrm{ord}_E\left(1 - \dfrac{y}{y'}\right)$.

(I-2) All of the following (I-2-a), (I-2-b), (I-2-c), and (I-2-d) are satisfied.

(I-2-a) $\min\left\{\mathrm{ord}_E\left(\dfrac{\pi^m}{x'}\right), \mathrm{ord}_E(F), \mathrm{ord}_E(G)\right\} = \mathrm{ord}_E(F)$, (I-2-b) $\operatorname{ord}_E(A) \le \operatorname{ord}_E(C)$,

(I-2-c) $\operatorname{ord}_E(F) \le \operatorname{ord}_E\left(1 - \frac{x}{x'}\right)$,

(I-2-d)
$$\min\left\{\operatorname{ord}_E\left(A^{-1}B\frac{\pi^n}{z'} + \frac{\pi^m}{x'}DF^{-1}\right), \operatorname{ord}_E(E - DF^{-1}G), \operatorname{ord}_E\left(\frac{\pi^n}{y'}\right)\right\}$$
$$\le \operatorname{ord}_E\left(\frac{z}{z'} - 1 - A^{-1}C\frac{\pi^n}{z'} - \left(\frac{x}{x'} - 1\right)DF^{-1}\right).$$

(I-3) All of the following (I-3-a), (I-3-b), (I-3-c), and (I-3-d) are satisfied.

(I-3-a) $\min\left\{\operatorname{ord}_E\left(\frac{\pi^m}{x'}\right), \operatorname{ord}_E(F), \operatorname{ord}_E(G)\right\} = \operatorname{ord}_E(G)$,

(I-3-b) $\operatorname{ord}_E(A) \le \operatorname{ord}_E(C)$,

(I-3-c) $\operatorname{ord}_E(G) \le \operatorname{ord}_E\left(1 - \frac{x}{x'}\right)$,

(I-3-d)
$$\min\left\{\operatorname{ord}_E\left(A^{-1}B\frac{\pi^n}{z'} + \frac{\pi^m}{x'}EG^{-1}\right), \operatorname{ord}_E(D - EFG^{-1}), \operatorname{ord}_E\left(\frac{\pi^n}{y'}\right)\right\}$$
$$\le \operatorname{ord}_E\left(\frac{z}{z'} - 1 - A^{-1}C\frac{\pi^n}{z'} - \left(\frac{x}{x'} - 1\right)EG^{-1}\right).$$

(II) If $x' \neq 0, z' \neq 0$, and $\operatorname{ord}_E(A) > \operatorname{ord}_E(B)$, then either the following (II-1), (II-2), or (II-3) holds.

(II-1) All of the following (II-1-a), (II-1-b), (II-1-c), and (II-1-d) are satisfied.

(II-1-a) $\min\left\{\operatorname{ord}_E\left(\frac{\pi^n}{z'}\right), \operatorname{ord}_E(D), \operatorname{ord}_E(E)\right\} = \operatorname{ord}_E\left(\frac{\pi^n}{z'}\right)$,

(II-1-b) $\operatorname{ord}_E(B) \le \operatorname{ord}_E(C)$,

(II-1-c) $z = z'$,

(II-1-d)
$$\min\left\{\operatorname{ord}_E\left(F + \frac{z'}{x'}AB^{-1}D\pi^{m-n}\right), \operatorname{ord}_E\left(G + \frac{z'}{x'}AB^{-1}E\pi^{m-n}\right),\right.$$
$$\left.\operatorname{ord}_E\left(\pi^n\left(1 - \frac{x}{x'}\right) + z'AB^{-1}\frac{\pi^m}{x'}\right), n + m - \operatorname{ord}_E(Bx')\right\}$$
$$\le \operatorname{ord}_E\left(\frac{x}{x'} - 1 - B^{-1}C\frac{\pi^m}{x'}\right).$$

(II-2) All of the following (II-2-a), (II-2-b), (II-2-c), and (II-2-d) are satisfied.

(II-2-a) $\min\left\{\operatorname{ord}_E\left(\frac{\pi^n}{z'}\right), \operatorname{ord}_E(D), \operatorname{ord}_E(E)\right\} = \operatorname{ord}_E(D)$,

(II-2-b) $\operatorname{ord}_E(B) \le \operatorname{ord}_E(C)$, (II-2-c) $\mathrm{ord}_E(D) \leq \mathrm{ord}_E\left(1 - \frac{z}{z'}\right)$,

(II-2-d) $$\min\left\{\mathrm{ord}_E\left(AB^{-1}\frac{\pi^m}{x'} + \frac{\pi^n}{z'}D^{-1}F\right), \mathrm{ord}_E(G - D^{-1}EF),\right.$$
$$\left. n + \mathrm{ord}_E\left(-\left(1 - \frac{x}{x'}\right) + D^{-1}F\right), n + m - \mathrm{ord}_E(Bx')\right\}$$
$$\leq \mathrm{ord}_E\left(\frac{x}{x'} - 1 - B^{-1}C\frac{\pi^m}{x'} - \left(\frac{z}{z'} - 1\right)D^{-1}F\right).$$

(II-3) All of the following (II-3-a), (II-3-b), (II-3-c), and (II-3-d) are satisfied.

(II-3-a) $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(D), \mathrm{ord}_E(E)\right\} = \mathrm{ord}_E(E)$,

(II-3-b) $\mathrm{ord}_E(B) \leq \mathrm{ord}_E(C)$,

(II-3-c) $\mathrm{ord}_E(E) \leq \mathrm{ord}_E\left(1 - \frac{z}{z'}\right)$,

(II-3-d) $$\min\left\{\mathrm{ord}_E\left(AB^{-1}\frac{\pi^m}{x'} + \frac{\pi^n}{z'}E^{-1}G\right), \mathrm{ord}_E(F - DE^{-1}G),\right.$$
$$\left. n + \mathrm{ord}_E\left(-\left(1 - \frac{x}{x'}\right) + E^{-1}G\right), n + m - \mathrm{ord}_E(Bx')\right\}$$
$$\leq \mathrm{ord}_E\left(\frac{x}{x'} - 1 - CB^{-1}\frac{\pi^m}{x'} - \left(\frac{z}{z'} - 1\right)E^{-1}G\right).$$

(III) If $\ell \neq 0$, $m \neq 0$, and $n = 0$, then the following (III-a) holds.

(III-a) $$\min\left\{\mathrm{ord}_E\left(\frac{\pi^m}{x'}\right), \mathrm{ord}_E(\pi^\ell - x')\right\} \leq \mathrm{ord}_E\left(\frac{x}{x'} - 1\right) \quad \text{if } x' \neq 0.$$

(IV) If $\ell \neq 0$ and $m = 0$, then either the following (IV-1), (IV-2), or (IV-3) holds.

(IV-1) All of the following (IV-1-a), (IV-1-b), and (IV-1-c) are satisfied.

(IV-1-a) $y' \neq 0$ and $z' \neq 0$,

(IV-1-b) $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$,

(IV-1-c) $$\min\left\{n, \mathrm{ord}_E\left((1 - z')\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell(1 - z') - y')\right\}$$
$$\leq \mathrm{ord}_E\left(z - 1 - (z' - 1)\frac{y}{y'}\right).$$

(IV-2) All of the following (IV-2-a), (IV-2-b), and (IV-2-c) are satisfied.

(IV-2-a) $y' \neq 0$ and $z' = 0$,

(IV-2-b) $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$,

(IV-2-c) $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y')\right\} \le \mathrm{ord}_E\left(\frac{y}{y'} - 1\right)$.

(IV-3) All of the following (IV-3-a) and (IV-3-b) are satisfied.

(IV-3-a) $y' = y = 0$,

(IV-3-b) $\mathrm{ord}_E(1 - z) = \mathrm{ord}_E(1 - z')$.

(V) If $\ell \neq 0, m \neq 0, n \neq 0$, and $z' = 0$, then either the following (V-1), (V-2), (V-3), (V-4), or (V-5) holds.

(V-1) All of the following (V-1-a), (V-1-b), and (V-1-c) are satisfied.

(V-1-a) $x' \neq 0, y' \neq 0$ and $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y')\right\} = \mathrm{ord}_E\left(\frac{\pi^n}{y'}\right)$,

(V-1-b) $y = y'$,

(V-1-c) $\min\left\{\mathrm{ord}_E\left(\frac{\pi^m}{x}\right), \mathrm{ord}_E\left(\pi^\ell - x' - \frac{x'}{x} - (\pi^\ell - y')\left(1 - \frac{x'}{x}\right)\right)\right\} \le \mathrm{ord}_E\left(1 - \frac{x'}{x}\right)$.

(V-2) All of the following (V-2-a), (V-2-b), (V-2-c), and (V-2-d) are satisfied.

(V-2-a) $x' \neq 0, y' \neq 0$ and $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y')\right\} = \mathrm{ord}_E(\pi^\ell - y')$,

(V-2-b) $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$,

(V-2-c) $\mathrm{ord}_E(\pi^\ell - y') \le \mathrm{ord}_E\left(\frac{y}{y'} - 1\right)$,

(V-2-d) $$\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\left(1 - \frac{x'}{x}\right) - \frac{\pi^n}{y'}\frac{\pi^\ell - x' - x'x^{-1}}{\pi^\ell - y'}\right), \mathrm{ord}_E\left(\frac{\pi^n(\pi^\ell - x' - x'x^{-1})}{\pi^\ell - y'}\right),\right.$$
$$\left.\mathrm{ord}_E\left(\frac{\pi^m}{x}\right)\right\} \le \mathrm{ord}_E\left(\frac{y}{y'}\left(1 - \frac{x}{x'}\right) - \left(\frac{y}{y'} - 1\right)\frac{\pi^\ell - x' - x'x^{-1}}{\pi^\ell - y'}\right).$$

(V-3) All of the following (V-3-a), (V-3-b), and (V-3-c) are satisfied.

