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Inversion Formula for the Discrete Radon Transform

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Abstract. We shall give a characterization of the discrete Radon transform of functions in the Schwartz space on $\mathbf{Z}^n$ and obtain various inversion formulas for the discrete Radon transform on $\mathbf{Z}^2$.

1. Introduction

The classical Radon transform was firstly defined on $\mathbf{R}^2$ by J. Radon [5] as the integral over a line L in $\mathbf{R}^2$:

$$Rf(L) = \int_L f(x)d\mu(x)\,,$$

where $d\mu(x)$ is the Euclidean measure on L. Each line L with the direction vector $\omega \in S^1$ is given by $L(\omega, t) = \{x \in \mathbf{R}^2 \mid x \cdot \omega = t\}$ where $t \in \mathbf{R}$ and $x \cdot \omega$ is the inner product of x and ω. Hence the set of all lines in $\mathbf{R}^2$ is parameterized as $S^1 \times \mathbf{R}/\{\pm 1\}$. The Radon transform R is related to the Fourier transform as the slice formula:

$$\widetilde{Rf}(L(\omega, \cdot))(\lambda) = \widetilde{f}(\lambda\omega)\,,$$

where the left hand side is the one-dimensional Fourier transform and the right hand side is the two-dimensional one. Hence we can recover f from Rf by using this relation. However, this inversion formula has a difficulty of convergence of inversion Fourier transforms. Another method to invert the Radon transform involves the dual Radon transform. We integrate $Rf(L(\omega, t))$ over S^1 and apply a fractional differential operator on $\mathbf{R}$ such as $\sqrt{-\Delta}$. The idea to recover a function on $\mathbf{R}^2$ from its integrals over all lines is generalized in different settings by various people. For an extensive survey we refer to Helgason's book [4].

In this paper, as analogue of the classical Radon transform on $\mathbf{R}^2$, we shall consider the discrete Radon transform on $\mathbf{Z}^n$, which was originally proposed by Strichartz [6] and was

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introduced by Abouelaz and Ihsane [1]. For a function f on $\mathbf{Z}^n$ the discrete Radon transform Rf is defined by the summation of $f(m)$ over m in a discrete hyperplane H in $\mathbf{Z}^n$:

$$Rf(H) = \sum_{m \in H} f(m)\,. \tag{1}$$

Let $\mathbf{G}$ be the set of all discrete hyperplanes H in $\mathbf{Z}^n$ (see §2). Then R transfers functions on $\mathbf{Z}^n$ to ones on $\mathbf{G}$. Some basic properties of the discrete Radon transform R were obtained in [1] and [2]. Especially, similarly as in the classical case, the slice formula and the inversion formula for R were established. Roughly speaking, the one-dimensional Fourier series of Rf is related with the n-dimensional Fourier series of f (see (4)). Since $\mathbf{G}$ is a discrete set, as the answer in [6], the inversion formula for R has a quite simple form without a dual Radon transform and a fractional differential operator (see (5)). However, concerning the Schwartz theorem, we have only a partial result. In the classical case, the image of the Radon transform of Schwartz space $\mathcal{S}(\mathbf{R}^2)$ is characterized as functions F in $\mathcal{S}(S^1 \times \mathbf{R})$ which have the property that for each $k = 0, 1, 2, \ldots,$

$$\int_{-\infty}^{\infty} F(\omega, t) t^k dt$$

can be written as a homogeneous kth degree polynomial of ω (see [3]). In our discrete case, if f is a rapidly decreasing function on $\mathbf{Z}^n$, then $Rf(H)$ is decreasing when H goes away from the parallel hyperplane through the origin and the sum of $Rf(H)$ over parallel hyperplanes satisfies the above property of a homogeneous polynomial. Hence R maps injectively the Schwartz space $\mathcal{S}(\mathbf{Z}^n)$ into a kind of Schwartz classes on $\mathbf{G}$ satisfying these properties. But, this map is not surjective.

The aim of this paper is to give a more precise characterization of the image of the discrete Radon transform of $\mathcal{S}(\mathbf{Z}^n)$ and obtain several new inversion formulas for the discrete Radon transform on $\mathbf{Z}^2$.

2. Notation

We briefly state some basic properties on the discrete Radon transform R on $\mathbf{Z}^n$. For more details we refer to [1] and [2].

Let $\mathcal{P}$ be the set of all $a = (a_1, a_2, \ldots, a_n) \in \mathbf{Z}^n$ such that the greatest common divisor $d(a_1, a_2, \ldots, a_n)$ equals 1. For each $a \in \mathcal{P}$ and $k \in \mathbf{Z}$, the set $H(a, k) = \{x \in \mathbf{Z}^n \mid ax = k\}$ forms a discrete hyperplane in $\mathbf{Z}^n$, where ax is the inner product of a and x. Then the set $\mathbf{G}$ of discrete hyperplanes on $\mathbf{Z}^n$ is parameterized as $\mathcal{P} \times \mathbf{Z}/\{\pm 1\}$ (see [1], §2). Hence R in (1) transfers a function $f(m)$ on $\mathbf{Z}^n$ to $Rf(H(a, k))$ on $\mathcal{P} \times \mathbf{Z}/\{\pm 1\}$. Let $l^p(\mathbf{Z}^n)$, $1 \le p < \infty$, denote the space of all functions f on $\mathbf{Z}^n$ with finite l^p-norm

$$\|f\|_p = \Big(\sum_{m \in \mathbf{Z}^n} |f(m)|^p\Big)^{1/p}$$

and $l^\infty(\mathbf{Z}^n)$ the one with finite l^∞-norm. Since $\bigcup_{k\in\mathbf{Z}} H(a,k) = \mathbf{Z}^n$ for all $a \in \mathcal{P}$, R is well-defined for $f \in l^1(\mathbf{Z}^n)$ and

$$\sum_{k\in\mathbf{Z}} |Rf(H(a,k))| \le \|f\|_1 . \tag{2}$$

The slice formula for the discrete Radon transform R is given as follows. For $f \in l^1(\mathbf{Z}^n)$ and $\varphi \in l^\infty(\mathbf{Z})$,

$$\begin{aligned} \sum_{k\in\mathbf{Z}} Rf(H(a,k))\varphi(k) &= \sum_{k\in\mathbf{Z}} \Big(\sum_{m\in H(a,k)} f(m) \Big) \varphi(k) \\ &= \sum_{m\in\mathbf{Z}^n} f(m)\varphi(am) . \end{aligned} \tag{3}$$

