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The Canonical Contact Structure on the Link of a Cusp Singularity

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Abstract. Caubel, Nemethi, and Popescu-Pampu in [2] proved that an oriented 3-manifold admits at most one positive contact structure which can be realized as the complex tangency along the link of a complex surface singularity. They call it the Milnor fillable contact structure. Lekili and Ozbagci in [10] showed that a Milnor fillable contact structure is universally tight. In particular, by Honda's classification [5], the link of a cusp singularity is contactomorphic to the positive contact structure associated to the Anosov flow on a Sol-manifold (see [1]). We describe the contact structure on the link of a cusp singularity in two different ways without using Honda's classification theorem. One description is based on the toric method introduced in Mori [15]. The other description is based on Hirzebruch's construction of the Hilbert modular cusps. Consequently, we give certain answers to the problems in Mori [14] concerning the relation between the cusp singularities and the simple elliptic singularities, and the higher dimensional extension of the local Lutz-Mori twist.

1. Introduction

Caubel, Nemethi, and Popescu-Pampu in [2] proved that an oriented 3-manifold admits at most one positive contact structure which can be realized as the complex tangency along the link of a complex surface singularity. They call it the Milnor fillable contact structure. Lekili and Ozbagci in [10] showed that a Milnor fillable contact structure is universally tight. In particular, by Honda's classification [5], the link of a cusp singularity is contactomorphic to the positive contact structure associated to the Anosov flow on a Sol-manifold (see [1]). We describe the contact structure on the link of a cusp singularity in two different ways without using Honda's classification theorem. A mapping torus T_A of a hyperbolic automorphism $A \in SL(2;\mathbf{Z})$ of the 2-torus is called a Sol-manifold. The suspension Anosov flow on T_A induces a positive contact structure $\ker(\beta_+ + \beta_-)$ and a negative contact structure $\ker(\beta_+ - \beta_-)$, where β_+ and β_- are the 1-forms defining the Anosov foliations. They form a bi-contact structure on T_A. Atsuhide Mori defined the Lutz-Mori tube, a generalization of the Lutz tube to higher dimensional contact geometry ([14]; see also [11]). He inserted a 5-dimensional Lutz-Mori tube along a codimension 2 contact submanifold in the standard S^5

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using an isolated surface singularity in $\mathbf{C}^3$ whose link is $(T_A, \ker(\beta_+ + \beta_-))$ for some hyperbolic matrix A. From Laufer [9] and Karras [6], we see that it is analytically equivalent to a cusp singularity $\left\{z_1^p + z_2^q + z_3^r + \lambda z_1 z_2 z_3 = 0\right\}$ ($\lambda \neq 0$, $p^{-1} + q^{-1} + r^{-1} < 1$) which is known as the T_{pqr} singularity. In this paper, we prove that the link of a T_{pqr} singularity is contactomorphic to $(T_A, \ker(\beta_+ + \beta_-))$ for some hyperbolic matrix A, in two ways.

1) The toric proof. This is an interpretation of the recent work of Ryo Furukawa (see §3). He constructed the examples of contact embeddings of hyperbolic mapping tori in the standard contact 5-sphere using the moment polytope of S^5. We can perturb the singularity link of T_{pqr} to his model as a contact submanifold of the standard contact 5-sphere. We also apply this method to the simple elliptic singularities, $\tilde{E}_6 : \left\{z_1^3 + z_2^3 + z_3^3 + \lambda_1 z_1 z_2 z_3 = 0\right\}$, $\tilde{E}_7 : \left\{z_1^2 + z_2^4 + z_3^4 + \lambda_2 z_1 z_2 z_3 = 0\right\}$ and $\tilde{E}_8 : \left\{z_1^2 + z_2^3 + z_3^6 + \lambda_3 z_1 z_2 z_3 = 0\right\}$ $({\lambda_1}^3 + 27 \neq 0, {\lambda_2}^4 - 64 \neq 0, {\lambda_3}^6 - 432 \neq 0)$. In Arnold's classification of hypersurface singularities, they are related with cusp singularities, T_{334}, T_{245} and T_{237}, respectively. However, the geometrical meaning of the relation was not clear. We give a new aspect to this relation through the moment polytope. Namely,

