

Minding's Theorem for Low Degrees of Differentiability

Josef F. DORFMEISTER and Ivan STERLING

Technische Universität München and St Mary's College of Maryland

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Abstract. We prove Minding's Theorem for C^2 -immersions with constant negative Gaussian curvature. As a Corollary we also prove Minding's Theorem for C^{1M} -immersions in the sense of [1].

1. Statement of Main Theorem

Let Ω denote a simply connected open set in \mathbf{R}^2 . We say that a function $f : \Omega \rightarrow \mathbf{R}^\ell$ is of class $C^k(\Omega)$ (written $f \in C^k(\Omega)$) if all its derivatives up to order k are continuous. We define $k = \infty$ ($k = \omega$) if the function is infinitely differentiable (resp. analytic). A coordinate chart on Ω is a pair $(\beta_{(t^1, t^2)}, U)$ where U is open in Ω and $\beta_{(t^1, t^2)} : U \rightarrow V \subset \mathbf{R}_{(t^1, t^2)}^2$ (always assumed to be at least $C^1(U)$ with non-vanishing Jacobian). A metric (always assumed to be positive definite) $g = \sum g_{ij} dt^i dt^j$ is said to be of class $C^k(V)$ in the coordinate chart $(\beta_{(t^1, t^2)}, U)$ if the coefficients $g_{ij} \in C^k(V)$. A coordinate chart $(\beta_{(x, y)}, U)$ is called isothermal with respect to a metric $g = g_{11} dx^2 + g_{12} dx dy + g_{21} dy dx + g_{22} dy^2$ if $g_{11} = g_{22}$ and $g_{12} (= g_{21}) = 0$. If $(\beta_{(x, y)}, U)$ is isothermal then $h^2 := g_{11} = g_{22}$ is called the conformal factor of $g = h^2(dx^2 + dy^2)$ with respect to $(\beta_{(x, y)}, U)$.

A $C^n(\Omega)$, $n \geq 1$, map $f : \Omega \rightarrow \mathbf{R}^\ell$ is called an immersion (always assumed to be regular) if $\text{rank}(df) \equiv 2$. The $C^{n-1}(\Omega)$ -metric induced on Ω by f is given by $f^* ds_{\mathbf{R}^\ell}^2$, where $ds_{\mathbf{R}^\ell}^2$ denotes the standard metric on \mathbf{R}^ℓ . If Ω_1 has a metric g_1 and Ω_2 has a metric g_2 then a $C^1(\Omega_1)$ -immersion $\phi : (\Omega_1, g_1) \rightarrow (\Omega_2, g_2)$ is called an isometry if $\phi^*(g_2) = g_1$. We will prove the following case of Minding's Theorem, which was previously known only for $n \geq 3$.

THEOREM 1.1. *Let $f : \Omega \xrightarrow{C^n} \mathbf{R}^3$ ($n \geq 2$) be a $C^n(\Omega)$ -immersion with the induced $C^{n-1}(\Omega)$ -metric $f^* ds_{\mathbf{R}^3}^2$ and Gaussian curvature $K \equiv -1$. Then there exists a $C^2(\Omega)$ isometry $\phi : (\Omega, f^* ds_{\mathbf{R}^3}^2) \rightarrow (\mathbf{H}^2, ds_{\mathbf{H}^2}^2)$, where $ds_{\mathbf{H}^2}^2$ denotes the standard metric on \mathbf{H}^2 .*

From Theorem 4.1 in Section 4, proved in [1], we immediately have the following Corollary 1.2. The C^{1M} surfaces (see definition in Section 4) which appear in Corollary 1.2 arise naturally in the loop group classification of $K \equiv -1$ surfaces [1]. For a further explanation of the terms used in Corollary 1.2 see Section 4.

COROLLARY 1.2. *Let $f = f_{asyche} : \Omega \xrightarrow{C^{1M}} \mathbf{R}^3$ be a C^{1M} -immersion in asymptotic Chebyshev coordinates. Assume $N = N_{asyche} := \frac{f_x \times f_y}{\sin \theta}$ is C^{1M} , $N_{xy} = N_{yx} = \cos \theta N$, $f_x = N \times N_x$, and $f_y = -N \times N_y$. (Note this implies $K \equiv -1$.) Then there exists a $C^2(\Omega)$ isometry $\phi : (\Omega, f^*ds_{\mathbf{R}^3}^2) \longrightarrow (\mathbf{H}^2, ds_{\mathbf{H}^2}^2)$.*

2. Preliminary Results

The proof of Theorem 1.1 will be as follows. By Theorem 2.1, it suffices to prove the theorem assuming isothermal coordinates, with a metric $h^2(dx^2 + dy^2)$ and $h \in C^1$. Using distributional derivatives we will prove that $u = \ln h$ is a weak solution to the equation $\Delta u = e^{2u}$. Using bootstrap results from elliptic PDE theory we will first show that $u \in C^2(\Omega)$ and then that $u \in C^\infty(\Omega)$. Theorem 1.1 will then follow by Liouville’s Theorem, Theorem 2.6, which will also imply that $u \in C^\omega(\Omega)$. Thus, u is analytic if we use isothermal coordinates.

