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A Remark on Regularity of Solutions to Wave Equations

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Abstract. We prove a generalization of the Strichartz estimate for the inhomogeneous wave equation $\Box u(t,x) = f(t,x)$ in the space-time $\boldsymbol{R}^{1+n}$. We estimate the solution in vector-valued homogeneous Besov spaces $\dot{B}^{\theta}_{q,2}(\boldsymbol{R}; \dot{B}^{\sigma}_{r,2}(\boldsymbol{R}^n))$. Such an estimate shows the time differentiability of the solution of fractional order.

1. Introduction

In this paper, we study the following Cauchy problem for the wave equation:

$$\Box u(t,x) = f(t,x)\,, \tag{1}$$

$$u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = u_1(x)\,. \tag{2}$$

Here, $(t,x) \in \boldsymbol{R}^{1+n}$ with $n \geq 2$ and $\Box = \partial_t^2 - \Delta$ is the usual d'Alembertian.

It is well-known that the solution to the wave equation satisfies the inequality called "Strichartz estimate", which shows space-time integrability of the solution [6, 10, 14, 7]. Similar estimates are also known for various kinds of wave and dispersive equations, e.g., the Klein-Gordon [2, 4, 10, 14, 13], the Schrödinger [5, 17, 14], and the Airy (linear part of the K-dV) [8] equations, and have played important roles in the analysis of related nonlinear problems.

We begin with the definition of function spaces. For a function f defined on $\boldsymbol{R}^n$, we define the Fourier transform of f by $\hat{f}(\xi) = (\mathscr{F} f)(\xi) = \int_{\boldsymbol{R}^n} e^{-ix\xi} f(x)dx$ and the inverse Fourier transform of $\hat{f}$ by $(\mathscr{F}^{-1}\hat{f})(x) = \int_{\boldsymbol{R}^n} e^{ix\xi}\hat{f}(\xi)\,\bar{d}\xi$ respectively, where $\bar{d}\xi = (2\pi)^{-n}d\xi$. Let $\omega = (-\Delta)^{1/2} = \mathscr{F}^{-1}|\xi|\mathscr{F}$. For $s \in \boldsymbol{R}$ and $1 \leq r \leq \infty$, the homogeneous Sobolev space $\dot{H}^s_r = \dot{H}^s_r(\boldsymbol{R}^n)$ is defined by $\dot{H}^s_r = \omega^{-s}L^r$, where $L^r = L^r(\boldsymbol{R}^n)$ denotes the usual Lebesgue space. If $r = 2$, we simply write $\dot{H}^s = \dot{H}^s_2$. To define the homogeneous Besov space $\dot{B}^s_r = \dot{B}^s_{r,2}(\boldsymbol{R}^n)$, we need the Littlewood-Paley decomposition. Let $\eta \in C_0^\infty(\boldsymbol{R})$ be a nonnegative even function satisfying $\operatorname{supp}\eta \subset \{\tau; 1/2 \leq |\tau| \leq 2\}$ and

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$\sum_{j=-\infty}^{\infty} \eta(\tau/2^j) = 1$ for $\tau \neq 0$. For any $j \in \boldsymbol{Z}$, we put $\varphi_j(x) = \int_{\boldsymbol{R}^n} e^{ix\xi} \eta(|\xi|/2^j)\, d\!\!{}^-\xi$. The homogeneous Besov space is then defined by

$$\dot{B}_r^s = \left\{ u \in \mathscr{S}'(\boldsymbol{R}^n);\ \|u;\, \dot{B}_r^s\| \equiv \left\{ \sum_{j=-\infty}^{\infty} 2^{2sj} \|\varphi_j \underset{(x)}{*} u;\, L^r\|^2 \right\}^{1/2} < \infty \right\}.$$

Precisely, we identify two tempered distributions if their difference is a polynomial since $\|u;\, \dot{B}_r^s\| = 0$ if and only if u is a polynomial. By the Plancherel theorem, we see that $\dot{H}^s = \dot{B}_2^s$. On the other hand, $\dot{H}_r^s \subset \dot{B}_r^s$ if $1 < r < 2$ and $\dot{H}_r^s \supset \dot{B}_r^s$ if $2 < r < \infty$. We refer the reader to [1, 15] for further details about these spaces.