THEOREM (Theorem 3.4). *Put*

$$A_{m,k} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_m \\ 0 & 1 \end{pmatrix} \in SL(2; \mathbf{Z}),$$

where $m \in \mathbf{Z}_{>0}$, $k = (k_1, \ldots, k_m) \in (\mathbf{Z}_{\geq 0})^m$. *The link of the isolated surface singularity* $\left\{f_{m,k}(z_1, z_2, z_3) = 0\right\}$ *is contactomorphic to* $(T_{A_{m,k}}, \ker(\beta_+ + \beta_-))$ *if* $k \neq 0$, *and to* $(T_{A_{m,0}}, \ker(dy + mzdx))$ *if* $k = 0$, *where*

$$\begin{aligned} f_{1,(k_1)}(z_1, z_2, z_3) &= z_1^2 + z_2^3 + z_3^{6+k_1} + z_1 z_2 z_3 \\ f_{2,(k_1,k_2)}(z_1, z_2, z_3) &= z_1^2 + z_2^{4+k_1} + z_3^{4+k_2} + z_1 z_2 z_3 \\ f_{3,(k_1,k_2,k_3)}(z_1, z_2, z_3) &= z_1^{3+k_1} + z_2^{3+k_2} + z_3^{3+k_3} + z_1 z_2 z_3 . \end{aligned}$$

2) The proof based on Hirzebruch's work. Given an algebraic number field of degree n over $\mathbf{Q}$, Hirzebruch constructed a Hilbert modular cusp ([4]). In the case where $n = 2$, he constructed the minimal resolution of a Hilbert modular cusp. This enables us to detect the contact structure on the link of a cusp singularity (§4).

We also argue the localization of the Lutz-Mori twist for the case where $n \geq 3$, and arrange the problem (§5).

2. Preliminary

2.1. Sol-manifolds and bi-contact structures. Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be the coordinates on the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ and $(\begin{pmatrix} x \\ y \end{pmatrix}, z)$ be the coordinates on $T^2 \times [0, 1]$. Let A be an element of $SL(2, \mathbf{Z})$ such that $\mathrm{tr}(A) > 2$. Then the matrix A has two positive eigenvalues a and a^{-1} and the corresponding eigenvectors v_+ and v_-, where $a > 1$ and $dx \wedge dy(v_+, v_-) = 1$.

DEFINITION 2.1 (Hyperbolic mapping tori). We define an equivalence relation $\sim$ on $T^2 \times [0, 1]$ by $(A \begin{pmatrix} x \\ y \end{pmatrix}, 0) \sim (\begin{pmatrix} x \\ y \end{pmatrix}, 1)$. The quotient $T_A = T^2 \times [0, 1]/ \sim$ is called a hyperbolic mapping torus.

DEFINITION 2.2 (Sol-manifolds). The Lie group Sol^3 is the split extension $1 \to \mathbf{R}^2 \to Sol^3 \to \mathbf{R} \to 1$ whose group structure is given by

$$(u, v; w) \cdot (u', v'; w') = (u + \exp w \cdot u', v + \exp(-w) \cdot v'; w + w') \text{ on } \mathbf{R}^2 \times \mathbf{R}.$$

There is a left invariant metric $\exp(-2w)du \otimes du + \exp(2w)dv \otimes dv + dw \otimes dw$ on Sol^3. Let Γ be a cocompact discrete subgroup of Sol^3. The compact quotient $M^3 = \Gamma \backslash Sol^3$ is called a Sol-manifold.

The discrete subgroups of Sol^3 are all of the forms $\Gamma = H \rtimes V = (\mathbf{Z} \times \mathbf{Z}) \rtimes \mathbf{Z}$. The quotient of $\mathbf{R}^2$ by the lattice H is T^2, while the quotient of $\mathbf{R}$ by V is a circle. The quotient $M^3 = \Gamma \backslash Sol^3$ is a T^2-bundle over S^1 with hyperbolic monodromy. Conversely, a hyperbolic mapping torus T_A is a Sol-manifold. The left invariant 1-forms $\exp(-w)du$ and $-\exp w dv$ on Sol^3 induce the 1-forms $\beta_+ = a^{-z} dx \wedge dy(v_+, \cdot)$ and $\beta_- = -a^z dx \wedge dy(v_-, \cdot)$ on T_A.