2.1. Isothermal Coordinates. The case $n = 2$ of the following version of Theorem (*) p.301 [2] is what we’ll need for the first part of the proof.

THEOREM 2.1 (Chern-Hartman-Wintner’s Existence of Isothermal Coordinates Theorem). *Let $f : \Omega \xrightarrow{C^n} \mathbf{R}^3$ ($n \geq 2$) be a C^n -immersion with the induced C^{n-1} -metric $f^*ds_{\mathbf{R}^3}^2$ and C^{n-2} Gaussian curvature K . Then there exist a $C^n(\Omega)$ coordinate chart (β, Ω) isothermal with respect to $f^*ds_{\mathbf{R}^3}^2$ with $C^{n-1}(\beta(\Omega))$ conformal factor h^2 .*

It is pointed out in [2] that for any two charts $\beta_{(x,y)}, \beta'_{(x',y')}$, isothermal with respect to any metric g , the transition function $\tau = \beta' \circ \beta^{-1}$ is analytic. In other words such mappings τ are of the form $x' + iy' = \tau(z)$ where τ is an analytic function of $z = x + iy$. The set of such charts therefore form a C^ω -atlas, which we denote by \mathcal{B} , and hence define a conformal structure making Ω a simply connected Riemann surface. Let \mathcal{A} denote the standard conformal structure on open subsets of \mathbf{C} . By the uniformization theorem (Ω, \mathcal{B}) is bi-holomorphic to (D, \mathcal{A}) , for some simply connected open subset $D \subset \mathbf{C}$ by some map $\pi : D \longrightarrow \Omega$. In our case, D could be chosen to be the unit disk. Then, by the Riemann mapping theorem, (D, \mathcal{A}) is bi-holomorphic to (Ω, \mathcal{A}) , by some $\rho : \Omega \longrightarrow D$. Finally by definition, $\beta := \rho^{-1} \circ \pi^{-1}$ is a global chart isothermal with respect to $f^*ds_{\mathbf{R}^3}^2$. We will choose such a global chart in the discussion below.

2.2. Distributions and Their Derivatives. In this subsection we review distributions and their derivatives. These will be used in the next subsection to show that our $u = \ln h$ is

a weak solution to $\Delta u = e^{2u}$. To be precise we use subscripts x, y for partial differentiation and subscripts \mathcal{X}, \mathcal{Y} for partial distributional differentiation.

Following [3] the set of test functions for our distributions will be $C_0^1(\Omega)$, the set of compactly supported continuously differentiable functions $v : \Omega \rightarrow \mathbf{R}$ or $v : \Omega \rightarrow \text{Mat}(2, 2, \mathbf{R})$. On $\text{Mat}(2, 2, \mathbf{R})$ we use the inner product $\langle A, B \rangle = \text{trace} A^t B$ and on \mathbf{R} the inner product $\langle a, b \rangle = ab$. A distribution is defined to be a linear map $T : C_0^1(\Omega) \rightarrow \mathbf{R}$. Every locally integrable function $f \in L_{loc}^1(\Omega)$ defines a distribution by $T_f(v) = \int_{\Omega} f v$. Such distributions are called regular and by an abuse of notation we use f to denote T_f . If T is a distribution and $h \in C^1(\Omega)$, then hT is the distribution

$$(hT)(v) = T(h^t v).$$

We define partial differentiation only for regular distributions. If $f = T_f$ is a regular distribution, then the distributional derivatives $\partial_{\mathcal{X}}$ and $\partial_{\mathcal{Y}}$ are defined by

$$\begin{aligned} (\partial_{\mathcal{X}} f)(v) &= - \int_{\Omega} \langle f, \partial_x v \rangle, \\ (\partial_{\mathcal{Y}} f)(v) &= - \int_{\Omega} \langle f, \partial_y v \rangle. \end{aligned}$$

We will need the following standard results which are proven by straightforward calculations.