Similarly we define spaces of vector-valued functions. For a Banach space E and $1 \leq q \leq \infty$, let $L^q(E) = L^q(\boldsymbol{R};\, E)$. We put $\phi_j(t) = \int_{\boldsymbol{R}} e^{it\tau} \eta(\tau/2^j)\, d\!\!{}^-\tau$ for any $j \in \boldsymbol{Z}$. For any $\theta \in \boldsymbol{R}$ and $1 \leq q \leq \infty$, we define the E-valued Besov space $\dot{B}_q^\theta(E) = \dot{B}_{q,2}^\theta(\boldsymbol{R};\, E)$ by

$$\dot{B}_q^\theta(E) = \left\{ u \in \mathscr{S}'(\boldsymbol{R};\, E);\ \|u;\, \dot{B}_q^\theta(E)\| \equiv \left\{ \sum_{j=-\infty}^{\infty} 2^{2\theta j} \|\phi_j \underset{(t)}{*} u;\, L^q(E)\|^2 \right\}^{1/2} < \infty \right\}.$$

For vector-valued Besov spaces, we refer the reader to [12].

Now we precisely state the Strichartz estimate. Following the notation in [6], we introduce the symbols

$$\beta(r) = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), \quad \gamma(r) = (n-1)\left(\frac{1}{2} - \frac{1}{r}\right), \quad \delta(r) = n\left(\frac{1}{2} - \frac{1}{r}\right),$$

and $\bar{r} = r/(r-1)$ for $1 \leq r \leq \infty$. The solution to (1)–(2) satisfies the following inequality:

$$\|u;\, L^q(\dot{B}_r^{s-\beta})\| + \|\partial_t u;\, L^q(\dot{B}_r^{s-\beta-1})\| \lesssim \|(u_0, u_1);\, \dot{H}^s \times \dot{H}^{s-1}\| + \|f;\, L^{\bar{q}}(\dot{B}_{\bar{r}}^{s+\beta-1})\|, \quad (3)$$

where $s \in \boldsymbol{R}$, $0 \leq 2/q = \gamma(r) \leq 1$ with $(n, q, r) \neq (3, 2, \infty)$, and $\beta = \beta(r)$. For the proof, see [6, 10, 14] when $q \neq 2$ and [7] when $q = 2$ respectively.

On the one hand, by the inequality (3) we can estimate fractional order spatial-derivatives of u; on the other hand, this inequality contains only the first order time-derivative of u. Since the wave equation is second order both in t and x, it is natural to expect that the solution should have the same regularity in both variables. Indeed, Brenner [2] has pointed out that such an estimate holds for the homogeneous Klein-Gordon equation. For the homogeneous wave equation, namely (1)–(2) with $f = 0$, the corresponding estimate $\|u;\, \dot{B}_q^\theta(\dot{B}_r^{s-\beta})\| \lesssim \|(u_0, u_1);\, \dot{H}^{s+\theta} \times \dot{H}^{s+\theta-1}\|$ easily follows from (3) as in the proof of Theorem 1 below. We can further expect that such an estimate holds for the inhomogeneous equation if the inhomogeneous term has appropriate regularity. Pecher [11] proved such an estimate for the inhomogeneous Schrödinger equation, and Uchizono-Wada [16] slightly improved his estimate. Their methods are also applicable for the wave equation, and the corresponding inequality is the following:

$$\|u;\dot B^\theta_q(\dot B^{s-\beta}_r)\|+\|\partial_t u;\dot B^\theta_q(\dot B^{s-\beta-1}_r)\|\lesssim\|(u_0,u_1);\dot H^{s+\theta}\times\dot H^{s+\theta-1}\|$$
$$+\|f;\dot B^\theta_{\bar q}(\dot B^{s+\beta-1}_{\bar r})\|+\sum_\pm\|f;L^{q_1(\theta\pm\varepsilon)}(\dot B^{s+\beta-1}_{r_1(\theta\pm\varepsilon)})\|\,. \tag{4}$$

Here, $\varepsilon>0$ is a sufficiently small number, and $q_1(\theta)$, $r_1(\theta)$ are defined by

$$1/q_1(\theta)=(2\theta-1)/q+1-\theta,\quad \delta(r_1(\theta))=1/q_1(\theta)-1+\theta-\beta\,.$$

However, the inequality (4) is not optimal from the scaling point of view. To see this, we consider the scale change

$$(u,f)\mapsto(u_\lambda,f_\lambda)\equiv(\lambda^{n/2-s}u(\lambda t,\lambda x),\lambda^{n/2-s+2}f(\lambda t,\lambda x))\,. \tag{5}$$

We can easily check that both (1) and (3) are invariant under this scaling. On the other hand, (4) is not scale-invariant, and therefore is not optimal.