DEFINITION 2.3 (Anosov flows). A non-singular flow ϕ_t on a closed oriented 3-manifold M^3 is an Anosov flow if the tangent bundle TM^3 has the ϕ_t-invariant decomposition for some Riemannian metric g on M^3 such that $TM^3 = T\phi \oplus E^{uu} \oplus E^{ss}$, where

$$\begin{aligned} T\phi &= \{\text{tangent vectors along the flow lines}\}, \\ E^{uu} &= \left\{v \in TM^3;\ \|(\phi_t)_* v\| \geq c^{-1} \exp(\lambda t)\|v\|, t > 0\right\}, \\ E^{ss} &= \left\{v \in TM^3;\ \|(\phi_t)_* v\| \leq c \exp(-\lambda t)\|v\|, t > 0\right\}, \end{aligned}$$

for some constants $c \geq 1$ and $\lambda > 0$. We call $E^s = T\phi \oplus E^{ss}$ (*resp.* $E^u = T\phi \oplus E^{uu}$) weakly stable (*resp. unstable*) plane field. We obtain two codimension 1 foliations, the unstable foliation F^u and the stable foliation F^s by integrating E^u and E^s respectively. They are called Anosov foliations.

DEFINITION 2.4 (Bi-contact structures [13]). A bi-contact structure on M^3 is a pair of a positive contact structure ξ_+ and a negative contact structure ξ_- on M^3 which are transverse to each other.

Mitsumatsu in [13] constructed a bi-contact structure associated to an algebraic Anosov flow on a closed 3-manifold. In particular, there is a bi-contact structure on a Sol-manifold associated to the suspension Anosov flow.

EXAMPLE 2.5 (Mitsumatsu [13]). Let M^3 be a Sol-manifold T_A and fix a Riemannian metric $g = \beta_+ \otimes \beta_+ + \beta_- \otimes \beta_- + dz \otimes dz$. Then the flow $\phi_t(x, y, z) = (x, y, z+t)$ is an Anosov flow with the stable foliation $F^s = \ker \beta_+ =< (\frac{\partial}{\partial z}), v_- >$ and the unstable foliation $F^u = \ker \beta_- =< (\frac{\partial}{\partial z}), v_+ >$. Moreover, $\xi_+ = \ker(\beta_+ + \beta_-)$ and $\xi_- = \ker(\beta_+ - \beta_-)$ form a bi-contact structure on T_A. We can see that the flow ϕ_t pushes the plane fields ξ_+ and ξ_- towards F^u (or F^s if you flow backward). More precisely,

$$\lim_{t \to +\infty} (\phi_t)_* \xi_+ = \lim_{t \to +\infty} (\phi_t)_* \xi_- = E^u$$

and

$$\lim_{t \to -\infty} (\phi_t)_* \xi_+ = \lim_{t \to -\infty} (\phi_t)_* \xi_- = E^s .$$

REMARK 2.6. The 1-forms $\beta_+ + \beta_-$ and $\beta_+ - \beta_-$ are induced by left invariant contact forms $\exp(-w)du - \exp w dv$ and $\exp(-w)du + \exp w dv$ on Sol^3, respectively. The universal covering of $(T_A, \ker(\beta_+ + \beta_-))$ is $(Sol^3, \ker(\exp(-w)du - \exp w dv))$, which is the standard positive contact structure on $\mathbf{R}^3$. Thus $(T_A, \ker(\beta_+ + \beta_-))$ is universally tight and it is the unique positive contact structure on T_A that is universally tight and minimally twisting ([5]). Similarly, $(T_A, \ker(\beta_+ - \beta_-))$ is the unique negative contact structure on T_A that is universally tight and minimally twisting.

2.2. Nil-manifolds

DEFINITION 2.7 (Nil-manifolds). The Lie group N is the central extension $1 \to \mathbf{R} \to N \to \mathbf{R}^2 \to 1$ whose group structure is given by

$$(u, v; w) \cdot (u', v'; w') = (u + u', v + v'; w + w' + uv') \text{ on } \mathbf{R}^2 \times \mathbf{R}.$$

Let Γ be a cocompact discrete subgroup of N. The compact quotient $M^3 = \Gamma \backslash N$ is called a Nil-manifold.

We note that N is the Heisenberg group of real matrices and a Nil-manifold is a parabolic mapping torus T_A, where $A = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ for some $m \in \mathbf{Z}$. On a Nil-manifold T_A, there is a left invariant positive contact form $\alpha_m = dy + mzdx$. The contact structure $(T_A, \ker \alpha_m)$ is the unique universally tight, minimally twisting structure ([5]). We also note that there is no Anosov flows on Nil-manifolds ([13]).