LEMMA 2.2. *If $W \in C^1(\Omega)$, then W_x and W_y are distributions and we have*

$$W_{x\mathcal{Y}} = W_{y\mathcal{X}}.$$

LEMMA 2.3. *If $P \in C^1(\Omega)$ and $L \in L_{loc}^1(\Omega)$, then PL is a regular distribution and*

$$\begin{aligned} \partial_{\mathcal{X}}(PL) &= (\partial_x P)L + P\partial_{\mathcal{X}}L, \\ \partial_{\mathcal{Y}}(PL) &= (\partial_y P)L + P\partial_{\mathcal{Y}}L. \end{aligned}$$

DEFINITION 2.4. A differentiable function u is called a weak solution to $\Delta u = g$ in Ω if

$$\int_{\Omega} u_x v_x + u_y v_y = - \int_{\Omega} g v \text{ for all } v \in C_0^1(\Omega).$$

2.3. Generalized Liouville Equation. Assume we have chosen coordinates isothermal with respect to $f^* ds_{\mathbf{R}^3}^2$ so our metric is of the form $h^2(dx^2 + dy^2)$. We now show that $u = \ln h$ is a weak solution to $\Delta u = e^{2u}$. Let the $C^1(\Omega)$ matrix function W be defined by

$$W = (f_x, f_y, N), \quad \text{where } N = \frac{f_x}{|f_x|} \times \frac{f_y}{|f_y|},$$

and define the $C^0(\Omega)$ matrices A and B by

$$A = W^{-1}W_x \quad \text{and} \quad B = W^{-1}W_y,$$

or

$$W_x = WA \quad \text{and} \quad W_y = WB.$$

If $\ell := -\langle N_x, f_x \rangle$, $m = -\langle N_x, f_y \rangle = -\langle N_y, f_x \rangle$, and $n = -\langle N_y, f_y \rangle$. Then

$$A = \begin{pmatrix} \frac{h_x}{h} & \frac{h_y}{h} & -\frac{\ell}{h^2} \\ -\frac{h_y}{h} & \frac{h_x}{h} & -\frac{m}{h^2} \\ \ell & m & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{h_y}{h} & -\frac{h_x}{h} & -\frac{m}{h^2} \\ \frac{h_x}{h} & \frac{h_y}{h} & -\frac{n}{h^2} \\ m & n & 0 \end{pmatrix}.$$

Thus $W \in C^1(\Omega)$, $A, B \in L^1_{loc}(\Omega)$, $W_x = WA$ and $W_y = WB$. By the product rule for distributions, Lemma 2.3, we have

$$W_x \mathcal{Y} = WA \mathcal{Y} + W_y A = WA \mathcal{Y} + WBA.$$

Similarly $W_y \mathcal{X} = WB \mathcal{X} + W_x B = WB \mathcal{X} + WAB$. Moreover, by the equality of mixed distributional derivatives, Lemma 2.2, we also have

$$W_x \mathcal{Y} = W_y \mathcal{X},$$

so

$$BA + A \mathcal{Y} = AB + B \mathcal{X}.$$

By definition, this means

$$(BA + A \mathcal{Y})(v) = (AB + B \mathcal{X})(v) \quad \text{for all } v \in C^1_0(\Omega).$$

Or

$$A \mathcal{Y}(v) - B \mathcal{X}(v) = (AB - BA)(v) \quad \text{for all } v \in C^1_0(\Omega).$$

Again, by definition, we have

$$-\int_{\Omega} (A^t v_y - B^t v_x) = \int_{\Omega} (AB - BA)^t v \quad \text{for all } v \in C^1_0(\Omega).$$

By choosing $v = w \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $w \in C^1_0(\Omega)$, we obtain

$$-\int_{\Omega} (A_{12} w_y - B_{12} w_x) = \int_{\Omega} (AB - BA)_{12} w \quad \text{for all } w \in C^1_0(\Omega) \text{ scalar.}$$

Substituting in A and B from above we have

$$-\int_{\Omega} \left(\frac{h_y}{h} w_y + \frac{h_x}{h} w_x \right) = -\int_{\Omega} \frac{\ell n - m^2}{h^2} w \quad \text{for all } w \in C^1_0(\Omega) \text{ scalar.}$$

In our notation this is equivalent to

$$\Delta \ln h = -\frac{\ell n - m^2}{h^2}.$$

Since $-1 \equiv K = \frac{\ell n - m^2}{h^4}$ it follows that $\Delta \ln h = h^2$ and $u = \ln h$ is a weak solution of $\Delta u = e^{2u}$.