Especially, letting $\varphi_\lambda(k) = e^{i\lambda k}$, $0 \le \lambda < 2\pi$, we see that

$$\widetilde{Rf}(H(a,\cdot))(\lambda) = \widetilde{f}(\lambda a) , \tag{4}$$

where the tildes denote the Fourier inverse transforms on $\mathbf{Z}$ and $\mathbf{Z}^n$, that is, the Fourier series on $\mathbf{T}$ and $\mathbf{T}^n$ respectively.

The inversion formula for R is given as follows. Let χ_N be the characteristic function of a discrete ball $B(N) = \{x \in \mathbf{Z}^n \mid \|x\| \le N\}$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbf{R}^n$. Then, for all $\varepsilon > 0$, there exists a sufficiently large N for which $\|f - f\chi_N\|_1 < \varepsilon$ and thus, by (2)

$$|R(f\chi_N)(H(a,k)) - Rf(H(a,k))| < \varepsilon$$

for all $a \in \mathcal{P}$. Let $a_j = (1, j, j^2, \dots, j^{n-1})$ for $j \in \mathbf{N}$. As shown in [1], if $j > N$, then $B(N) \cap H(a_j, 0) = \{0\}$ and thus, $R(f\chi_N)(H(a,0)) = f\chi_N(0) = f(0)$. Therefore, combining the above inequality, we have the following inversion formula: For $f \in l^1(\mathbf{Z}^n)$,

$$\lim_{j\to\infty} Rf(a_j, 0) = f(0) . \tag{5}$$

When $n = 2$, we can prove that $B(N) \cap H(a,0) = \{0\}$ if $\|a\| > N$ † . Hence, it follows that for $f \in l^1(\mathbf{Z}^2)$,

$$\lim_{\|a\|\to\infty} Rf(a,0) = f(0) .$$

3. Schwartz space

Let $\mathcal{S}(\mathbf{Z}^n)$ be the Schwartz space on $\mathbf{Z}^n$ consisting of all functions f on $\mathbf{Z}^n$ such that

$$p_N(f) = \sup_{m\in\mathbf{Z}^n} (1 + \|m\|^2)^N |f(m)| < \infty$$

† This is not true when $n > 2$. For example, let $n = 3$ and $a_p = (p, 2, -(p+2))$ for a prime number p. Then $(1,1,1) \in H(a_p, 0)$ for all a_p.

for all $N = 0, 1, 2, \ldots$. Then $\{p_N \mid N = 0, 1, 2, \ldots\}$ is a family of semi-norms of $\mathcal{S}(\mathbf{Z}^n)$. We note that, for $f \in \mathcal{S}(\mathbf{Z}^n)$ and $N \in \mathbf{N}$, if $2N > n$, then

$$\begin{aligned}
&(1+k^2)^N Rf(H(a,k)) \\
\leq &\sum_{\{m \mid am=k\}} |f(m)|(1+|am|^2)^N \\
\leq &\sum_{\{m \mid am=k\}} |f(m)|(1+\|a\|^2)^N(1+\|m\|^2)^N \\
\leq &p_{2N}(f) \sum_{m\in\mathbf{Z}^n} (1+\|m\|^2)^{-N} \cdot (1+\|a\|^2)^N.
\end{aligned}$$

Therefore, it follows that

$$|Rf(H(a,k))| \leq c_N p_{2N}(f)\Big(\frac{1+\|a\|^2}{1+k^2}\Big)^N, \tag{6}$$

where c_N is independent of a and k. In what follows, for simplicity, we assume that $n = 2$. In the case of general n, the same arguments are easily applicable. For $a = (a_1, a_2) \in \mathcal{P}$, $H(a,k)$, $k \in \mathbf{Z}$, are discrete hyperplanes with the same direction and they cover $\mathbf{Z}^2$. For each $k \in \mathbf{Z}$, we choose $m \in H(a,k)$ that is nearest to the origin. We denote it by $m_0(a,k)$ and set

$$D(a) = \{m_0(a,k) \mid k \in \mathbf{Z}\},$$

where we take $m_0(a,0) = 0$. Clearly, we see that

$$H(a,k) = \{m_0 + la_0 \mid l \in \mathbf{Z}\},$$

where $a_0 = (-a_2, a_1)$ and $m_0 = m_0(a,k) \in D(a)$. Then it follows that

$$\begin{aligned}
Rf(H(a,k)) &= \sum_{m\in H(a,k)} f(m) \\
&= f(m_0) + \sum_{0<|l|\leq 4\frac{\|m_0\|}{\|a\|}} f(m_0+la_0) + \sum_{|l|>4\frac{\|m_0\|}{\|a\|}} f(m_0+la_0) \\
&= f(m_0) + I_1 + I_2.
\end{aligned}$$