(V-3-a) $x' \neq 0$ and $y' = 0$,

(V-3-b) $y = 0$,

(V-3-c) $\min\left\{\mathrm{ord}_E\left(\frac{\pi^m}{x}\right), \mathrm{ord}_E(\pi^\ell - x)\right\} \le \mathrm{ord}_E\left(1 - \frac{x'}{x}\right)$.

(V-4) All of the following (V-4-a), (V-4-b), and (V-4-c) are satisfied.

(V-4-a) $x' = 0$ and $y' \neq 0$,

(V-4-b) $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$,

(V-4-c) $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y')\right\} \leq \mathrm{ord}_E\left(1 - \frac{y}{y'}\right)$.

(V-5) The following is satisfied.

$$x' = x = 0 \ \text{ and } \ y = y' = 0 .$$

(VI) If $\ell \neq 0$, $m \neq 0$, $x' = 0$, and $z' \neq 0$, then either the following (VI-1), (VI-2), (VI-3), or (VI-4) holds.

(VI-1) All of the following (VI-1-a), (VI-1-b), and (VI-1-c) are satisfied.

(VI-1-a) $y' \neq 0$ and $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y'), \mathrm{ord}_E(z')\right\} = \mathrm{ord}_E\left(\frac{\pi^n}{y'}\right)$,

(VI-1-b) $y = y'$,

(VI-1-c) $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(y'), \mathrm{ord}_E(\pi^m - z')\right\} \leq \mathrm{ord}_E\left(1 - \frac{z}{z'}\right)$.

(VI-2) All of the following (VI-2-a), (VI-2-b), (VI-2-c), and (VI-2-d) are satisfied.

(VI-2-a) $y' \neq 0$ and $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y'), \mathrm{ord}_E(z')\right\} = \mathrm{ord}_E(\pi^\ell - y')$,

(VI-2-b) $\mathrm{ord}_E(\pi^\ell - y') \leq \mathrm{ord}_E\left(\frac{y}{y'} - 1\right)$,

(VI-2-c) $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E\left(\pi^m - \frac{z'\pi^\ell}{\pi^\ell - y'}\right)\right\} \leq \mathrm{ord}_E\left(\frac{z}{z'} - 1 + \frac{y - y'}{\pi^\ell - y'}\right)$,

(VI-2-d) $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$.

(VI-3) All of the following (VI-3-a), (VI-3-b), (VI-3-c), and (VI-3-d) are satisfied.

(VI-3-a) $y' \neq 0$ and $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y'), \mathrm{ord}_E(z')\right\} = \mathrm{ord}_E(z')$,

(VI-3-b) $\mathrm{ord}_E(z') \leq \mathrm{ord}_E\left(\frac{y}{y'} - 1\right)$,

(VI-3-c) $\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E\left(\frac{\pi^n}{y'}\frac{1}{z'}(\pi^m - z')\right), \mathrm{ord}_E\left(-y' + (\pi^\ell - y')\frac{1}{z'}(\pi^m - z')\right)\right\}$

$\leq \mathrm{ord}_E\left(\frac{z}{z'} - 1 + \left(\frac{y}{y'} - 1\right)\frac{1}{z'}(\pi^m - z')\right)$,

(VI-3-d) $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$.

(VI-4) All of the following (VI-4-a) and (VI-4-b) are satisfied.

$$\text{(VI-4-a)}\quad y = y' = 0\,,$$
$$\text{(VI-4-b)}\quad \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(\pi^m - z')\right\} \le \mathrm{ord}_E\left(\frac{z}{z'} - 1\right).$$

(VII) If $\ell = 0, m \neq 0, n \neq 0, x' \neq 0, 1, y' \neq 0$, and $z' = 0$, then the following (VII-a) and (VII-b) hold.

$$\text{(VII-a)}\quad \mathrm{ord}_E(y) = \mathrm{ord}_E(y'),\quad \mathrm{ord}_E(1-y) = \mathrm{ord}_E(1-y')\,,$$
$$\text{(VII-b)}\quad \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}(1-y')\right), \mathrm{ord}_E\left(\frac{\pi^m}{x}(1-y')\right), \mathrm{ord}_E\left(\frac{\pi^m}{1-x'}(1-y')\right), n\right\}$$
$$\le \mathrm{ord}_E\left(1 - y - \frac{y}{y'}\frac{x'}{x}\frac{1-x}{1-x'}(1-y')\right).$$

(VIII) If $\ell = 0, m \neq 0, n \neq 0, x' \neq 0, 1, y' = 0$, and $z' = 0$, then the following holds.

$$\text{(VIII-a)}\quad y = 0\,.$$

(IX) If $\ell = 0, m \neq 0, n \neq 0$, and $x' = 0$, then either the following (IX-1), (IX-2), (IX-3), or (IX-4) holds.
(IX-1) All of the following (IX-1-a), (IX-1-b), and (IX-1-c) are satisfied.

$$\text{(IX-1-a)}\quad y' \neq 0 \text{ and } z' \neq 0\,,$$
$$\text{(IX-1-b)}\quad \mathrm{ord}_E(y) = \mathrm{ord}_E(y')\,,$$
$$\text{(IX-1-c)}\quad \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}(1-y')\right), n, \mathrm{ord}_E\left(\pi^m(1-y') - z'\right)\right\}$$
$$\le \mathrm{ord}_E\left(y - 1 - \frac{z}{z'}(y'-1)\right).$$

(IX-2) All of the following (IX-2-a) and (IX-2-b) are satisfied.

$$\text{(IX-2-a)}\quad y' \neq 0 \text{ and } z' = 0\,,$$
$$\text{(IX-2-b)}\quad \mathrm{ord}_E(y) = \mathrm{ord}_E(y'),\quad \mathrm{ord}_E(1-y) = \mathrm{ord}_E(1-y')\,.$$

(IX-3) All of the following (IX-3-a), (IX-3-b), and (IX-3-c) are satisfied.

$$\text{(IX-3-a)}\quad y' = 0 \text{ and } z' \neq 0\,,$$
$$\text{(IX-3-b)}\quad y = 0\,,$$
$$\text{(IX-3-c)}\quad \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(\pi^m - z')\right\} \le \mathrm{ord}_E\left(\frac{z}{z'} - 1\right).$$

(IX-4) The following is satisfied.

$$\text{(IX-4-a)} \quad y = y' = 0 \ \text{ and } \ z = z' = 0\,.$$

(X) If $\ell = 0, m \neq 0, n \neq 0$, and $x' = 1$, then either the following (X-1) or (X-2) holds.
(X-1) All of the following (X-1-a), (X-1-b), and (X-1-c) are satisfied.

$$\text{(X-1-a)} \quad z' \neq 0\,,$$
$$\text{(X-1-b)} \quad \mathrm{ord}_E(1-y) = \mathrm{ord}_E(1-y')\,,$$
$$\text{(X-1-c)} \quad \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}y'\right), \mathrm{ord}_E(\pi^m y' - z'), n\right\} \leq \mathrm{ord}_E\left(\frac{z}{z'}y' - y\right).$$

(X-2) All of the following (X-2-a) and (X-2-b) are satisfied.

$$\text{(X-2-a)} \quad z' = 0\,,$$
$$\text{(X-2-b)} \quad \mathrm{ord}_E(y) = \mathrm{ord}_E(y'), \quad \mathrm{ord}_E(1-y) = \mathrm{ord}_E(1-y')\,.$$

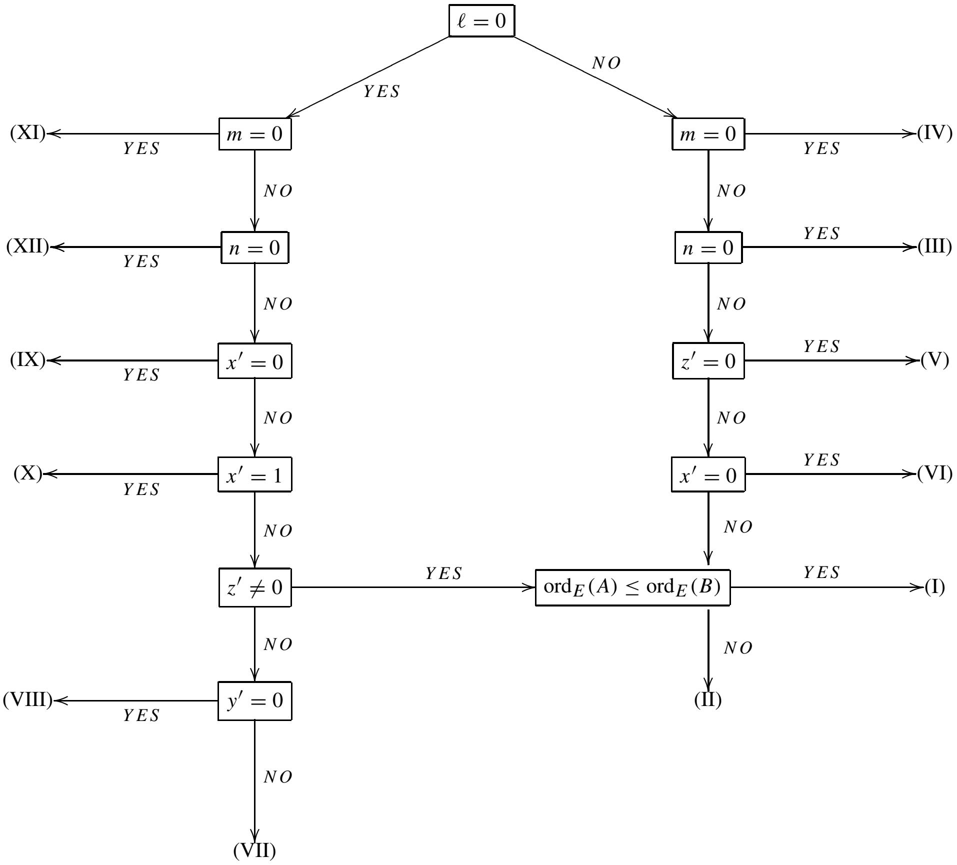
(XI) If $\ell = 0$ and $m = 0$, then the following (XI-a) and (XI-b) hold.

$$\text{(XI} - \text{a)} \quad \mathrm{ord}_E(y) = \mathrm{ord}_E(y')\,,$$
$$\text{(XI} - \text{b)} \quad \mathrm{ord}_E(1-y-z) = \mathrm{ord}_E(1-y'-z')\,.$$

(XII) $\ell = 0, m \neq 0$, and $n = 0$.