2.4. Generalized Dirichlet Problem for Liouville Equation

THEOREM 2.5. *If $u : \Omega \rightarrow \mathbf{R}$, $u \in C^1(\Omega)$ and u is a weak solution to $\Delta u = e^{2u}$, then $u \in C^\infty(\Omega)$.*

PROOF. Let $p \in \Omega$ and choose a small open ball Ω' with $p \in \Omega'$. Let $b := u|_{\partial\Omega'}$ and $a := u|_{\overline{\Omega'}}$. Consider the equation $\Delta w = e^{2a}$ with Dirichlet condition $w|_{\partial\Omega'} = b$. Note that $e^{2a} \in C^1(\Omega')$ and $b \in C^1(\partial\Omega')$. Thus Theorem 4.3 in [3] implies that w exists, is unique and $w \in C^2(\Omega')$. Now both w and u solve (u as a weak solution) $\Delta u = e^{2u}$ on Ω' and $w \equiv u$ on $\partial\Omega'$. Thus Theorem 8.3 in [3] implies $w \equiv u$ on Ω' also. In particular $u \in C^2(\Omega')$. Now we can repeatedly apply Theorem 6.17 in [3] and obtain via bootstrapping that $u \in C^\infty(\Omega')$. \square

2.5. Liouville's Theorem. We will use the version of Liouville's Theorem given in [4].

THEOREM 2.6. *u solves $\Delta u = e^{2u}$ on Ω if and only if*

$$u = \frac{1}{4} \ln \frac{4|\phi'|^2}{(1 - |\phi|^2)^2}$$

where ϕ is holomorphic (with respect to $z = x + iy$) with $|\phi| < 1$ and $\phi' \neq 0$. Furthermore the developing map ϕ gives an isometric immersion of $(\Omega, e^{2u}|dz|^2)$ into $(\mathbf{H}^2, ds_{\mathbf{H}^2}^2)$.

3. Proof of Theorem 1.1

PROOF. By Theorem 2.1 we can assume our coordinates are isothermal with respect to the induced metric

$$f^* ds_{\mathbf{R}^3}^2 = h^2(dx^2 + dy^2).$$

The curvature is

$$K = -1.$$

If we define u by $u = \ln h$, then we have that u is a weak solution to

$$\Delta u = e^{2u}.$$

It then follows from Theorem 2.5 that $u \in C^\infty(\Omega)$. Our desired isometry is then guaranteed by Theorem 2.6 where $z = x + iy$. \square

4. Proof of Corollary 1.2

For a C^1 -immersion $f : \Omega \rightarrow \mathbf{R}^3$, the induced metric may be only C^0 with respect to the given coordinates. Furthermore in general K may not be defined. However it was shown in [1] that if the conditions of Corollary 1.2 hold, then by a change of coordinates, the immersion f becomes C^2 . Hence in the new coordinates the induced metric is C^1 , K is C^0 and all the conclusions of Theorem 1.1 hold.

More precisely we proved the following Theorem 4.1 in [1]. We say that a C^1 -function $f : \Omega \rightarrow \mathbf{R}$ is C^{1M} if its mixed partials exist, are continuous, and are equal. A C^1 -immersion $f : \Omega \rightarrow \mathbf{R}^3$ is C^{1M} if its components are, it is asymptotic if all the parameter curves are asymptotic, and it is called Chebyshev if $\langle f_x, f_x \rangle \equiv \langle f_y, f_y \rangle \equiv 1$. Here we are assuming θ , the angle from f_x to f_y , satisfies $0 < \theta < \pi$. The subscript “graph” is used because the type of coordinates used are often called graph coordinates.

THEOREM 4.1. *Let Ω be a rectangle and $f = f_{asyche} : \Omega \xrightarrow{C^{1M}} \mathbf{R}^3$ be a regular C^{1M} -immersion in asymptotic Chebyshev coordinates. Assume $N = N_{asyche} := \frac{f_x \times f_y}{\sin \theta}$ is C^{1M} , $N_{xy} = N_{yx} = \cos \theta N$, $f_x = N \times N_x$, and $f_y = -N \times N_y$. (Note this implies $K \equiv -1$.) Then there exists a C^1 -diffeomorphism $\rho : \Omega \rightarrow \Omega$ such that $f_{graph} = f_{asyche} \circ \rho$ is a regular C^2 -immersion.*

References

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Present Addresses:

JOSEF F. DORFMEISTER
 ZENTRUM MATHEMATIK,
 TECHNISCHE UNIVERSITÄT MÜNCHEN,
 D-85747 GARCHING BEI MÜNCHEN, GERMANY.
e-mail: dorfm@ma.tum.de

IVAN STERLING
 MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT,
 ST MARY’S COLLEGE OF MARYLAND,
 ST MARY’S CITY, MD 20686–3001, USA.
e-mail: isterling@smcm.edu