2.3. Simple elliptic singularities and cusp singularities. In this subsection, we remind the definitions of a simple elliptic singularity and a cusp singularity. Moreover, we confirm that a simple elliptic singularity in $\mathbf{C}^3$ is $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, and a cusp singularity in $\mathbf{C}^3$ is the T_{pqr} singularity.

DEFINITION 2.8 (Normal surface singularities). Let $(X, 0)$ be an isolated surface singularity germ. We say that $(X, 0)$ is normal if every bounded holomorphic function on $X^* = X - 0$ extends to a holomorphic function at 0.

DEFINITION 2.9 (Resolutions, exceptional sets). Let $(X, 0)$ be a normal surface singularity. Then there exists a non-singular complex surface $\tilde{X}$ and a proper analytic map $\pi : \tilde{X} \to X$ satisfying the following conditions (1) and (2).

(1) $E = \pi^{-1}(0)$ is a divisor in $\tilde{X}$, i.e., a union of 1-dimensional compact curves in $\tilde{X}$; and
(2) the restriction of π to $\pi^{-1}(X^*)$ is a biholomorphic map between $\tilde{X} - E$ and X^*.

The surface $\tilde{X}$ is called a resolution of the singularity of X, and $\pi : \tilde{X} \to X$ is the resolution map. The divisor E is called the exceptional set. The divisor E is good if it satisfies the following two conditions:

(3) Each irreducible component E_i of E is non-singular; and
(4) E has normal crossings, i.e., E_i intersects E_j, $i \neq j$, in at most one point, where they meet transversally, and no three of them intersect at the same point.

DEFINITION 2.10 (Minimal resolutions). A resolution $\pi : \tilde{X} \to X$ is minimal if any resolution $\pi' : \tilde{X}' \to X$, there is a proper analytic map $p : \tilde{X}' \to X$ such that $\pi' = \pi \circ p$.

By Castelnuovo's criterion, minimality of a resolution is equivalent to the condition that the exceptional set contains no non-singular rational curves with self-intersection -1.

DEFINITION 2.11 (Singularity links). Let X be a complex analytic variety of dimension at least 2 with only one singular point x. Let $r : X \to [0, \infty)$ be a real analytic function on X such that $r^{-1}(0) = x$, and r is strictly pluri-subharmonic ($spsh$) on $X - \{x\}$. For $\varepsilon > 0$ sufficiently small, r has no critical points in $r^{-1}(0, \varepsilon]$. We define a link of germ (X, x) to be $r^{-1}(\varepsilon)$. Any two links of (X, x) are diffeomorphic.

DEFINITION 2.12 (Simple elliptic singularities). Let $(X, 0)$ be a normal surface singularity and $\pi : \tilde{X} \to X$ be the minimal resolution of $(X, 0)$. We say that $(X, 0)$ is a simple elliptic singularity if the exceptional set $E = \pi^{-1}(0)$ is a non-singular elliptic curve.

The boundary of the tubular neighborhood of such an elliptic curve is a parabolic mapping torus. Thus the link of a simple elliptic singularity is a Nil-manifold. K. Saito gave a description of simple elliptic singularities which can be surface singularities in $\mathbf{C}^3$.

THEOREM 2.13 (Saito [17]). *If a simple elliptic singularity is embedded in* $\mathbf{C}^3$, *it is analytically equivalent to* $\tilde{E}_6$, $\tilde{E}_7$ *and* $\tilde{E}_8$.

Since they satisfy $E^2 = -3, -2$, and -1 respectively, their links are the mapping tori of $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

DEFINITION 2.14 (Cusp singularities). Let $(X, 0)$ be a normal surface singularity and $\pi : \tilde{X} \to X$ be the minimal resolution of $(X, 0)$. We say that $(X, 0)$ is a cusp singularity if the exceptional set $E = \pi^{-1}(0)$ is a cycle of non-singular rational curves, i.e., E_i intersects E_{i+1} transversally at one point for $1 \leq i \leq n$ and E has no other crossings, where $E = \bigcup_{i=1}^n E_i$ and E_{n+1} means E_1.

The graph manifold obtained by plumbing along such a cyclic weighted graph is a hyperbolic mapping torus. Thus the link of a cusp singularity is a Sol-manifold. The following theorem gives embeddings of cusp singularities into $\mathbf{C}^3$.