REMARK 1. We can check the statements (I), (II), ..., (XII) by calculating p-adic valuations of quantities described by using (ℓ, m, n, x, y, z) and (ℓ, m, n, x', y', z'). The following Table 1 is the algorithm of our main Theorem 1. This table can be used when we check whether two Λ_E-modules $M(\ell, m, n; x, y, z)$ and $M(\ell, m, n; x', y', z')$ are isomorphic or not.

TABLE 1



$A = \dfrac{\pi^n}{z'}\dfrac{x}{x'}y'$ and $B = \dfrac{\pi^m}{x'}y' - z'$, which is defined before the statement of (I).

4. Proof of Theorem 1

In this section, we prove our main Theorem 1. We fix notation. Let $M_{mn}(E)$ be the set of $m \times n$ matrices with entries in E and $GL_m(\mathcal{O}_E)$ be the group of $m \times m$ matrices over $\mathcal{O}_E$ that are invertible. For A and $B \in M_{mn}(E)$, we write $A \sim B$ if there is $P \in GL_m(\mathcal{O}_E)$ such that $PA = B$. This is an equivalence relation on $M_{mn}(E)$.

First we give necessary conditions for the two modules $M(\ell, m, n; x, y, z)$ and $M(\ell, m, n; x', y', z')$ to be isomorphic.

PROPOSITION 5. *Let* $(\ell, m, n; x, y, z)$ *and* $(\ell, m, n; x', y', z')$ *be admissible. Assume that* $M(\ell, m, n; x, y, z) \cong M(\ell, m, n; x', y', z')$ *as* Λ_E*-modules. Then we have* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ *and* $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$.

PROOF. We assume that $M(\ell, m, n; x, y, z) \cong M(\ell, m, n; x', y', z')$ as Λ_E-modules. Then we have (7) and (9) by Proposition 3. If $\mathrm{ord}_E(x) > \mathrm{ord}_E(x')$, we get $\mathrm{ord}_E(a_3x - a_2x') = \mathrm{ord}_E(x') \geq m$ by (7). Since $(\ell, m, n; x', y', z')$ is admissible, this implies $x' = 0$. This contradicts $\mathrm{ord}_E(x) > \mathrm{ord}_E(x')$. By the same reason, $\mathrm{ord}_E(x) < \mathrm{ord}_E(x')$ does not hold. Therefore, we have $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$. In the same way, we get $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$. □

Further in the case of $\ell = 0$, we have the following

LEMMA 2. *Let* $[M]_E, [M']_E \in \mathcal{M}^E_{f(T)}$. *We put* $M = M(0, m, n; x, y, z)$, $M' = M(0, m, n; x', y', z')$. *Then the following two statements are equivalent*:

(i) *We have* $M \cong M'$ *as* Λ_E*-modules.*

(ii) *There exist* $a_1, a_2, a_3 \in \mathcal{O}_E^\times$ *satisfying* (6), (7), (8), (9) *in Proposition* 3 *and*

$$a_3(1 - x) \equiv a_1(1 - x') \mod \pi^m. \tag{10}$$

In particular, if (i) *holds and* $(0, m, n; x, y, z)$, $(0, m, n; x', y', z')$ *are admissible, we have*

$$\mathrm{ord}_E(1 - x) = \mathrm{ord}_E(1 - x').$$

PROOF. The conditions (10) and (5) are equivalent under the condition (7). Hence we get the conclusion. □

PROOF (Proof of Theorem 1). By the Table 1 in Remark 1, for given two 6-tuples $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$, we have only to apply one statement among (I), (II), ..., and (XII). Using the following Propositions 6, 7, 8, we can prove Theorem 1 in the case of (I), (III), and (VII). By the same method as these Propositions, we can prove for the rest cases. This implies that our Theorem 1 holds. □

Let $[M(\ell, m, n; x, y, z)] \in \mathcal{M}^E_{f(T)}$. We fix non-negative integers ℓ, m, and n.

PROPOSITION 6. *Let* $(\ell, m, n; x, y, z)$ *and* $(\ell, m, n; x', y', z')$ *be admissible. Assume that* $x' \neq 0, z' \neq 0$ *if* $\ell \neq 0$ *and that* $x' \neq 0, 1, z' \neq 0$ *if* $\ell = 0$. *Suppose that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, *and* $\mathrm{ord}_E(A) \leq \mathrm{ord}_E(B)$, *where* A, B *are defined before the statement* (I). *Suppose also that* $\mathrm{ord}_E(1 - x) = \mathrm{ord}_E(1 - x')$ *if* $\ell = 0$. *Then the following are equivalent*:

(i) *We have* $M(\ell, m, n; x, y, z) \cong M(\ell, m, n; x', y', z')$ *as* Λ_E*-modules.*

(ii) *The statement* (I) *holds for* $(\ell, m, n; x, y, z)$ *and* $(\ell, m, n; x', y', z')$.

PROOF. First we prove (i) ⇒ (ii). Let $A, B, C, D, E, F, G \in \mathcal{O}_E$ be the elements defined before the statement (I). We note that these elements are all in $\mathcal{O}_E$. By Proposition 3, we have units $a_1, a_2, a_3 \in \mathcal{O}_E^\times$ satisfying

$$a_2 - a_1 = \pi^\ell v, \tag{11}$$

$$a_3 - a_1 - (a_2 - a_1)\pi^{-\ell}x' = \pi^m w\,, \tag{12}$$

$$1 - a_1 - (a_2 - a_1)\pi^{-\ell}y' - \left\{a_3 - a_1 - (a_2 - a_1)\pi^{-\ell}x'\right\}\pi^{-m}z' = \pi^n\eta\,, \tag{13}$$

$$a_3 x - a_2 x' = \pi^m \xi_x\,, \tag{14}$$

$$y - a_2 y' - \xi_x z' = \pi^n \xi_y\,, \tag{15}$$

$$z - a_3 z' = \pi^n \xi_z \tag{16}$$

for some v, w, η, ξ_x, ξ_y and $\xi_z \in \mathcal{O}_E$. By the equations (11), (14) and (16), we have

$$\begin{aligned} a_1 &= \left(\frac{z}{z'} - \frac{\pi^n}{z'}\xi_z\right)\frac{x}{x'} - \frac{\pi^m}{x'}\xi_x - \pi^\ell v\,,\\ a_2 &= \left(\frac{z}{z'} - \frac{\pi^n}{z'}\xi_z\right)\frac{x}{x'} - \frac{\pi^m}{x'}\xi_x\,,\\ a_3 &= \frac{z}{z'} - \frac{\pi^n}{z'}\xi_z\,. \end{aligned}$$

By the equations (12), (13), (15), we have

$$\frac{\pi^n}{z'}\left(\frac{x}{x'} - 1\right)\xi_z + \frac{\pi^m}{x'}\xi_x + (\pi^\ell - x')v - \pi^m w = \frac{z}{z'}\left(\frac{x}{x'} - 1\right), \tag{17}$$

$$\frac{\pi^n}{z'}\frac{x}{x'}\xi_z + \frac{\pi^m}{x'}\xi_x + (\pi^\ell - y')v - z'w - \pi^n\eta = \frac{z}{z'}\frac{x}{x'} - 1\,, \tag{18}$$

$$\frac{\pi^n}{z'}\frac{x}{x'}y'\xi_z + \left(\frac{\pi^m}{x'}y' - z'\right)\xi_x - \pi^n\xi_y = \frac{z}{z'}\frac{x}{x'}y' - y\,. \tag{19}$$

By the equations (17), (18) and (19), we obtain

$$\begin{pmatrix} -\frac{\pi^n}{z'}\left(1 - \frac{x}{x'}\right) & \frac{\pi^m}{x'} & \pi^\ell - x' & -\pi^m & 0 & 0\\ \frac{\pi^n}{z'}\frac{x}{x'} & \frac{\pi^m}{x'} & \pi^\ell - y' & -z' & -\pi^n & 0\\ \frac{\pi^n}{z'}\frac{x}{x'}y' & \frac{\pi^m}{x'}y' - z' & 0 & 0 & 0 & -\pi^n \end{pmatrix}\begin{pmatrix} \xi_z\\ \xi_x\\ v\\ w\\ \eta\\ \xi_y \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{z}{z'}\left(1 - \frac{x}{x'}\right)\\ \frac{z}{z'}\frac{x}{x'} - 1\\ \frac{z}{z'}\frac{x}{x'}y' - y \end{pmatrix}.$$

Therefore, the augmented matrix for the system of the equations (17), (18) and (19) is

$$\begin{pmatrix} -\frac{\pi^n}{z'}\left(1 - \frac{x}{x'}\right) & \frac{\pi^m}{x'} & \pi^\ell - x' & -\pi^m & 0 & 0 & -\frac{z}{z'}\left(1 - \frac{x}{x'}\right)\\ \frac{\pi^n}{z'}\frac{x}{x'} & \frac{\pi^m}{x'} & \pi^\ell - y' & -z' & -\pi^n & 0 & \frac{z}{z'}\frac{x}{x'} - 1\\ A & B & 0 & 0 & 0 & -\pi^n & C \end{pmatrix}. \tag{20}$$