In the present paper, we aim at improving Pecher's estimate and obtaining a scale-invariant estimate for the fractional time-derivative of the solution to (1)–(2). In order to state our theorem, we introduce the space $\dot Z^s_{q,r}$, which is the completion of $\dot{\mathscr S}(\boldsymbol R^{1+n})=\{u\in\mathscr S(\boldsymbol R^{1+n});\int_{\boldsymbol R^n}x^\alpha u(t,x)\,dx=0$ for any $\alpha\in\boldsymbol Z^n_+\}$ by the norm

$$\|u;\dot Z^s_{q,r}\|=\left\{\sum_{j=-\infty}^{\infty}2^{2sj}\|\varphi_j\underset{(x)}{*}u;L^q(L^r)\|^2\right\}^{1/2}\sim\left\{\sum_{j=-\infty}^{\infty}\|\varphi_j\underset{(x)}{*}u;L^q(\dot B^s_r)\|^2\right\}^{1/2}.$$

The main theorem in this paper is the following:

THEOREM 1. *Let $s\in\boldsymbol R$ and $0<\theta<1$. Let $1\le q,r,q_1,r_1\le\infty$ satisfy $0\le 2/q=\gamma(r)\le1$ with $(n,q,r)\ne(3,2,\infty)$, $r_1\le2$ and $1/q_1-\delta(r_1)=1-\theta+\beta$ with $\beta=\beta(r)$. Then for any $(u_0,u_1)\in\dot H^{s+\theta}\times\dot H^{s+\theta-1}$ and $f\in\dot B^\theta_{\bar q}(\dot B^{s+\beta-1}_{\bar r})\cap\dot Z^{s+\beta-1}_{q_1,r_1}$, the solution to* (1)–(2) *satisfies the estimate*

$$\|u;\dot B^\theta_q(\dot B^{s-\beta}_r)\|+\|\partial_t u;\dot B^\theta_q(\dot B^{s-\beta-1}_r)\|$$
$$\lesssim\|(u_0,u_1);\dot H^{s+\theta}\times\dot H^{s+\theta-1}\|+\|f;\dot B^\theta_{\bar q}(\dot B^{s+\beta-1}_{\bar r})\cap\dot Z^{s+\beta-1}_{q_1,r_1}\|\,. \tag{6}$$

REMARK 1. (i) We can easily check that both sides of (6) is invariant under the transform (5) with s replaced by $s+\theta$.

(ii) Minkowski's inequality shows $\dot Z^{s+\beta-1}_{q_1,r_1}\supset L^{q_1}(\dot B^{s+\beta-1}_{r_1})$ if $q_1\le2$, and $\dot Z^{s+\beta-1}_{q_1,r_1}\subset L^{q_1}(\dot B^{s+\beta-1}_{r_1})$ if $q_1\ge2$.

(iii) If $\theta\le1/2$, then $q_1(\theta)\le2$. Therefore, putting $(q_1,r_1)=(q_1(\theta),r_1(\theta))$, we have $\dot Z^{s+\beta-1}_{q_1,r_1}\supset L^{q_1}(\dot B^{s+\beta-1}_{r_1})\supset\bigcap_\pm L^{q_1(\theta\pm\varepsilon)}(\dot B^{s+\beta-1}_{r_1(\theta\pm\varepsilon)})$. Therefore (4) follows from (6). On the other hand, (4) does not follow from (6) when $\theta>1/2$, although (6) is a scale-invariant refinement of (4). Even in this case, a slight modification of the proof of Theorem 1 gives a simpler, alternative proof of (4) (see the remark at the end of §2).

(iv) The space $\dot{Z}^s_{1,2}$ (precisely the inhomogeneous counterpart of this space) was introduced by Chemin and Lerner [3] for the analysis of the Navier-Stokes equations.

(v) Similar estimate also holds for the Schrödinger equation [9].