As for I_1, since $|f(m)| \leq p_N(f)(1+\|m\|^2)^{-N} \leq p_N(f)(1+\|m_0\|^2)^{-N}$,

$$\begin{aligned}
|I_1| &\leq p_N(f)(1+\|m_0\|^2)^{-N} 4\frac{\|m_0\|}{\|a\|} \\
&\leq \frac{c}{\|a\|(1+\|m_0\|)^{2N-1}},
\end{aligned}$$

where c is independent of a and k. As for I_2, we note that $\|a_0\| = \|a\|$ and $2lm_0a_0 \geq -2|l|\|m_0\|\|a_0\| \geq -\frac{|l|^2\|a\|^2}{2}$. Hence

$$\|m\|^2 = \|m_0\|^2 + l^2\|a\|^2 + 2lm_0a \geq \|m_0\|^2 + \frac{l^2\|a\|^2}{2}$$

and thus,

$$|f(m)| \leq p_N(f)\Big(1 + \|m_0\|^2 + \frac{l^2\|a\|^2}{2}\Big)^{-N}.$$

Therefore, if $N > 1$, then

$$\begin{aligned}|I_2| &\leq cp_N(f) \sum_{|l|>4\frac{\|m_0\|}{\|a\|}} \Big(1 + \|m_0\|^2 + \frac{l^2\|a\|^2}{2}\Big)^{-N} \\ &\leq 2cp_N(f)\int_{4\frac{\|m_0\|}{\|a\|}}^{\infty} \Big(1 + \|m_0\|^2 + \frac{x^2\|a\|^2}{2}\Big)^{-N} dx \\ &\leq \frac{c}{\|a\|(1+\|m_0\|)^{2N-1}},\end{aligned}$$

where c is independent of a and k. Hence we can deduce that $Rf(H(a,k))$ has a decomposition

$$Rf(H(a,k)) = f(m_0) + g(a,k) \tag{7}$$

and for each $N = 0, 1, 2, \ldots,$

$$g(a,k) \leq \frac{c}{\|a\|(1+\|m_0\|)^N}, \tag{8}$$

where c is independent of a and k. Moreover, noting (3) and (4), we see that

$$\begin{aligned}\widetilde{Rf}(H(a,\cdot))(\lambda) &= \sum_{m\in\mathbf{Z}^n} f(m)e^{i\lambda am} \\ &= \sum_{m_0\in D(a)} f(m_0)e^{i\lambda am_0} + \sum_{m\in D(a)^c} f(m)e^{i\lambda am} \\ &= \widetilde{f|_{D(a)}}(\lambda a) + \widetilde{f|_{D(a)^c}}(\lambda a).\end{aligned}$$

On the other hand, from (7) we see that

$$\begin{aligned}\widetilde{Rf}(H(a,\cdot))(\lambda) &= \sum_{k\in\mathbf{Z}} \big(f(m_0(a,k))e^{i\lambda k} + g(a,k)e^{i\lambda k}\big) \\ &= \widetilde{f|_{D(a)}}(\lambda a) + \widetilde{g}(a,\cdot)(\lambda).\end{aligned}$$

Hence we can obtain that

$$\widetilde{g}(a,\cdot)(\lambda) = \widetilde{f|_{D(a)^c}}(\lambda a). \tag{9}$$

We now consider a characterization of the image of the discrete Radon transforms $Rf(H(a,k))$ of $f \in \mathcal{S}(\mathbf{Z}^2)$.

PROPOSITION 3.1. *Let $F(a,k)$ and $f(m)$ be functions on $\mathbf{G}$ and $\mathbf{Z}^2$ respectively. We suppose that*

$$F(a,k) = f(m_0) + g(a,k)\,,$$

where $m_0 = m(a,k) \in D(a)$ and $g(a,k)$ is a function on $\mathbf{G}$, which satisfies that for each $N = 0, 1, 2, \dots,$

$$g(a,k) \leq \frac{c}{\|a\|(1+\|m_0\|)^N}\,,$$

where c is independent of a and k. Then

$$\lim_{\|a\|\to\infty} F(a, am) = f(m) \tag{10}$$

for all $m \in \mathbf{Z}^2$. Especially, the above decomposition of F is unique. Furthermore, F satisfies that for each $N = 0, 1, 2, \dots,$

$$|F(a,k)| \leq c\Big(\frac{1+\|a\|^2}{1+k^2}\Big)^N, \tag{11}$$

where c is independent of a and k, if and only if $f \in \mathcal{S}(\mathbf{Z}^2)$.

PROOF. We fix $m \in \mathbf{Z}^2$ and consider $a \in \mathcal{P}$ such that $\|a\| > 2\|m\|$[‡)]. Then it easily follows that

$$m_0(a, am) = m$$

and thus,

$$|F(a, am) - f(m)| \leq \frac{c}{\|a\|(1+\|m\|)^N}\,, \tag{12}$$

where c is independent of a and m, Therefore,

$$\lim_{\|a\|\to\infty} F(a, am) = f(m)\,.$$

We suppose that $F(a,k)$ satisfies (11). Without loss of generality, we may suppose that, for $m = (m_1, m_2)$, $m_2 \neq 0$ and $|m_2| \geq |m_1|$. Let $a_j = (1, j) \in \mathcal{P}$ and $\|a_j\| > 2\|m\|$. Then for each $N = 0, 1, 2, \dots,$

$$\begin{aligned}(1+\|m\|^2)^N |F(a_j, a_j m)| &\leq c(1+\|m\|^2)^N \Big(\frac{1+\|a_j\|^2}{1+(a_j m)^2}\Big)^N \\ &\leq c(1+m_1^2+m_2^2)^N \Big(\frac{1+1+j^2}{1+(m_1+jm_2)^2}\Big)^N\end{aligned}$$

‡) See REMARK 3.2.