By elementary row operations, the matrix in (20) is equivalent to

$$\begin{pmatrix} -\frac{\pi^n}{z'}\left(1-\frac{x}{x'}\right) & \frac{\pi^m}{x'} & \pi^\ell - x' & -\pi^m & 0 & 0 & -\frac{z}{z'}\left(1-\frac{x}{x'}\right) \\ \frac{\pi^n}{z'} & 0 & x'-y' & \pi^m - z' & -\pi^n & 0 & \frac{z}{z'}-1 \\ A & B & 0 & 0 & 0 & -\pi^n & C \end{pmatrix}$$
$$\sim \begin{pmatrix} A & B & 0 & 0 & 0 & -\pi^n & C \\ \frac{\pi^n}{z'} & 0 & D & E & -\pi^n & 0 & \frac{z}{z'}-1 \\ -\frac{\pi^n}{z'}\left(1-\frac{x}{x'}\right) & \frac{\pi^m}{x'} & \pi^\ell - x' & -\pi^m & 0 & 0 & -\frac{z}{z'}\left(1-\frac{x}{x'}\right) \end{pmatrix}$$
$$\sim \begin{pmatrix} A & B & 0 & 0 & 0 & -\pi^n & C \\ \frac{\pi^n}{z'} & 0 & D & E & -\pi^n & 0 & \frac{z}{z'}-1 \\ 0 & \frac{\pi^m}{x'} & F & G & -\pi^n\left(1-\frac{x}{x'}\right) & 0 & \frac{x}{x'}-1 \end{pmatrix}. \quad (21)$$

By the matrix (21), we get $A\xi_z + B\xi_x - \pi^n\xi_y = C$. Since $\xi_x, \xi_y, \xi_z \in \mathcal{O}_E$ and $\mathrm{ord}_E(A) \le \mathrm{ord}_E(B)$, we have $\min\{\mathrm{ord}_E(A), n\} \le \mathrm{ord}_E(C)$. Further we have $\mathrm{ord}_E(A) \le \mathrm{ord}_E(C)$. Indeed, if $\mathrm{ord}_E(y') \ge \mathrm{ord}_E(z')$, we have $\mathrm{ord}_E(B) = \mathrm{ord}_E(z') < n$, since we assume that $(\ell, m, n; x, y, z), (\ell, m, n; x', y', z')$ are admissible. If $\mathrm{ord}_E(y') < \mathrm{ord}_E(z')$, we have $\mathrm{ord}_E(A) < n$. Thus we get $\mathrm{ord}_E(A) \le \mathrm{ord}_E(C)$. We will prove that the statement (I) holds for $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$. First we note that either (I-1-a), (I-2-a) or (I-3-a) holds. We suppose that (I-2-a) holds. By the matrix (21), we have $\frac{\pi^m}{x'}\xi_x + Fv + Gw - \pi^n(1-\frac{x}{x'})\eta = \frac{x}{x'} - 1$. Since we suppose (I-2-a), $\min\{\mathrm{ord}_E(\frac{\pi^m}{x'}), \mathrm{ord}_E(F), \mathrm{ord}_E(G)\} = \mathrm{ord}_E(F)$. This implies $\mathrm{ord}_E(F) \le \mathrm{ord}_E(\frac{x}{x'} - 1)$. Thus we get the condition (I-2-c). Since $\mathrm{ord}_E(A) < n$, $\mathrm{ord}_E(F) < m$, we have $A \neq 0$ and $F \neq 0$. By elementary row operations for (21), we have

$$\begin{pmatrix} 1 & A^{-1}B & 0 & 0 & 0 & -A^{-1}\pi^n & A^{-1}C \\ 0 & -A^{-1}B\frac{\pi^n}{z'} & D & E & -\pi^n & A^{-1}\frac{\pi^{2n}}{z'} & -A^{-1}C\frac{\pi^n}{z'} + \frac{z}{z'} - 1 \\ 0 & \frac{\pi^m}{x'}F^{-1} & 1 & GF^{-1} & -\pi^n\left(1-\frac{x}{x'}\right)F^{-1} & 0 & \left(\frac{x}{x'}-1\right)F^{-1} \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & A^{-1}B & 0 & 0 & 0 & -A^{-1}\pi^n & A^{-1}C \\ 0 & \frac{\pi^m}{x'}F^{-1} & 1 & GF^{-1} & -\pi^n\left(1-\frac{x}{x'}\right)F^{-1} & 0 & \left(\frac{x}{x'}-1\right)F^{-1} \\ 0 & U & 0 & E - GF^{-1}D & S & A^{-1}\frac{\pi^{2n}}{z'} & T \end{pmatrix},$$

where $T = -A^{-1}C\frac{\pi^n}{z'} + \frac{z}{z'} - 1 - (\frac{x}{x'} - 1)F^{-1}D$, $S = -\pi^n + \pi^n(1 - \frac{x}{x'})F^{-1}D$ and $U = -A^{-1}B\frac{\pi^n}{z'} - \frac{\pi^m}{x'}F^{-1}D$. By the matrix above, we have

$$U\xi_x + (E - GF^{-1}D)w + S\eta + A^{-1}\frac{\pi^{2n}}{z'}\xi_y = T\,.$$

This implies that $\min\{\mathrm{ord}_E(U), \mathrm{ord}_E(E - DF^{-1}G), \mathrm{ord}_E(S), \mathrm{ord}_E(A^{-1}\frac{\pi^{2n}}{z'})\} \le \mathrm{ord}_E(T)$. Since we have $\mathrm{ord}_E(A^{-1}\frac{\pi^{2n}}{z'}) = \mathrm{ord}_E(\frac{\pi^n}{y'})$, this is the condition (I-2-d). (I-2-b) was already obtained after (21). Therefore (I-2) holds. We can prove the case of (I-1) and that of (I-3) by the same method. Thus we have obtained (ii).

We next prove (ii) $\Rightarrow$ (i). Then either (I-1), (I-2) or (I-3) holds. We suppose that (I-2) holds. By the condition (I-2-d), there exist integers $\xi_x, w, \eta, \xi_y \in \mathcal{O}_E$ satisfying

$$U\xi_x + (E - DF^{-1}G)w + S\eta + A^{-1}\frac{\pi^{2n}}{z'}\xi_y = T\,.$$

We put

$$\begin{aligned} v &= \left(\frac{x}{x'} - 1\right)F^{-1} - \frac{\pi^m}{x'}F^{-1}\xi_x - GF^{-1}w + \pi^n\left(1 - \frac{x}{x'}\right)F^{-1}\eta\,,\\ \xi_z &= A^{-1}C - A^{-1}B\xi_x + A^{-1}\pi^n\xi_y\,. \end{aligned}$$

By (I-2-a), (I-2-b), (I-2-c), we have $v, \xi_z \in \mathcal{O}_E$. By the converse operation of the proof of (i) $\Rightarrow$ (ii), $\xi_x, \xi_y, \xi_z, w, \eta$ and v satisfy (17), (18) and (19). We also set

$$\begin{aligned} a_1 &= \left(\frac{z}{z'} - \frac{\pi^n}{z'}\xi_z\right)\frac{x}{x'} - \frac{\pi^m}{x'}\xi_x - \pi^\ell v\,,\\ a_2 &= \left(\frac{z}{z'} - \frac{\pi^n}{z'}\xi_z\right)\frac{x}{x'} - \frac{\pi^m}{x'}\xi_x\,,\\ a_3 &= \frac{z}{z'} - \frac{\pi^n}{z'}\xi_z\,. \end{aligned}$$

Then a_1, a_2, a_3 satisfy (11), (12), (13), (14), (15), (16). In the case where $\ell \neq 0$, we can check $a_1, a_2, a_3 \in \mathcal{O}_E^\times$ since $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, and $z' \neq 0$. In the case of $\ell = 0$, we have

$$a_1 = \frac{z}{z'}\frac{1-x}{1-x'} - \frac{\pi^n}{z'}\frac{1-x}{1-x'}\xi_z + \frac{\pi^m}{1-x'}\xi_x - \frac{\pi^m}{1-x'}w\,.$$

We note that we have $\mathrm{ord}_E(\frac{\pi^m}{1-x}) > 0$ since $x \in S_m$. Thus we have $a_1 \in \mathcal{O}_E^\times$. By the same method, we can show $a_2, a_3 \in \mathcal{O}_E^\times$. Then a_1, a_2, a_3 satisfy equalities (4), (5), (6), (7), (8), (9). By Proposition 3, we obtain (i). If (I-1) or (I-3) holds, we can prove (i) by the same method. □

Next we treat the case where $\ell \neq 0$ and $n = 0$. In this case, we have $y = z = 0$ for any admissible $(\ell, m, n; x, y, z)$.

PROPOSITION 7. *Suppose that* $(\ell, m, 0; x, 0, 0)$ *and* $(\ell, m, 0; x', 0, 0)$ *are admissible. Suppose also that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ *and* $\ell \neq 0$. *Then the following are equivalent*:

(i) *We have* $M(\ell, m, 0; x, 0, 0) \cong M(\ell, m, 0; x', 0, 0)$ *as* Λ_E*-modules.*

(ii) *The statement* (III) *holds for* $(\ell, m, 0; x, 0, 0)$ *and* $(\ell, m, 0; x', 0, 0)$.

PROOF. We prove (i) $\Rightarrow$ (ii). By Proposition 3, we have units $a_1, a_2, a_3 \in \mathcal{O}_E^\times$ satisfying

$$\begin{aligned} &a_2 \equiv a_1 \mod \pi^\ell, \\ &1 - a_1 - (a_2 - a_1)\pi^{-\ell}x' \equiv 0 \mod \pi^m, \\ &x \equiv a_2 x' \mod \pi^m. \end{aligned}$$

By Proposition 4.5 and Lemma 4.6 in [8], this is equivalent to say that $N(\ell, n, x) \cong N(\ell, n, x')$, where $N(\ell, n, x) = \langle (1,1,1),\ (0, \pi^\ell, x),\ (0,0,\pi^n)\ \rangle_{\mathcal{O}_E} \subset \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma)$. By Theorem 2, this implies that (I') or (II') holds. This is the same as statement (III). Hence we have (ii).

Next we suppose (ii). Then $M(x,0,0) \cong M(x',0,0)$ by Theorem 2. Thus we have (i). □

From now on, we treat the case where $\ell = 0$ and $z' = 0$. Let $(0, m, n; x, y, z)$ and $(0, m, n; x', y', 0)$ be admissible. If $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, then we have $z = 0$.