2. Proof of Theorem

We put $v_\pm = \partial_t \omega^{-1} u \pm iu$ and rewrite (1)–(2) to the following first-order system:

$$\partial_t v_\pm \mp i\omega v_\pm = \omega^{-1} f\,, \tag{7}$$

$$v_\pm(0) = v_{0,\pm} = \omega^{-1} u_1 \pm i u_0\,. \tag{8}$$

Furthermore, by the propagator $U_\pm(t) = \exp(\pm it\omega)$, we convert (7)–(8) to the integral equations

$$v_\pm(t) = U_\pm(t) v_{0,\pm} + \int_0^t U_\pm(t-t')\omega^{-1} f(t')dt'\,. \tag{9}$$

Since $\dot{B}^s_r = \omega^{-s}\dot{B}^0_r$, without loss of generality we may assume $s = 0$. Therefore the inequality (3) is clearly equivalent to

$$\|v_\pm;\, L^q(\dot{B}^{-\beta}_r)\| \lesssim \|v_{0,\pm};\, L^2\| + \|f;\, L^{\bar{q}}(\dot{B}^{\beta-1}_{\bar{r}})\|\,, \tag{10}$$

and Theorem 1 follows from the proposition below:

PROPOSITION 1. *Let* $0 < \theta < 1$. *Let* $1 \le q, r, q_1, r_1 \le \infty$ *satisfy* $0 \le 2/q = \gamma(r) \le 1$ *with* $(n, q, r) \ne (3, 2, \infty)$, $r_1 \le 2$ *and* $1/q_1 - \delta(r_1) = 1 - \theta + \beta$ *with* $\beta = \beta(r)$. *Then for any* $v_{0,\pm} \in \dot{H}^\theta$ *and* $f \in \dot{B}^\theta_{\bar{q}}(\dot{B}^{\beta-1}_{\bar{r}}) \cap \dot{Z}^{\beta-1}_{q_1,r_1}$, *the solution to* (7)–(8) *satisfies the estimate*

$$\|v_\pm;\, \dot{B}^\theta_q(\dot{B}^{-\beta}_r)\| \lesssim \|v_{0,\pm};\, \dot{H}^\theta\| + \|f;\, \dot{B}^\theta_{\bar{q}}(\dot{B}^{\beta-1}_{\bar{r}}) \cap \dot{Z}^{\beta-1}_{q_1,r_1}\|\,. \tag{11}$$

PROOF. We only prove the estimate for v_+, so that we omit the subscript "$\pm$". We can estimate the homogeneous and the inhomogeneous terms separately, and the homogeneous estimate is quite easy. Indeed, the Fourier transform shows $\phi_j *_{(t)} U(t)v_0 = U(t)\varphi_j *_{(x)} v_0$, and hence we obtain by (10)

$$\begin{aligned}\|U(t)v_0;\, \dot{B}^\theta_q(\dot{B}^{-\beta}_r)\|^2 &= \sum_{j=-\infty}^{\infty} 2^{2\theta j}\|U(t)\varphi_j \underset{(x)}{*} v_0;\, L^q(\dot{B}^{-\beta}_r)\|^2 \\ &\lesssim \sum_{j=-\infty}^{\infty} 2^{2\theta j}\|\varphi_j \underset{(x)}{*} v_0;\, L^2\|^2 \sim \|v_0;\, \dot{H}^\theta\|^2\,.\end{aligned} \tag{12}$$

We proceed to the inhomogeneous estimate. In what follows we assume $v_0 = 0$, so that

$$v(t) = \int_0^t U(t-t')\omega^{-1} f(t')dt'\,.$$

Then we see by the Fourier transform

$$\hat{v}(t,\xi) = \int_0^t e^{i(t-t')|\xi|}|\xi|^{-1}\hat{f}(t',\xi)dt' = \int_0^t e^{i(t-t')|\xi|}|\xi|^{-1}dt' \int_{-\infty}^{\infty} e^{it'\tau}\tilde{f}(\tau,\xi)\,d\tau$$
$$= \int_{-\infty}^{\infty} \frac{e^{it\tau}-e^{it|\xi|}}{i|\xi|(\tau-|\xi|)}\tilde{f}(\tau,\xi)\,d\tau\,.$$

Here, $\tilde{f}(\tau,\xi)$ denotes the Fourier transform of f in the space-time. Therefore, $v = v_1 - v_2$ with

$$\hat{v}_1(t,\xi) = \text{p.v. -}\int_{-\infty}^{\infty} \frac{e^{it\tau}}{i|\xi|(\tau-|\xi|)}\tilde{f}(\tau,\xi)\,d\tau\,,$$
$$\hat{v}_2(t,\xi) = e^{it|\xi|}\,\text{p.v. -}\int_{-\infty}^{\infty} \frac{1}{i|\xi|(\tau-|\xi|)}\tilde{f}(\tau,\xi)\,d\tau \equiv e^{it|\xi|}\hat{\psi}_0(\xi)\,.$$