$$\leq c(1+m_2^2)^N\Big(\frac{1+j^2}{1+(j-1)^2m_2^2}\Big)^N$$
$$\leq c\,,$$

where c is independent of a_j and m. Hence, by multiplying $(1+\|m\|^2)^N$ to the both sides of (12) replaced a and N by a_j and $2N$ respectively, and then, by letting j go to ∞, it follows that

$$\sup_{m\in\mathbf{Z}^2}(1+\|m\|^2)^N|f(m)|<\infty\,.$$

Conversely, we suppose that $f\in\mathcal{S}(\mathbf{Z}^n)$. Since $m_0=m_0(a,k)$ is in $H(a,k)$ and thus, lies on the line m_0+ta_0, $t\in\mathbf{R}$, and $a\perp a_0$. Hence

$$\|m_0\|\geq\frac{|m_0a|}{\|a\|}=\frac{|k|}{\|a\|}$$

and thus,

$$\frac{1}{1+\|m_0\|^2}\leq\frac{1+\|a\|^2}{1+k^2}\,.$$

Therefore, the desired result follows from the decomposition of F. □

REMARK 3.2. As pointed in §2, when $n>2$, the inversion formula (10) in Proposition 3.1 is not true. The one replaced a by a_j holds.

PROPOSITION 3.3. *Let $F(a,k)=f(m_0)+g(a,k)$, where $m_0=m(a,k)\in D(a)$. We suppose that f belongs to $\mathcal{S}(\mathbf{Z}^2)$ and $\widetilde{g}(a,\cdot)(\lambda)=\widetilde{f|_{D(a)^c}}(\lambda a)$. Then for each $N=0,1,2,\dots,$*

$$|g(a,k)|\leq\frac{c}{\|a\|(1+\|m_0\|)^N}\,,$$

where c is independent of a and k.

PROOF. We note that

$$\begin{aligned}g(a,k)&=\int_{\mathbf{T}}\widetilde{f|_{D(a)^c}}(\lambda a)e^{-i\lambda k}d\lambda\\&=\sum_{m\in D(a)^c,ma=k}f(m)\\&=\sum_{m\neq m_0(a,k)\in H(a,k)}f(m)\,.\end{aligned}$$

As in the calculation that yields (8), g satisfies the desired estimate. □

We now define $\mathcal{S}_*(\mathbf{G})$ as follows.

DEFINITION 3.4. Let $\mathcal{S}_*(\mathbf{G})$ be the space of all $F(a,k)$ on $\mathbf{G}$ being of the form

$$F(a,k) = f(m_0) + g(a,k)\,, \tag{13}$$

where $f \in \mathcal{S}(\mathbf{Z}^2)$, $m_0 = m_0(a,k) \in D(a)$ and g satisfies that for $\lambda \in \mathbf{T}$,

$$\widetilde{g}(a,\cdot)(\lambda) = \widetilde{f|_{D(a)^c}}(\lambda a)\,. \tag{14}$$

According to Propositions 3.1 and 3.3, if $F \in \mathcal{S}_*(\mathbf{G})$, then the decomposition $F(a,k) = f(m_0) + g(a,k)$ is unique and the following properties hold: For each $N = 0, 1, 2, \ldots$,

$$|g(a,k)| \le \frac{c}{\|a\|(1+\|m_0\|)^N}\,,$$
$$|F(a,k)| \le c\Big(\frac{1+\|a\|^2}{1+k^2}\Big)^N,$$
$$\lim_{\|a\|\to\infty} F(a,am) = f(m)^{\ddagger},$$

where c is independent of a and k. We define for all $N = 0, 1, 2, \ldots$,

$$q_N(F) = \sup_{a\in\mathcal{P},k\in\mathbf{Z}} \Big(\frac{1+\|a\|^2}{1+k^2}\Big)^{-N} |F(a,k)|\,.$$

Then $\{q_N \mid N = 0, 1, 2, \ldots\}$ is a family of semi-norms of $\mathcal{S}_*(\mathbf{G})$.

Our main theorem is the following.

THEOREM 3.5. *R is a bijective continuous map from $S(\mathbf{Z}^2)$ to $S_*(\mathbf{G})$.*

PROOF. From (6), (7), (8), (9), and Propositions 3.1, it follows that R is an injective continuous map from $S(\mathbf{Z}^n)$ to $S_*(\mathbf{G})$. We shall prove that R is surjective. Let F be in $S_*(\mathbf{G})$ and $F = f + g$ denote the decomposition (13) of F in Definition 3.4. Since $f \in S(\mathbf{Z}^n)$, $Rf \in S_*(\mathbf{G})$ and thus, $H = F - Rf$ belongs to $S_*(\mathbf{G})$. By noting (7), the unique decomposition (13) of H is of the form $H = 0 + g'$. Hence $\widetilde{g'}(a,\cdot) = 0$ for all $a \in \mathcal{P}$ by (14). Then $g' = 0$ and thus, $F = Rg$. □

REMARK 3.6. The relation (14) in Definition 3.4 is used to prove that, if $f = 0$, then $F(a,\cdot) = 0$ for all $a \in \mathcal{P}$. Since $Rf(H(a,k))$ satisfies (3), we may replace the relation by the following condition: Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $\{v_k \mid k \in \mathbf{Z}\}$ a complete orthonormal system of $\mathcal{H}$. Then F and f satisfy

$$\sum_{k\in\mathbf{Z}} F(a,k)v_k = \sum_{m\in\mathbf{Z}^n} f(m)v_{am}$$

for all $a \in \mathcal{P}$. Actually, if $f = 0$, then $F(a,\cdot) = 0$ for all $a \in \mathcal{P}$.

‡See REMARK 3.2.

4. Inversion formula

In the following, let $n = 2$. In addition to (5) we shall obtain several methods of recovering f from Rf. For $a, b \in \mathbf{N}$, let $[a, b]$ denote the sets of integers p such that $a \le p \le b$. For a set $S \subset \mathbf{Z}^2$, let χ_S denote the characteristic function of S and $|S|$ the cardinality of S.