PROPOSITION 8. *Suppose that* $(0, m, n; x, y, 0)$ *and* $(0, m, n; x', y', 0)$ *are admissible. Suppose also that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$, $x' \neq 0, 1$, *and* $y' \neq 0$. *Then the following are equivalent*:

(i) *We have* $M(0, m, n; x, y, 0) \cong M(0, m, n; x', y', 0)$ *as* Λ_E*-modules.*

(ii) *The statement* (VII) *holds for* $(0, m, n; x, y, 0)$ *and* $(0, m, n; x', y', 0)$.

PROOF. First we assume (i). By Lemma 2, we have units $a_1, a_2, a_3 \in \mathcal{O}_E^\times$ satisfying (10), (6), (7), (8), and (9). By (8), we have $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$. Further using (6) and (8), we get

$$1 - y \equiv a_1(1 - y') \mod \pi^n. \tag{22}$$

Hence we have (VII-a). We show (VII-b). By (10), (7), (8), we obtain

$$a_1 = \left\{ \left(\frac{y}{y'} - \frac{\pi^n}{y'}\xi_y \right) \frac{x'}{x} + \frac{\pi^m}{x}\xi_x \right\} \frac{1-x}{1-x'} - \frac{\pi^m}{1-x'}w', \tag{23}$$

$$a_2 = \frac{y}{y'} - \frac{\pi^n}{y'}\xi_y, \tag{24}$$

$$a_3 = \left(\frac{y}{y'} - \frac{\pi^n}{y'}\xi_y \right) \frac{x'}{x} + \frac{\pi^m}{x}\xi_x \tag{25}$$

for some $\xi_x, \xi_y, w' \in \mathcal{O}_E$. By (22), we have $1 - y - a_1(1-y') = \pi^n \eta$ for some $\eta \in \mathcal{O}_E$. This implies that

$$-\frac{\pi^n}{y'} \frac{1-x}{1-x'} \frac{x'}{x}(1-y')\xi_y + \frac{\pi^m}{x} \frac{1-x}{1-x'}(1-y')\xi_x + \frac{\pi^m}{1-x'}(1-y')w' + \pi^n \eta$$

$$= 1 - y - \frac{y}{y'}\frac{x'}{x}\frac{1-x}{1-x'}(1-y')\,. \tag{26}$$

This implies that (VII-b). Conversely, we suppose that (ii) holds. By (VII-b), there exist ξ_x, ξ_y, w', and $\eta \in \mathcal{O}_E$ satisfying (26). We put a_1, a_2, and a_3 as (23), (24), and (25), respectively. Since $(0, m, n; x, y, 0)$, $(0, m, n; x', y', 0)$ are admissible and (VII-a) holds, $a_2, a_3 \in \mathcal{O}_E^{\times}$. Using $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$, we have $a_1 \in \mathcal{O}_E^{\times}$. It is easy to check that a_1, a_2, and a_3 satisfy (10), (5), (6), (7), (8), and (9). By Lemma 2, we get (i). □

As an example, we classify all the elements of $\mathcal{M}_{f(T)}$ in the case of $E = \mathbf{Q}_p$ and $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\delta) = \mathrm{ord}_p(\delta-\alpha) = \mathrm{ord}_p(\beta-\delta) = \mathrm{ord}_p(\alpha-\gamma)=1$, where we write $\mathcal{M}_{f(T)}$ for $\mathcal{M}_{f(T)}^{\mathbf{Q}_p}$ and ord_p for $\mathrm{ord}_{\mathbf{Q}_p}$. This example was also treated by C.Franks. We note that there is no distinguished polynomial which has this property in the case of $p = 2$ and 3. In the following, we take $R = \{0, 1, \ldots, p-1\}$, which is a set of complete representatives in $\mathbf{Z}_p$ of the elements of the residue field $\mathbf{Z}_p/(p)$.

COROLLARY 1. *Let $p \geq 5$. Let $f(T)$ be the same polynomial as* (3) *in Section* 2 *and $E = \mathbf{Q}_p$. Assume that* $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\delta) = \mathrm{ord}_p(\delta-\alpha) = \mathrm{ord}_p(\beta-\delta) = \mathrm{ord}_p(\alpha-\gamma) = 1$. *Then we have* $\sharp\mathcal{M}_{f(T)} = 2p + 36$.

We note that this corollary holds for any totally ramified extensions of $\mathbf{Q}_p$.

PROOF (Sketch of the proof of Corollary 1). For fixed non-negative integers ℓ, m, and n, we put

$$\mathcal{M}_{f(T)}^{E}(\ell, m, n) := \left\{[M(\ell, m, n; x, y, z)] \in \mathcal{M}_{f(T)}^{E} \,\middle|\, x, y, z \in \mathbf{Z}_p\right\}.$$

By Proposition 5, we have

$$\mathcal{M}_{f(T)}^{E} = \coprod_{\ell}\coprod_{n}\coprod_{m}\mathcal{M}_{f(T)}^{E}(\ell, m, n)\,. \tag{27}$$

Using the conditions of Lemma 1, we have $0 \leq \ell \leq 1$, $0 \leq m \leq 2$, and $0 \leq n \leq 3$. Indeed, by (a), we have $0 \leq \ell \leq \mathrm{ord}_p(\beta-\alpha) = 1$. If $\mathrm{ord}_p(x) \geq 2$, we have $m \leq 1$ by (b). If $\mathrm{ord}_p(x) \leq 1$, we obtain $m \leq 2$ by (d). These imply $0 \leq m \leq 2$. We can prove that $0 \leq n \leq 3$ by Lemma 1. In fact, by (f), we have $n \leq 3$ in the case of $\mathrm{ord}_p(z) \leq 2$. We suppose $\mathrm{ord}_p(z) \geq 3$. In the case of $\mathrm{ord}_p(y) \leq 1$, we have $n \leq 2$ by (e). If $\mathrm{ord}_p(y) \geq 2$, we have $n \leq 1$ by (c). Thus we get $0 \leq n \leq 3$. We denote $M(\ell, m, n; x, y, z)$ by $M(x, y, z)$ for the fixed triple ℓ, m, and n. Then we get the following:

$$\mathcal{M}_{f(T)}^{E}(0,0,0) = \{[M(0,0,0)]\}\,,$$

$$\mathcal{M}_{f(T)}^{E}(0,0,1) = \left\{\begin{array}{l}[M(0,2,p-1)], [M(0,1,1)], [M(0,0,0)], [M(0,0,1)],\\ [M(0,0,2)], [M(0,1,0)], [M(0,2,0)]\end{array}\right\},$$

$$\mathcal{M}^E_{f(T)}(0,1,0) = \{[M(0,0,0)],[M(1,0,0)],[M(2,0,0)]\},$$

$$\mathcal{M}^E_{f(T)}(0,1,1) = \left\{ \begin{array}{l} [M(2,2,0)],\dots,[M(p-1,2,0)],[M(p-2,4,0)], \\ [M(1,1,0)],[M(1,2,0)],[M(2,1,0)],[M(1,0,0)], \\ [M(0,0,0)],[M(0,1,0)],[M(0,2,0)],[M(2,0,0)] \end{array} \right\},$$

$$\mathcal{M}^E_{f(T)}(0,1,2) = \left\{ \begin{array}{l} \left[M\left(0,0,\frac{\delta-\alpha}{\gamma-\alpha}p\right)\right],\left[M\left(0,p,\frac{\delta-\alpha}{\gamma-\alpha}p\right)\right],\left[M\left(1,1+p,\frac{\delta-\beta}{\gamma-\beta}p\right)\right], \\ \left[M\left(1,1,\frac{\delta-\beta}{\gamma-\beta}p\right)\right],\left[M\left(\frac{\beta-\delta}{\beta-\alpha},\frac{\beta-\gamma}{\beta-\alpha},p\right)\right] \end{array} \right\},$$

$$\mathcal{M}^E_{f(T)}(1,0,0) = \{[M(0,0,0)]\},$$

$$\mathcal{M}^E_{f(T)}(1,0,1) = \{[M(0,0,0)],[M(0,0,1)],[M(0,0,2)]\},$$

$$\mathcal{M}^E_{f(T)}(1,0,2) = \left\{\left[M\left(0,\frac{\delta-\alpha}{\beta-\alpha}p,0\right)\right],\left[M\left(0,\frac{\delta-\alpha}{\beta-\alpha}p,p\right)\right]\right\},$$

$$\mathcal{M}^E_{f(T)}(1,1,0) = \{[M(0,0,0)]\},$$

$$\mathcal{M}^E_{f(T)}(1,1,1) = \left\{[M(0,0,0)],\left[M\left(0,\frac{\gamma-\alpha}{\beta-\alpha},1\right)\right]\right\},$$

$$\mathcal{M}^E_{f(T)}(1,1,2) = \left\{\left[M\left(0,0,\frac{\delta-\alpha}{\gamma-\alpha}p\right)\right],\left[M\left(0,pu,\frac{\delta-\alpha}{\gamma-\alpha}p\left(1-\frac{\beta-\alpha}{\delta-\alpha}u\right)\right)\right]\middle| u=1,\dots,p-1\right\},$$

$$\mathcal{M}^E_{f(T)}(1,2,0) = \left\{\left[M\left(\frac{\gamma-\alpha}{\beta-\alpha}p,0,0\right)\right]\right\},$$

$$\mathcal{M}^E_{f(T)}(1,2,1) = \left\{\left[M\left(\frac{\gamma-\alpha}{\beta-\alpha}p,0,0\right)\right]\right\},$$

$$\mathcal{M}^E_{f(T)}(1,2,2) = \left\{\left[M\left(\frac{\gamma-\alpha}{\beta-\alpha}p,\frac{\delta-\alpha}{\beta-\alpha}p,0\right)\right]\right\},$$

$$\mathcal{M}^E_{f(T)}(1,2,3) = \left\{\left[M\left(\frac{\gamma-\alpha}{\beta-\alpha}p,\frac{\delta-\alpha}{\beta-\alpha}p,\frac{(\delta-\alpha)(\delta-\beta)}{(\gamma-\alpha)(\gamma-\beta)}p^2\right)\right]\right\}.$$

The following table is the number of $\mathcal{M}^E_{f(T)}(\ell,m,n)$ for each (ℓ,m,n). We pick up the case of $(\ell,m,n) = (1,0,0)$, $(0,1,1)$ and determine $\mathcal{M}^E_{f(T)}(1,0,0)$ and $\mathcal{M}^E_{f(T)}(0,1,1)$, using our Theorem 1. The rest cases are proved by the same method as the case of $(1,0,0)$ and that of $(0,1,1)$. First, we consider the former $(\ell,m,n) = (1,0,0)$. This is the simplest case. Since we have $x,y,z \in S_0$, we get $x = y = z = 0$. Thus we obtain the conclusion.