By the residue theorem,

$$\text{p.v. -}\int_{-\infty}^{\infty} \frac{e^{it\tau}}{i(\tau-|\xi|)}\,d\tau = \frac{1}{2}\,\text{sign}(t)e^{it|\xi|}\,, \tag{13}$$

and hence we obtain

$$v(t) = \frac{1}{2}\int_{-\infty}^{\infty} \text{sign}(t-t')U(t-t')\omega^{-1}f(t')dt' - U(t)\psi_0\,.$$

Therefore, using the formula $\phi_j *_{(t)} U(t) = U(t)\varphi_j *_{(x)}$ and the inequality (10), we can show

$$\|v;\, \dot{B}^{\theta}_{q}(\dot{B}^{-\beta}_{r})\|^2 = \sum_{j=-\infty}^{\infty} 2^{2\theta j}\|\phi_j \underset{(t)}{*} v;\, L^q(\dot{B}^{-\beta}_{r})\|^2$$
$$\lesssim \sum_{j=-\infty}^{\infty} 2^{2\theta j}\|\phi_j \underset{(t)}{*} f;\, L^{\bar{q}}(\dot{B}^{\beta-1}_{\bar{r}})\|^2 + \sum_{j=-\infty}^{\infty} 2^{2\theta j}\|\varphi_j \underset{(x)}{*} \psi_0;\, L^2\|^2\,.$$

The first term of the right-hand side is $\|f;\, \dot{B}^{\theta}_{\bar{q}}(\dot{B}^{\beta-1}_{\bar{r}})\|^2$. To estimate the second term, we further decompose

$$\hat{\varphi}_j(\xi)\hat{\psi}_0(\xi) = \text{p.v. -}\int_{-\infty}^{\infty} \frac{\hat{\varphi}_j(\xi)\hat{\chi}_j(\tau)}{i|\xi|(\tau-|\xi|)}\tilde{f}(\tau,\xi)\,d\tau + \int_{-\infty}^{\infty} \frac{\hat{\varphi}_j(\xi)(1-\hat{\chi}_j(\tau))}{i|\xi|(\tau-|\xi|)}\tilde{f}(\tau,\xi)\,d\tau$$
$$\equiv \hat{\psi}_{1j}(\xi) + \hat{\psi}_{2j}(\xi)\,,$$

where $\hat{\chi}_j(\tau) = \sum_{k=j-2}^{j+2}\hat{\phi}_k(\tau)$. Similarly as above, the formula (13) yields

$$\psi_{1j} = -\frac{1}{2}\int_{-\infty}^{\infty} \text{sign}(t)U(-t)\varphi_j \underset{(x)}{*} \chi_j \underset{(t)}{*} \omega^{-1}f(t)dt\,.$$

We estimate ψ_{1j} by a duality argument. To that end, we take an arbitrary function $w \in L^2$. Since $U(t)$ is a unitary operator in L^2, from the inequality (10) we see

$$\begin{aligned}\left|(\psi_{1j}, w)_{L^2}\right| &\leq \frac{1}{2}\int_{-\infty}^{\infty}\left|(\varphi_j \underset{(x)}{*} \chi_j \underset{(t)}{*} \omega^{-1} f(t), U(t)w)_{L^2}\right| dt \\ &\leq \frac{1}{2}\|\chi_j \underset{(t)}{*} \omega^{-1} f; L^{\bar{q}}(\dot{B}^{\beta}_{\bar{r}})\| \, \|U(t)w; L^q(\dot{B}^{-\beta}_r)\| \\ &\lesssim \|\chi_j \underset{(t)}{*} \omega^{-1} f; L^{\bar{q}}(\dot{B}^{\beta}_{\bar{r}})\| \, \|w; L^2\| .\end{aligned}$$

Since w is arbitrary, we obtain $\|\psi_{1j}; L^2\| \lesssim \|\chi_j *_{(t)} f; L^{\bar{q}}(\dot{B}^{\beta-1}_{\bar{r}})\|$, which yields

$$\sum_{j=-\infty}^{\infty} 2^{2\theta j}\|\psi_{1j}; L^2\|^2 \lesssim \|f; \dot{B}^{\theta}_{\bar{q}}(\dot{B}^{\beta-1}_{\bar{r}})\|^2 .$$