4.1. Mean inversion formula. Let Q be a direction function on $\mathbf{Z}^2$ which depends only on directions:

$$Q(0) = 0, \quad Q(x) = Q(a) = Q(-a) \text{ for } x = la, \tag{15}$$

where $a \in \mathcal{P}$ and $l \ne 0 \in \mathbf{Z}$. Suppose that $\|Q\|_\infty < \infty$ and $\sum_{a\in\mathcal{P}} Q(a) < \infty$. Then for $f \in l^1(\mathbf{Z}^2)$, since $Q(0) = 0$, it follows that

$$\begin{aligned} Q * f(m) &= \sum_{m'\in\mathbf{Z}^2} Q(m - m')f(m') \\ &= \frac{1}{2}\sum_{a\in\mathcal{P}}\sum_{l\ne 0\in\mathbf{Z}} Q(a)f(m + la) \\ &= \frac{1}{2}\sum_{a\in\mathcal{P}} Q(a_0)Rf(H(a, am)) - \frac{1}{2}f(m)\sum_{a\in\mathcal{P}} Q(a)\,, \end{aligned}$$

where $a \perp a_0$ and $a_0 \in \mathcal{P}$. Hence we can obtain the following.

THEOREM 4.1. *Let Q be a direction function on $\mathbf{Z}^2$ and suppose that $\|Q\|_\infty < \infty$ and $\sum_{a\in\mathcal{P}} Q(a) \ne 0 < \infty$. Then for $f \in l^1(\mathbf{Z}^2)$,*

$$f(m) = \frac{1}{\sum_{a\in\mathcal{P}} Q(a)}\Big(\sum_{a\in\mathcal{P}} Q(a_0)Rf(H(a, am)) - 2Q * f(m)\Big)\,. \tag{16}$$

COROLLARY 4.2. *Let $\{Q_i\}$, $i \in \mathbf{N}$, be a sequence of direction functions on $\mathbf{Z}^2$ satisfying*

(a) $\|Q_i\|_\infty < C$ *for all* i,

(b) $\sum_{a\in\mathcal{P}} Q_i(a) \to \infty$ *if* $i \to \infty$.

Then for $f \in l^1(\mathbf{Z}^2)$,

$$f(m) = \lim_{i\to\infty} \frac{1}{\sum_{a\in\mathcal{P}} Q_i(a)} \sum_{a\in\mathcal{P}} Q_i(a_0)Rf(H(a, am))\,.$$

For example, if we take $Q_i(x) = \chi_{B(i)}(a)$ for $x = la$, then we see that

$$f(m) = \lim_{i\to\infty} \frac{1}{|B(i)\cap\mathcal{P}|} \sum_{a\in B(i)\cap\mathcal{P}} Rf(H(a, am))\,.$$

PROOF. Since $\|Q_i * f\|_\infty \le \|Q_i\|_\infty \|f\|_1 \le C\|f\|_1$, $\big(\sum_{a\in\mathcal{P}} Q_i(a)\big)^{-1} Q_i * f \to 0$ if $i \to \infty$ by (b). Hence the desired formula follows from (16). □

COROLLARY 4.3. *Let $\{Q_i\}$, $i \in \mathbf{N}$, be a sequence of direction functions on $\mathbf{Z}^2$. Furthermore, we suppose that*

(a) $\|Q_i\|_\infty < C$ *for all* i,

(b) $\operatorname{supp} Q_i \subset B(r_i)^c$ *where* $r_i \to \infty$ *if* $i \to \infty$,

(c) $\lim_{i\to\infty} \sum_{a\in\mathcal{P}} Q_i(a) > 0$.

Then for $f \in l^1(\mathbf{Z}^2)$,

$$f(m) = \lim_{i\to\infty} \frac{1}{\sum_{a\in\mathcal{P}} Q_i(a)} \sum_{a\in\mathcal{P}} Q_i(a_0) Rf(H(a, am)) .$$

For example, if we take a finite subset S_i in $B(r_i)^c$ and let $Q_i(x) = \chi_{S_i}(a)$ for $x = la$, then

$$f(m) = \lim_{i\to\infty} \frac{1}{|S_i|} \sum_{a\in S_i} Rf(H(a, am)) .$$

PROOF. Since $Q_i(0) = 0$, $|Q_i * f(m)| \le \|Q_i\|_\infty \sum_{m'\in B(r_i)^c} |f(m+m')| \le C \sum_{m'\in B(r_i)^c} |f(m+m')|$. Hence, $|Q_i * f(m)| \to 0$ if $i \to \infty$, because $f \in l^1(\mathbf{Z}^2)$ and (b). Therefore, the desired formula follows from (16) and (c). □

REMARK 4.4. These corollaries are generalizations of the formulas obtained in [2], Theorem 2.1 (a) and (b). In Corollary 4.3, if we take $S_i = \{a_i\}$, $a_i \in \mathcal{P}$, then we can deduce (5).