Next we consider the case of $(\ell,m,n) = (0,1,1)$. This is one of the most complicated cases. In this case, if $(0,1,1;x,y,z)$ is admissible, then we have $z = 0$. Indeed we suppose that $(0,1,1;x,y,z)$ is admissible. Then x,y,z satisfy (a), (b), (c), (d), (e), (f) in Lemma 1. We have $\mathrm{ord}_E(zx) \geq 1$ by (e). We have also $\mathrm{ord}_E(z) \geq 1$ by (c). Since $z \in S_1$, we have $z = 0$. We classify all the elements of $\mathcal{M}^E_{f(T)}(0,1,1)$. We note that $(0,1,1;x,y,0)$ is

(ℓ, m, n)	$\sharp\mathcal{M}^E_{f(T)}(\ell, m, n)$
(0, 0, 0)	1
(0, 0, 1)	7
(0, 1, 0)	3
(0, 1, 1)	$p+7$
(0, 1, 2)	5
(1, 0, 0)	1
(1, 0, 1)	3
(1, 0, 2)	2
(1, 1, 0)	1
(1, 1, 1)	2
(1, 1, 2)	p
(1, 2, 0)	1
(1, 2, 1)	1
(1, 2, 2)	1
(1, 2, 3)	1

admissible for any $x, y \in S_1$. Let $(0, 1, 1; x', y', 0)$ be admissible. We consider the following two cases:

$$\begin{cases} \text{(i)} & x' \in \{0, 1\} \ \text{ or } \ y' \in \{0, 1\}\,, \\ \text{(ii)} & x' \notin \{0, 1\} \ \text{ and } \ y' \notin \{0, 1\}\,. \end{cases}$$

(i) We suppose that $x' \in \{0, 1\}$ or $y' \in \{0, 1\}$. Then we have

$$\begin{aligned} M(x, y, 0) \cong M(x', y', 0) \Leftrightarrow \mathrm{ord}_E(x) &= \mathrm{ord}_E(x'), \mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')\,, \\ \mathrm{ord}_E(y) &= \mathrm{ord}_E(y') \text{ and } \mathrm{ord}_E(1-y) = \mathrm{ord}_E(1-y')\,. \end{aligned}$$

Indeed, using the Table 1 in Remark 1, the 6-tuple $(0, 1, 1; x', y', 0)$ corresponds to (VII), (VIII), (XI), or (X). Therefore the isomorphism classes of $M(x, y, 0)$ satisfying (i) are

$$\left\{ \begin{array}{l} [M(0,0,0)], [M(0,1,0)], [M(0,2,0)], [M(1,0,0)], \\ [M(1,1,0)], [M(1,2,0)], [M(2,0,0)], [M(2,1,0)] \end{array} \right\}.$$

(ii) We suppose that $x' \notin \{0, 1\}$ and $y' \notin \{0, 1\}$. Then we have the following

LEMMA 3. *We suppose* (ii). *Then we have*

$$\begin{aligned} M(x, y, 0) \cong M(x', y', 0) \Leftrightarrow x \neq 0, 1, y \neq 0, 1 \ \text{ and } \\ \frac{1-x}{x}\frac{y}{1-y} \equiv \frac{1-x'}{x'}\frac{y'}{1-y'} \bmod p\,. \end{aligned}$$

Further we have

$$\frac{1-x'}{x'}\frac{y'}{1-y'} \bmod p \equiv \begin{cases} 2-\frac{2}{k} \bmod p \\ \qquad \text{if } (x',y')=(k,2)\,, \\ 2 \bmod p \\ \qquad \text{if } (x',y')=(p-2,4)\,. \end{cases}$$

PROOF. Since we suppose (ii), the 6-tuple $(0,1,1;x',y',0)$ corresponds to (VII) by the Table 1. Since we assume that $x' \neq 0,1$ and $y' \neq 0,1$, the condition (VII-a) says that $x \neq 0,1$ and $y \neq 0,1$. By the same reason, the condition (VII-b) says that

$$\frac{1-x}{x}\frac{y}{1-y} \equiv \frac{1-x'}{x'}\frac{y'}{1-y'} \bmod p\,.$$

Thus we get the former. It is easy to show the latter. □

Using Lemma 3, the isomorphism classes of $M(x,y,0)$ satisfying (ii) are

$$\{[M(p-2,4,0)],[M(k,2,0)]\,|\; 2 \le k \le p-1\}\,.$$

Therefore we obtain $\sharp\mathcal{M}^{E}_{f(T)}(0,1,1) = p+7$,

$$\mathcal{M}^{E}_{f(T)}(0,1,1) = \left\{\begin{array}{l} [M(2,2,0)],\dots,[M(p-1,2,0)],[M(p-2,4,0)], \\ [M(1,1,0)],[M(1,2,0)],[M(2,1,0)],[M(1,0,0)], \\ [M(0,0,0)],[M(0,1,0)],[M(0,2,0)],[M(2,0,0)] \end{array}\right\}.$$

□

As a preparation for the next section, we prove some propositions. First by Proposition 5.1 in [8], we have

PROPOSITION 9. *For a distinguished polynomial $f(T) \in \mathbf{Z}_p[T]$, let E be the splitting field of $f(T)$ over $\mathbf{Q}_p$. Then the natural map*

$$\Psi : \mathcal{M}^{\mathbf{Q}_p}_{f(T)} \longrightarrow \mathcal{M}^{E}_{f(T)} \quad ([M]_{\mathbf{Q}_p} \longmapsto [M \otimes_{\Lambda} \Lambda_E]_E)$$

is injective.

In order to determine isomorphism classes of modules, we will use the higher Fitting ideals in the next section. For a commutative ring R and a finitely presented R-module M, we consider the following exact sequence

$$R^m \xrightarrow{f} R^n \to M \to 0\,,$$

where m and n are positive integers. For an integer $0 \le i < n$, the i-th Fitting ideal $\mathrm{Fitt}_{i,R}(M)$ of M is defined to be the ideal of R generated by all $(n-i)\times(n-i)$ minors of the matrix corresponding to f. This definition does not depend on the choice of the exact sequence above (see [10]).

PROPOSITION 10. *Let E be the splitting field of $f(T)$ over $\mathbf{Q}_p$. Let $[M]_E \in \mathcal{M}^E_{f(T)}$ and $M = M(\ell, m, n; x, y, z)$. Then we have*

$$\mathrm{Fitt}_{1,\Lambda_E}(M) \mod (T-\delta) = ((\delta-\alpha)(\delta-\beta)(\delta-\gamma)\pi^{-n}),$$

$$\mathrm{Fitt}_{1,\Lambda_E}(M) \mod (T-\gamma) = \begin{cases} ((\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)z\pi^{-m-n}) & \text{if } z \neq 0, \\ ((\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)\pi^{-m}) & \text{if } z = 0. \end{cases}$$

PROOF. By the action of T, we have

$$\begin{aligned} T(1,1,1,1) &= (\alpha,\beta,\gamma,\delta) \\ &= \alpha(1,1,1,1) + (\beta-\alpha)\pi^{-\ell}(0,\pi^{\ell},x,y) + \{\gamma-\alpha-(\beta-\alpha)\pi^{-\ell}x\}\pi^{-m}(0,0,\pi^m,z) \\ &\quad + \Big[(\delta-\alpha)-(\beta-\alpha)\pi^{-\ell}y - \{(\gamma-\alpha)-(\beta-\alpha)\pi^{-\ell}x\}\pi^{-m}z\Big]\pi^{-n}(0,0,0,\pi^n), \\ T(0,\pi^{\ell},x,y) &= (0,\beta\pi^{\ell},\gamma x,\delta y) \\ &= \beta(0,\pi^{\ell},x,y) + (\gamma-\beta)x\pi^{-m}(0,0,\pi^m,z) \\ &\quad + \{(\delta-\beta)y-(\gamma-\beta)x\pi^{-m}z\}\pi^{-n}(0,0,0,\pi^n), \\ T(0,0,\pi^m,z) &= (0,0,\gamma\pi^m,\delta z) \\ &= \gamma(0,0,\pi^m,z) + (\delta-\gamma)z\pi^{-n}(0,0,0,\pi^n), \\ T(0,0,0,\pi^n) &= \delta(0,0,0,\pi^n). \end{aligned}$$