We next consider ψ_{2j}. We put

$$K(t,x) = \iint_{\boldsymbol{R}^{1+n}} e^{it\tau + ix\xi} \frac{\hat{\varphi}_0(|\xi|)(1-\hat{\chi}_0(\tau))}{i(\tau - |\xi|)} \, d\!\!\bar{}\tau \, d\!\!\bar{}\xi .$$

On the support of the integrand, we have $|\tau - |\xi|| \geq 1/4$. Therefore, the integration by parts shows that $(1+|t|+|x|)^m K(t,x) \in L^\infty(\boldsymbol{R}^{1+n})$ for any nonnegative integer m. We also introduce the rescaled function $K_j(t,x) = 2^{nj}K(2^j t, 2^j x)$. Then

$$\psi_{2j}(x) = \iint_{\boldsymbol{R}^{1+n}} K_j(-t, x-x') \sum_{k=j-2}^{j+2} \varphi_k \underset{(x)}{*} \omega^{-1} f(t,x') \, dt dx' .$$

For q_1, r_1 in the assumption of the proposition, we can choose $1 \leq q_0, r_0 \leq \infty$ such that $1/q_0 + 1/q_1 = 1/r_0 + 1/r_1 - 1/2 = 1$. Applying the Hölder and the Young inequalities, we see

$$\begin{aligned}\|\psi_{2j}; L^2\| &\leq \|K_j; L^{q_0}(L^{r_0})\| \sum_{k=j-2}^{j+2} \|\varphi_k \underset{(x)}{*} \omega^{-1} f; L^{q_1}(L^{r_1})\| \\ &= 2^{(n-1/q_0 - n/r_0)j}\|K; L^{q_0}(L^{r_0})\| \sum_{k=j-2}^{j+2} \|\varphi_k \underset{(x)}{*} \omega^{-1} f; L^{q_1}(L^{r_1})\| \\ &\sim 2^{(\beta-\theta)j} \sum_{k=j-2}^{j+2} 2^{-k}\|\varphi_k \underset{(x)}{*} f; L^{q_1}(L^{r_1})\| ,\end{aligned}$$

since $n - 1/q_0 - n/r_0 = 1/q_1 - \delta(r_1) - 1 = \beta - \theta$. Therefore, we obtain

$$\sum_{j=-\infty}^{\infty} 2^{2\theta j}\|\psi_{2j}; L^2\|^2 \lesssim \sum_{j=-\infty}^{\infty} 2^{2(\beta-1)j}\|\varphi_j \underset{(x)}{*} f; L^{q_1}(L^{r_1})\|^2 = \|f; \dot{Z}^{\beta-1}_{q_1,r_1}\|^2 .$$

Collecting these estimates, we can prove the proposition. □

REMARK 2. Let $\varepsilon > 0$ be a sufficiently small number, and let $1/q_{1,\pm} - \delta(r_{1,\pm}) = 1 - (\theta \pm \varepsilon) + \beta$. Then we see

$$\sum_{j=-\infty}^{\infty} 2^{2\theta j} \|\psi_{2j}; L^2\|^2 \lesssim \sum_{j=-\infty}^{-1} 2^{2(\beta-1+\varepsilon)j} \|\varphi_j \underset{(x)}{*} f; L^{q_{1,-}}(L^{r_{1,-}})\|^2 + \sum_{j=0}^{\infty} 2^{2(\beta-1-\varepsilon)j} \|\varphi_j \underset{(x)}{*} f; L^{q_{1,+}}(L^{r_{1,+}})\|^2$$
$$\lesssim \|f; L^{q_{1,-}}(\dot{B}^{\beta-1}_{r_{1,-}})\|^2 + \|f; L^{q_{1,+}}(\dot{B}^{\beta-1}_{r_{1,+}})\|^2 .$$

Therefore we obtain

$$\|v_\pm; \dot{B}^\theta_q(\dot{B}^{-\beta}_r)\| \lesssim \|v_{0,\pm}; \dot{H}^\theta\| + \|f; \dot{B}^\theta_{\tilde{q}}(\dot{B}^{\beta-1}_{\tilde{r}})\| + \sum_\pm \|f; L^{q_{1,\pm}}(\dot{B}^{\beta-1}_{r_{1,\pm}})\| .$$

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