4.2. Discrete Fourier inversion formula. We introduce an inversion formula using the discrete Fourier transform.

Step1. We first suppose that $\operatorname{supp} f \subset [-N, N]^2$. Since $|am| \le (|a_1| + |a_2|)N = |a|N$, where $|a| = |a_1| + |a_2|$, for $a = (a_1, a_2) \in \mathcal{P}$ and $m \in [-N, N]^2$, the support of $Rf(H(a, k))$ with respect to k is in $[-|a|N, |a|N]$. We recall the discrete Fourier transform on $[-N, N]^2$ and its inversion formula: For $t = (t_1, t_2)$, $0 \le t_1, t_2 \le 2N$, the discrete Fourier transform $F(t)$ of $f(n)$ is given by

$$F(t) = \frac{1}{(2N+1)^2} \sum_{n_1,n_2=0}^{2N} f((n_1, n_2) - (N, N)) e^{-i\frac{2(n_1t_1+n_2t_2)\pi}{2N+1}}$$

and for $n = (n_1, n_2)$, $-N \le n_1, n_2 \le N$, $f(m)$ is recovered as

$$\begin{aligned} f(n) &= \sum_{t_1,t_2=0}^{2N} F(t) e^{i\frac{2((n_1+N)t_1+(n_2+N)t_2)\pi}{2N+1}} \\ &= \sum_{t\in[0,2N]^2} F(t)(-1)^{|t|} e^{i\frac{(2nt-|t|)\pi}{2N+1}} . \end{aligned} \tag{17}$$

Step2. We apply the slice formula (3). For each $a = (a_1, a_2) \in \mathcal{P}$ with $a_1, a_2 \ge 0$, and $l \in \mathbf{Z}$,

$$\begin{aligned} &\sum_{k=-|a|N}^{|a|N} Rf(H(a,k)) e^{-i\frac{2(k+|a|N)l\pi}{2N+1}} \\ &= \sum_{m\in\mathbf{Z}^2} f(m) e^{-i\frac{2(am+|a|N)l\pi}{2N+1}} \\ &= \sum_{m\in\mathbf{Z}^2} f((m_1,m_2)-(N,N)) e^{-i\frac{2l(m_1a_1+m_2a_2)\pi}{2N+1}} \\ &=(2N+1)^2 F(la) . \end{aligned} \tag{18}$$

Step3. We combine (17) and (18). Let $\mathcal{P}(N) = \mathcal{P} \cap [0, 2N]^2$ and $\mathbf{Z}_+$ the set of the positive integers. We denote $t = (t_1, t_2) \ne (0, 0)$, $0 \le t_1, t_2 \le 2N$, as

$$t = (t_1, t_2) = l_t a_t ,$$

where $a_t \in \mathcal{P}(N)$ and $l_t \in \mathbf{Z}_+$. When $t = (0, 0)$, we let $l_t = 0$ and a_t is arbitrary. Then, replacing $F(t) = F(l_t a_t)$ in (17) with (18), we see that

$$\begin{aligned} f(n) =& \frac{1}{(2N+1)^2} \sum_{t\in[0,2N]^2} \Big(\sum_{k=-|a_t|N}^{|a_t|N} Rf(H(a_t,k)) e^{-i\frac{2(k+|a_t|N)l_t\pi}{2N+1}} \Big) \\ &\times (-1)^{|t|} e^{i\frac{(2nt-|t|)\pi}{2N+1}} \\ =& \frac{1}{(2N+1)^2} \sum_{t\in[0,2N]^2} \Big(\sum_{k=-|a_t|N}^{|a_t|N} Rf(H(a_t,k)) e^{-i\frac{2(k-na_t)l_t\pi}{2N+1}} \Big) . \end{aligned}$$

For $a \in \mathcal{P}(N)$ let $L(a, N) = \max\{l \in \mathbf{N} \mid la \in [0, 2N]^2\}$. We recall that, when $t = (0, 0)$, $l_t = 0$, and that $\sum_{k\in\mathbf{Z}} Rf(H(a_t, k)) = \sum_{m\in\mathbf{Z}^2} f(m)$. Hence, we can rewrite the previous equation as

$$f(n) = \frac{1}{(2N+1)^2} \Big(\sum_{a\in\mathcal{P}(N)} \Big(\sum_{k=-|a|N}^{|a|N} Rf(H(a,k)) \sum_{l=1}^{L(a,N)} e^{-i\frac{2(k-na)l\pi}{2N+1}} \Big) \Big) .$$

Step4. Let f be an arbitrary function in $l^1(\mathbf{Z}^2)$. Then it is easy to see that

$$\frac{1}{(2N+1)^2}\Big|\sum_{a\in\mathcal{P}(N)}\Big(\sum_{k=-|a|N}^{|a|N} R(f-f\chi_N)(H(a,k))\sum_{l=1}^{L(a,N)} e^{-i\frac{2(k-na)l\pi}{2N+1}}\Big)\Big|$$

$$\leq\frac{1}{(2N+1)^2}\sum_{a\in\mathcal{P}(N)} L(a,N)\sum_{k=-|a|N}^{|a|N}|R(f-f\chi_N)(H(a,k))|$$

$$\leq\frac{N^2}{(2N+1)^2}\sum_{\|m\|>N}|f(m)|\,.$$

Since the last term goes to 0 if $N\to\infty$, we can obtain the following.

THEOREM 4.5. *For each $N\in\mathbf{N}$, let $\mathcal{P}(N)=\mathcal{P}\cap[0,2N]^2$ and for each $a\in\mathcal{P}(N)$, let $L(a,N)=\max\{l\in\mathbf{N}\mid la\in[0,2N]^2\}$. Then for $f\in l^1(\mathbf{Z}^2)$,*

$$f(n)=\lim_{N\to\infty}\frac{1}{(2N+1)^2}\sum_{a\in\mathcal{P}(N)}\Big(\sum_{k=-|a|N}^{|a|N} Rf(H(a,k))\sum_{l=1}^{L(a,N)} e^{-i\frac{2(k-na)l\pi}{2N+1}}\Big).$$

4.3. Algorithmic inversion formula. We introduce a method to recover $f(0,0)$ from $Rf(H(a,k))$ by an algorithmic process. We first note that, if f is supported on $[-N,N]^2$and $\|a\|>N$, then

$$Rf(H(a,0))=f(0,0)\,,$$

because $[-N,N]^2\cap H(a,0)=\{(0,0)\}$ (see Fig. 1).