Then we get the following matrix

$$\begin{pmatrix} T-\alpha & -(\beta-\alpha)\pi^{-\ell} & -\{\gamma-\alpha-(\beta-\alpha)\pi^{-\ell}x\}\pi^{-m} & a_{14} \\ 0 & T-\beta & -(\gamma-\beta)x\pi^{-m} & a_{24} \\ 0 & 0 & T-\gamma & -(\delta-\gamma)z\pi^{-n} \\ 0 & 0 & 0 & T-\delta \end{pmatrix},$$

where $a_{24} = -\{(\delta-\beta)y-(\gamma-\beta)x\pi^{-m}z\}\pi^{-n}$ and $a_{14} = -\big[(\delta-\alpha)-(\beta-\alpha)\pi^{-\ell}y-\{(\gamma-\alpha)-(\beta-\alpha)\pi^{-\ell}x\}\pi^{-m}z\big]\pi^{-n}$. We prove the former part. By the definition of Fitting ideals, we obtain

$$\begin{aligned} &\mathrm{Fitt}_{1,\Lambda_E}(M) \mod (T-\delta) \\ &= (\widetilde{a_{41}}, (\delta-\alpha)(\delta-\beta)(\delta-\gamma)z\pi^{-n}, (\delta-\alpha)(\delta-\beta)(\delta-\gamma)\pi^{-n}y) \mod (T-\delta), \end{aligned}$$

where

$$\widetilde{a_{41}} = \det \begin{pmatrix} -(\beta-\alpha)\pi^{-\ell} & -\{\gamma-\alpha-(\beta-\alpha)\pi^{-\ell}x\}\pi^{-m} & a_{14} \\ T-\beta & -(\gamma-\beta)x\pi^{-m} & a_{24} \\ 0 & T-\gamma & -(\delta-\gamma)z\pi^{-n} \end{pmatrix} .$$

Since we have

$$\widetilde{a_{41}} \bmod (T-\delta) = (\delta-\alpha)(\delta-\beta)(\delta-\gamma)\pi^{-n} \bmod (T-\delta) ,$$

we obtain the conclusion. We can also prove the latter equation by the same method above. □

PROPOSITION 11. *Let $f(T) = g(T)(T-\delta)$, where $\delta \in p\mathbf{Z}_p$ and $g(T) \in \mathbf{Z}_p[T]$ is an Eisenstein polynomial of degree* 3. *Let E be the splitting field of $g(T)$ over $\mathbf{Q}_p$. Let $[M]_{\mathbf{Q}_p} \in \mathcal{M}^{\mathbf{Q}_p}_{f(T)}$ and $[M \otimes \Lambda_E] = [M(\ell, m, n; x, y, z)] \in \mathcal{M}^{E}_{f(T)}$. Assume that* $\mathrm{ord}_E(\delta-\alpha) = \mathrm{ord}_E(\delta-\beta) = \mathrm{ord}_E(\delta-\gamma) = 1$, *and*

$$M/TM \cong \mathbf{Z}/p^i\mathbf{Z} \oplus \mathbf{Z}/p^j\mathbf{Z} \qquad (i, j \in \mathbf{Z}_{\geq 1}) .$$

Then we have $n = 0$.

PROOF. We have $\mathrm{Fitt}_{1,\Lambda_{\mathbf{Q}_p}}(M) \neq \Lambda_{\mathbf{Q}_p}$, since $\mathrm{Fitt}_{1,\mathbf{Z}_p}(M/TM) = (p^{\min\{i,j\}})$. By our assumption, $g(T)$ is an Eisenstein polynomial. Hence we have $\mathrm{Fitt}_{1,\Lambda_E}(M \otimes \Lambda_E) \mod (T-\delta) = (\pi^{3i})$ for some $i \geq 1$. Using Proposition 10, we obtain $\mathrm{Fitt}_{1,\Lambda_E}(M \otimes \Lambda_E) \mod (T-\delta) = (\pi^{3-n})$. This implies that $3i = 3-n$. Thus we have $n = 0$. □

5. Numerical Examples

In this section, we introduce numerical examples. Let $p = 3$. We suppose that $k = \mathbf{Q}(\sqrt{-12453})$ or $\mathbf{Q}(\sqrt{-78730})$. In this case, p does not split in k and we have $\lambda_p(k) = 4$, where $\lambda_p(k)$ is the Iwasawa λ-invariant with respect to the cyclotomic $\mathbf{Z}_p$-extension. Let k_∞/k be the cyclotomic $\mathbf{Z}_p$-extension of k. For each $n \geq 0$, we denote by k_n the intermediate field of k_∞/k such that k_n is the unique cyclic extension over k of degree p^n. Let A_n be the p-Sylow subgroup of the ideal class group of k_n. We put $X = \varprojlim A_n$, where the inverse limit is taken with respect to the relative norms. Then X becomes a $\mathbf{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$-module. Since there is an isomorphism of rings between $\Lambda = \mathbf{Z}_p[[T]]$ and $\mathbf{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$, which depends on the choice of a topological generator of $\mathrm{Gal}(k_\infty/k)$, X becomes a finitely generated torsion Λ -module. Let $f(T)$ be the distinguished polynomial which generates $\mathrm{char}(X)$. It is known that X is a free $\mathbf{Z}_p$-module, so $[X]_{\mathbf{Q}_p} \in \mathcal{M}^{\mathbf{Q}_p}_{f(T)}$ and we can apply our theorem to the Iwasawa module X.

We can calculate the polynomial $f(T) \bmod p^n$ for small n numerically. Let χ be the Dirichlet character associated to k, ω be the Teichimüler character and f_0 be the least common multiple of p and conductor of χ. By the Iwasawa main conjecture, there exists a power series

$g_{\chi^{-1}\omega}(T) \in \Lambda$ such that

$$\mathrm{char}(X) = (g_{\chi^{-1}\omega}(T)) .$$

Here, $g_{\chi^{-1}\omega}(T)$ is the p-adic L-function constructed by Iwasawa. We can approximate $g_{\chi^{-1}\omega}(T)$ such as

$$g_{\chi^{-1}\omega}(T) \equiv -\frac{1}{2f_0p^n} \sum_{0<a<f_0p^n,(a,f_0p^n)=1} a\chi\omega^{-1}(a)(1+T)^{i_n(a)} \mod \omega_n ,$$

where $i_n(a)$ is the unique integer such that $a\omega^{-1}(a) \equiv (1+p)^{i_n(a)} \bmod p^{n+1}$ and $0 \le i_n(a) < p^n$. By Weierstrass preparation theorem ([14], Theorem 7.3), there exists $u_{\chi^{-1}\omega} \in \Lambda^\times$ such that $g_{\chi^{-1}\omega}(T) = f(T)u_{\chi^{-1}\omega}(T)$. Thus we can get $f(T)$ approximately ([14], Proposition 7.2). For the detail about computation of $g_{\chi^{-1}\omega}(T)$, see [1] and [4]. We computed $f(T)$ by Mizusawa's program Iwapoly.ub ([9], Research, Programing, Approximate Computation of Iwasawa Polynomials by UBASIC), and referred Fukuda's table for the λ-invariants of imaginary quadratic fields [3].

For a non-negative integer n, we put $\omega_n = \omega_n(T) = (1+T)^{p^n} - 1$. In order to determine the structure of X, we use the following fact. In our case, exactly one prime is ramified in k_∞/k and it is totally ramified. So there are Λ-isomorphisms

$$X/\omega_n X \cong A_n \tag{28}$$

for any non-negative integers ([14], Proposition 13.22). We determine the Λ-isomorphism class of X by the information on the structures of A_n for some $n \ge 0$.

EXAMPLE 1. Let $k = \mathbf{Q}(\sqrt{-12453})$. In this case, we have $A_0 \cong \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ (cf. [11]). We have

$$f(T) \equiv (T^3 + 204T^2 + 567T + 426)(T + 525) \mod 3^6 .$$

By Hensel's Lemma, there exist $\delta \in \mathbf{Z}_p$ and an irreducible polynomial $g(T) \in \mathbf{Z}_p[T]$ such that

$$f(T) = g(T)(T - \delta) ,$$

where $\delta \equiv 204 \bmod 3^5$ and $g(T) \equiv T^3 + 204T^2 + 81T + 183 \bmod 3^5$. Let E be the minimal splitting field of $g(T)$ and $g(T) = (T-\alpha)(T-\beta)(T-\gamma)$, where $\alpha, \beta, \gamma \in E$. Then $[E : \mathbf{Q}_p] = 3$ and the ramification index is 3 in $E/\mathbf{Q}_p$. Indeed, let $d(g)$ be the discriminant of $g(T)$. Then we have $d(g) \equiv (-1)\cdot 3^4 \cdot 13 \cdot 104 \equiv -162 \bmod 3^5$. Thus we have $\sqrt{d(g)} \in \mathbf{Q}_p$. This implies that $[E : \mathbf{Q}_p] = 3$ and $E/\mathbf{Q}_p$ is a totally ramified extension. Further we have $\mathrm{ord}_E(\alpha-\beta) = \mathrm{ord}_E(\beta-\gamma) = \mathrm{ord}_E(\gamma-\alpha) = 2$, $\mathrm{ord}_E(\alpha-\delta) = \mathrm{ord}_E(\beta-\delta) = \mathrm{ord}_E(\gamma-\delta) = 1$, $\mathrm{ord}_E(\alpha) = \mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 1$ and $\mathrm{ord}_E(\delta) = 3$. Let $[X \otimes_\Lambda \Lambda_E] = [M(\ell, m, n; x, y, z)] \in \mathcal{M}^E_{f(T)}$. By Proposition 11, we have $n = 0$. Therefore we may assume that $[X \otimes_\Lambda \Lambda_E] = [M(\ell, m, 0; x, 0, 0)] = [N(\ell, m, x) \oplus \langle(0,0,0,1)\rangle_{\mathbf{Z}_p}]$, where $N(\ell, m, x)$ are defined before Theorem 2. Since we have $X/TX \otimes \mathcal{O}_E \cong A_0 \otimes \mathcal{O}_E \cong \mathcal{O}_E/(\pi^3) \oplus \mathcal{O}_E/(\pi^3)$, $N(\ell, m, x)/TN(\ell, m, x)$ is a cyclic module. Then N becomes a Λ_E-cyclic module by Nakayama's Lemma. Using Proposition 5.2 in [8], we have $N(\ell, m, x) = N(2, 4, u\pi^2)$, where $u = \dfrac{\gamma - \alpha}{\beta - \alpha}$. Hence we obtain $X \otimes_\Lambda \Lambda_E \cong M(2, 4, 0; u\pi^2, 0, 0)$.