In the following, we shall consider an algorithm by which $f(0,0)$ is recovered from $Rf(H(a,k))$ with $\|a\|\leq N$.

Step1. For each $j=0,1,2,\dots$, we first define a set $V(N,j)$ of points in $[-N,N]^2$ and a set $E(N,j)$ of hyperplanes on $\mathbf{Z}^2$ inductively. The case of $N=3$ is referred to Example

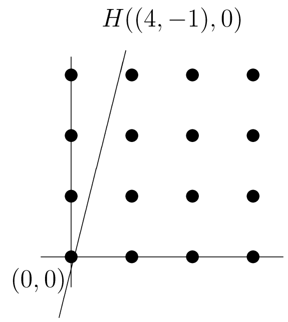

FIGURE 1. $N=3$ and $a=(4,-1)$

4.6. Let

$$E(N,0) = \emptyset,\ \ V(N,0) = [-N,N]^2.$$

Let $E(N,1)$ be the set of hyperplanes $H(a,ap)$ through $p \in V(N,0)$ such that

$$(i)\quad H(a,ap) \cap V(N,0) = \{p\}$$

$$(ii)\ \ \|a\| \text{ is minimum among } H(a,ap) \text{ satisfying } (i)$$

and $V(N,1) = V(N,0) - \{p \mid H(a,ap) \in E(N,1)\}$. Furthermore, inductively, we define $E(N, j+1)$ as the set of hyperplanes $H(a,ap)$ through $p \in V(N,j)$ such that

$$(i)\quad H(a,ap) \cap V(N,j) = \{p\}$$

$$(ii)\ \ \|a\| \text{ is minimum among } H(a,ap) \text{ satisfying } (i)$$

and $V(N, j+1) = V(N,j) - \{p \mid H(a,ap) \in E(N, j+1)\}$.

We note that there exists j_N for which $V(N, j_N - 1)$ is not contained in $[-(N-1), N-1]^2$, but

$$V(N, j_N) \subset [-(N-1), N-1]^2.$$

EXAMPLE 4.6. Let $N = 3$. In the following figures we denote the area only in the first quadrant.

$$V(3,j)_+ = V(3,j) \cap [0,3]^2,$$

$$E(3,j)_+ = \{H(a,ap) \in E(3,j) \mid p \in [0,3]^2\}.$$

In the first line of Fig. 2, we let $a = (1,1)$ and $p = (3,3)$. Then $H(a,6) \cap V(3,0) = \{p\}$ and $\|a\| = \sqrt{2}$. Hence it follows that

$$E(3,1)_+ = \{H((1,1),6)\}$$

$$V(3,1)_+ = [0,3]^2 - \{(3,3)\}.$$

In the second line in Fig. 2, we let $a = (1,2), (2,1)$ and $p = (2,4), (4,2)$ respectively. Then $H(a,10) \cap V(3,1) = \{p\}$ and $\|a\| = \sqrt{5}$. Hence it follows that

$$E(3,2)_+ = \{H((1,2),10), H((2,1),10)\}$$

$$V(3,2)_+ = V(3,1)_+ - \{(2,4),(4,2)\}.$$

Finally, we see that $V(3,4)_+ \subset [0,2]^2$ and thus, $j_3 = 4$.

Step2. We define an operator Shave_N for a function f on $[-N,N]^2$. We note that, if f is supported in $V(N,j)$, then $Rf(H(a,ap)) = f(p)$ for each $H(a,ap) \in E(N,j)$. We define a set of functions $\{f_j\}$ inductively as

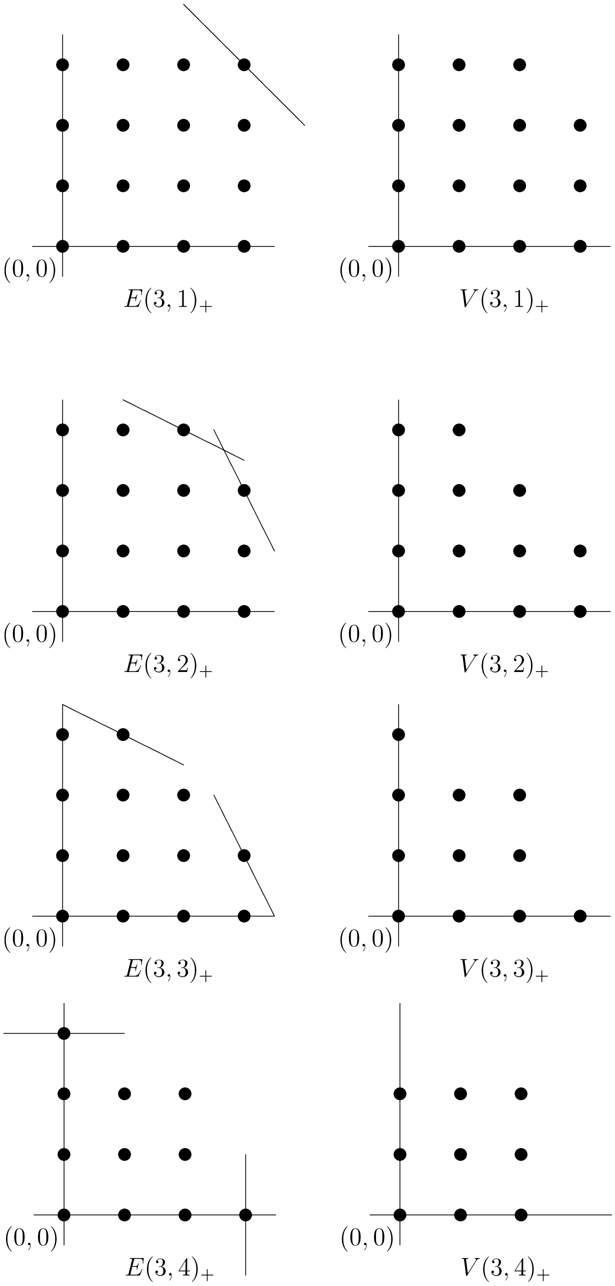

FIGURE 2.

$$f_1 = f - \sum_{H(a,ap)\in E(N,1)} Rf(H(a,ap))\delta_p\,,$$
$$f_{j+1} = f_j - \sum_{H(a,ap)\in E(N,j+1)} Rf_j(H(a,ap))\delta_p$$

and put

$$\mathrm{Shave}_N(f) = f_{j_N}\,.$$

Clearly, $\mathrm{Shave}_N(f)$ is supported on $[-(N-1), N-1]^2$.