EXAMPLE 2. Let $k = \mathbf{Q}(\sqrt{-78730})$. In this case, we have $A_0 \cong \mathbf{Z}/9\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ (cf. [11]). We have

$$f(T) \equiv (T^2 + 4068T + 5817)(T + 3189)(T + 888) \mod 3^8 .$$

By Hensel's Lemma, there exist γ and $\delta \in \mathbf{Z}_p$ and an irreducible polynomial $g(T) \in \mathbf{Z}_p[T]$ such that

$$f(T) = g(T)(T - \gamma)(T - \delta) ,$$

where $\gamma \equiv 84 \mod 3^5$, $\delta \equiv 213 \mod 3^5$ and $g(T) \equiv T^2 + 180T + 228 \mod 3^5$. Let E be the minimal splitting field of $g(T)$ and $g(T) = (T - \alpha)(T - \beta)$, where $\alpha, \beta \in E$. Since $g(T)$ is an Eisenstein polynomial, the extension $E/\mathbf{Q}_p$ is a totally ramified extension. Therefore, we have $\mathrm{ord}_E(\alpha) = \mathrm{ord}_E(\beta) = 1$, $\mathrm{ord}_E(\gamma) = \mathrm{ord}_E(\delta) = 2$, $\mathrm{ord}_E(\gamma - \delta) = 2$, $\mathrm{ord}_E(\alpha-\beta) = \mathrm{ord}_E(\beta-\gamma) = \mathrm{ord}_E(\beta-\delta) = \mathrm{ord}_E(\alpha-\delta) = \mathrm{ord}_E(\gamma-\alpha) = 1$. By Proposition 10, we obtain $\mathrm{Fitt}_{1,\Lambda_E}(X\otimes\Lambda_E) \mod (T-\delta) = (\pi^{4-n})$. Since we have $A_0 \cong \mathbf{Z}/9\mathbf{Z}\oplus\mathbf{Z}/3\mathbf{Z}$, we obtain $\mathrm{Fitt}_{1,\Lambda}(X) \mod (T-\delta) \neq \Lambda$. We put $\mathrm{Fitt}_{1,\Lambda}(X) \mod (T-\delta) = (p^i)$ for some $i \geq 1$. Then we have $(\pi^{4-n}) = (\pi^{2i})$. This implies $4 - n = 2i$. Clearly, we have $n = 0$ or $n = 2$. Using Proposition 10, we get

$$\mathrm{Fitt}_{1,\Lambda_E}(X \otimes \Lambda_E) \mod (T - \gamma) = \begin{cases} (\pi^{\mathrm{ord}_E(z)+4-m-n}) & \text{if } z \neq 0 , \\ (\pi^{4-m}) & \text{if } z = 0 . \end{cases}$$

Therefore we may consider the only three cases

$$(\natural) \begin{cases} n = 2 \ \textit{and} \ m = \mathrm{ord}_E(z) , \\ n = 2 \ \textit{and} \ z = 0 , \\ n = 0 . \end{cases}$$

The isomorphism classes of Λ_E-module $M(\ell, m, n; x, y, z)$ satisfying $(\natural)$are

$$\left\{ \begin{array}{l} [M(0,1,2;0,0,\pi)], [M(0,1,2;0,\pi,\pi)], [M(0,1,2;1,1,\pi)] , \\ [M(0,1,2;1,1+\pi,\pi)], [M(0,1,2;2,2,\pi)], [M(0,1,2;2,2+\pi,\pi)] , \\ [M(0,1,2;2,2+2\pi,\pi)], [M(1,0,2;0,0,1)], [M(1,0,2;0,\pi,2)] , \\ [M(1,0,2;0,0,1+\pi)], [M(1,1,2;0,\pi,2\pi)], [M(1,1,2;0,0,\pi)] , \\ [M(1,0,2;0,2\pi,0)], [M(1,2,2;2\pi,2\pi,0)] \end{array} \right\}$$

$$\cup \ \{ [N \oplus \Lambda_E/(T - \delta)\Lambda_E] \mid [N] \in \mathcal{M}^E_{(T-\alpha)(T-\beta)(T-\gamma)} \}$$

$$\cup \; \{ \, [M(0,0,2;0,y,z)] \mid \mathrm{ord}_E(z) = 0\} \, . \tag{29}$$

It is easy to see that $M = N \oplus \Lambda_E/(T-\delta)\Lambda_E$ does not satisfy $M/TM \cong \mathcal{O}_E/\pi^4\mathcal{O}_E \oplus \mathcal{O}_E/\pi^2\mathcal{O}_E$ if $N \not\cong N(1,2,u\pi)$, where $u = \dfrac{\gamma-\alpha}{\beta-\alpha}$. We note that $N(1,2,u\pi) \cong \Lambda_E/(T-\alpha)(T-\beta)(T-\gamma)\Lambda_E$ by Proposition 5.2 in [8]. We can also check $M/TM \not\cong \mathcal{O}_E/\pi^4\mathcal{O}_E \oplus \mathcal{O}_E/\pi^2\mathcal{O}_E$ for $[M] \in \{[M(0,0,2;0,y,z)] \mid \mathrm{ord}_E(z) = 0\}$ and $[M(0,1,2;0,0,\pi)]$, $[M(0,1,2;1,1,\pi)]$ and $[M(1,1,2;0,0,\pi)]$.

Now we investigate the structure of A_1 as a $\mathrm{Gal}(k_1/k)$-module. We have an isomorphism $A_1 \cong \mathbf{Z}/27\mathbf{Z} \oplus \mathbf{Z}/9\mathbf{Z} \oplus \mathbf{Z}/9\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$. Furthermore, Pari-Gp gives explicit generators which give this isomorphism. Let $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ and $\mathfrak{a}_4$ be the generators Pari-Gp computed. (We do not write down $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ and $\mathfrak{a}_4$ because they are complicated.) Let σ be a generator of $\mathrm{Gal}(k_1/k)$. By Pari-Gp, we compute

$$\begin{aligned}
(\sigma-1)\mathfrak{a}_1 &= 6\mathfrak{a}_1 - \mathfrak{a}_2 + \mathfrak{a}_3 \, ,\\
(\sigma-1)\mathfrak{a}_2 &= 3\mathfrak{a}_2 + 4\mathfrak{a}_3 \, ,\\
(\sigma-1)\mathfrak{a}_3 &= 9\mathfrak{a}_1 + 6\mathfrak{a}_2 + 6\mathfrak{a}_3 \, ,\\
(\sigma-1)\mathfrak{a}_4 &= 6\mathfrak{a}_2 \, .
\end{aligned}$$

There is a topological generator $\tilde{\sigma} \in \mathrm{Gal}(k_\infty/k)$ such that $\tilde{\sigma}$ is an extension of σ. By this topological generator, we have an isomorphism

$$\mathbf{Z}_p[[\mathrm{Gal}(k_\infty/k)]] \cong \Lambda = \mathbf{Z}_p[[T]] \;\; \text{such that } \tilde{\sigma} \leftrightarrow 1+T.$$

We regard X as a Λ-module by this isomorphism. Because $\mathbf{Z}_p[\mathrm{Gal}(k_1/k)] \cong \Lambda/\omega_1\Lambda$, we get

$$\begin{aligned}
\overline{T}\mathfrak{a}_1 &= 6\mathfrak{a}_1 - \mathfrak{a}_2 + \mathfrak{a}_3 \, ,\\
\overline{T}\mathfrak{a}_2 &= 3\mathfrak{a}_2 + 4\mathfrak{a}_3 \, ,\\
\overline{T}\mathfrak{a}_3 &= 9\mathfrak{a}_1 + 6\mathfrak{a}_2 + 6\mathfrak{a}_3 \, ,\\
\overline{T}\mathfrak{a}_4 &= 6\mathfrak{a}_2 \, ,
\end{aligned}$$

where $\overline{T} = T \bmod \omega_1$. Now we have

$$\begin{cases}
\overline{(T^2-12T)}\mathfrak{a}_1 + \overline{(T-12)}\mathfrak{a}_2 &= 0\, ,\\
\overline{(4T-24)}\mathfrak{a}_1 - \overline{(T-7)}\mathfrak{a}_2 &= 0\, ,\\
\overline{6}\mathfrak{a}_2 - \overline{T}\mathfrak{a}_4 &= 0\, ,\\
\overline{27}\mathfrak{a}_1 &= 0\, ,\\
\overline{9T}\mathfrak{a}_1 &= 0\, ,\\
\overline{9}\mathfrak{a}_2 &= 0\, ,\\
\overline{3}\mathfrak{a}_4 &= 0\, .
\end{cases} \tag{30}$$

Therefore, we can calculate the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$;

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E) \mod 9 = (T, 3) \mod (\omega_1, 9)\,, \tag{31}$$

where $\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E)$ is the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$ as a $\Lambda_E/\omega_1\Lambda_E$-module. Then $M(0,1,2;0,\pi,\pi)$, $M(1,0,2;0,0,1)$, $M(1,0,2;0,0,1+\pi)$, $M(1,1,2;0,\pi,2\pi)$ do not satisfy (31). Therefore we get

$$\begin{aligned} X \otimes_\Lambda \Lambda_E \cong\, & M(0,1,2;2,2+\pi,\pi),\ M(0,1,2;1,1+\pi,\pi),\ M(1,0,2;0,\pi,2)\,, \\ & M(0,1,2;2,2,\pi), M(0,1,2;2,2+2\pi,\pi),\ M(1,0,2;0,2\pi,0)\,, \\ & M(1,2,2;2\pi,2\pi,0), \text{or } M(1,2,0;u\pi,0,0)\,. \end{aligned}$$

Further, using the relations above (30), we get

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(\overline{(T-\gamma)A_1 \otimes \mathcal{O}_E}) \mod 9 = (T, 3) \mod (\omega_1, 9)\,, \tag{32}$$

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(\overline{(T-\delta)A_1 \otimes \mathcal{O}_E}) \mod 9 = (T, 3) \mod (\omega_1, 9)\,. \tag{33}$$

Then only $M(1,0,2;0,\pi,2)$ satisfies (32) and (33). Hence we obtain $X \otimes_\Lambda \Lambda_E \cong M(1,0,2;0,\pi,2)$.

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