Step3. We replace N by $N-1$ and apply the previous arguments to $\mathrm{Shave}_N(f)$, which is supported on $[-(N-1), N-1]^2$. Furthermore, we repeat the process successively. Then we can easily deduce that

$$\begin{aligned} f(0,0)\delta_{(0,0)} &= \mathrm{Shave}_1 \circ \mathrm{Shave}_2 \circ \cdots \circ \mathrm{Shave}_{N-1} \circ \mathrm{Shave}_N(f) \\ &= f - \sum_{\substack{p\in[-N,N]^2,\\ p\neq(0,0)}} d_N(Rf, p)\delta_p\,, \end{aligned} \tag{19}$$

where $d_N(Rf, p)$ is a linear combination of $Rf(H)$ with

$$H \in \bigcup_{n=1}^{N} \bigcup_{j=1}^{j_n} E(n, j)\,.$$

Therefore, for $p \neq (0,0) \in [-N, N]^2$, it follows that

$$f(p) = d_N(Rf, p)\,.$$

For $p = (0,0)$, we take the discrete Radon transform over $H((0,1),0)$. Since $H((0,1),0) = \{(q,0) \mid q \in \mathbf{Z}\}$, it follows that

$$f(0,0) = Rf(H(0,1),0) - \sum_{q=-N,\ q\neq 0}^{N} d_N(Rf, (q,0))\,. \tag{20}$$

REMARK 4.7. We suppose that all a of $H(a, ap)$ in $\bigcup_{1\le j\le j_N-1} E(N, j)$ are of the forms $(1, \pm l)$ or $(\pm l, 1)$, $l \geq 1$, and l_0 their maximum. Then we can deduce that $1 + 2 + \cdots + l_0 + l_0 \geq N$ (see Fig. 3 for the case of $N = 9$). Therefore, since $E(N, j_N) = \{H((1,0), \pm N), H((0,1), \pm N)\}$, the maximum $\|a\|$ of $H(a, ap)$ in $\bigcup_{1\le j\le j_N} E(N, j)$ is $O(\sqrt{N})$. Hence, in the formulas (19) and (20) we use $Rf(H(a,k))$ with $\|a\| = O(\sqrt{N})$.

Step4. Let f be an arbitrary function in $\mathcal{S}(\mathbf{Z}^2)$. We extend the definition of the operator Shave_N to f, that is, we apply (19) to f. We note that, in the process to define Shave_N, each hyperplane $H(a, ap)$ is used to vanish the value $f(p)$ at p when f is supported in $[-N, N]^2$.

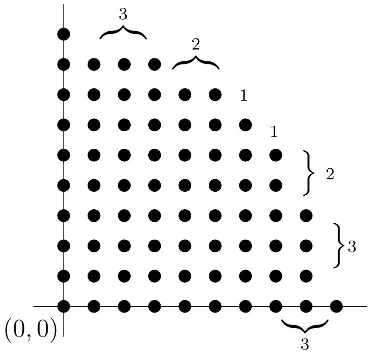

FIGURE 3. $V(9, 16)_+$, $j_9 = 17$ and $l_0 = 3$

Therefore, the total number of discrete hyperplanes which appear to define Shave_N is at most $O(N^2)$ and thus, the total number of hyperplanes appeared in (19) is $O(N^3)$. Since $f - f\chi_N$ is supported in $B(N)^c$ and is rapidly decreasing, it is easy to see that for each $q \in \mathbf{N}$, $|f(m)| \le C_q(1 + \|m\|^2)^{-q}$ and thus, $\|R(f - f\chi_N)\|_\infty \le C_q \sum_{m \in B(N)^c}(1 + \|m\|^2)^{-q} \le cN^{-2(q-1)}$. Hence, if $q > 2$, then

$$\|\mathrm{Shave}_1 \circ \cdots \circ \mathrm{Shave}_N(f - f\chi_N)\|_\infty \le C_q N^3 N^{-2(q-1)},$$

and this goes to 0 if $N \to \infty$. Hence, it follows that

$$\begin{aligned} f(0,0)\delta_{(0,0)} &= (f\chi_N)(0,0)\delta_{(0,0)} \\ &= \mathrm{Shave}_1 \circ \mathrm{Shave}_2 \circ \cdots \circ \mathrm{Shave}_{N-1} \circ \mathrm{Shave}_N(f\chi_N) \\ &= \lim_{N\to\infty} \mathrm{Shave}_1 \circ \mathrm{Shave}_2 \circ \cdots \circ \mathrm{Shave}_{N-1} \circ \mathrm{Shave}_N(f)\,. \end{aligned}$$

Therefore, we can obtain the following.

THEOREM 4.8. *Let notations be as above. For $f \in \mathcal{S}(\mathbf{Z}^2)$,*

$$f(p) = \lim_{N\to\infty} d_N(R(f), p), \quad p \ne (0,0),$$

$$f(0,0) = Rf(H(0,1), 0) - \lim_{N\to\infty} \sum_{q=-N,\ q\ne 0}^{N} d_N(R(f), (q,0))\,.$$

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