THEOREM 2.15 (Laufer [9]). *T_{pqr} singularity is a cusp singularity whose link K is diffeomorphic to the hyperbolic mapping torus T_A, where*

$$A = \begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix}.$$

THEOREM 2.16 (Karras [6]). *If a cusp singularity is embedded in $\mathbf{C}^3$, then it is analytically equivalent to T_{pqr} singularity for some $p, q, r \in \mathbf{Z}_{\geq 2}$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.*

REMARK 2.17. Laufer [8] also proved that cusp singularities are taut, that is, any two cusp singularities whose links are orientation preserving homeomorphic are analytically equivalent. On the other hand, the simple elliptic singularities are not taut. For example, two singularities $\{z_1^3 + z_2^3 + z_3^3 = 0\}$ and $\{z_1^3 + z_2^3 + z_3^3 + z_1 z_2 z_3 = 0\}$ have the common resolution graph. However they are not analytically equivalent to each other.

The canonical contact structure on the link K is the restriction $(K, \xi_0|K)$ of the standard contact structure (S^5_ε, ξ_0). In the following two sections, we investigate the structure of $(K, \xi_0|K)$.

3. The toric proof

3.1. The moment polytope of S^5. Let $(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3)$ be the polar coordinates on $S^5 \subset \mathbf{C}^3$, where

$$(z_1, z_2, z_3) = (r_1 \exp(2\pi i \theta_1), r_2 \exp(2\pi i \theta_2), r_3 \exp(2\pi i \theta_3)) \in \mathbf{C}^3,$$
$$S^5 = \{r_1^2 + r_2^2 + r_3^2 = 1\}.$$

The standard contact form on S^5 is $\alpha_0 = r_1^2 d\theta_1 + r_2^2 d\theta_2 + r_3^2 d\theta_3$. Let $\phi \colon S^5 \to \mathbf{R}^3$ be the projection, where

$$\phi(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3) = (r_1^2, r_2^2, r_3^2).$$

Then the image $\phi(S^5) = \{x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$ is a regular triangle in $\mathbf{R}^3$. It is the moment polytope $\triangle$. The intersection $L = S^5 \cap \{z_1 z_2 z_3 = 0\}$ is the preimage of the union of the three edges. We put $L_{z_1} = S^5 \cap \{z_1 = 0\}$, $L_{z_2} = S^5 \cap \{z_2 = 0\}$ and $L_{z_3} = S^5 \cap \{z_3 = 0\}$. Then $L = L_{z_1} \cup L_{z_2} \cup L_{z_3}$ and each L_{z_i} is the standard contact 3-sphere. Mori [15] gave an example of a contact embedding of the overtwisted contact structure on S^3 associated to the negative Hopf band. The key point of his construction was that he connected the rotation around the barycenter of the triangle $\triangle$ with the non-integrability of the induced contact structure. Ryo Furukawa extended this principle to the rotation around a weighted excenter of the triangle. As a result, he obtained the following examples of contact embeddings.

EXAMPLE 3.1 (Furukawa). The standard contact structure ξ on $L(p, p-1)$. The link of the singularity $\left\{z_1^p - z_2 z_3 = 0\right\}$ gives a contact embedding of ξ. The intersection $K = S^5 \cap \left\{z_1^p - z_2 z_3 = 0\right\}$ is determined by

$$\begin{cases} r_1^p = r_2 r_3 \,, \\ r_1^2 + r_2^2 + r_3^2 = 1 \,, \\ p\theta_1 - \theta_2 - \theta_3 = 0 \,. \end{cases}$$

Hence the image $\phi(K)$ is the curve $\left\{x_1^p = x_2 x_3\right\}$ on $\triangle$ and K is the slice section $\{p\theta_1 - \theta_2 - \theta_3 = 0\}$ over $\phi(K)$. On the other hand, we can obtain the standard contact structure on $L(p, p-1)$ by collapsing the boundaries of $T_p = (T^2 \times I, \ker(f(z)dx + g(z)dy))$ to circles along the characteristic foliations, where (f, g) is a curve rotating clockwise around $(0, 0)$ from $(1, 0)$ to $(-1, -p)$. Thus, if $c : [0, 1] \to \triangle$ is a C^∞-curve satisfying that $c(0) = (0, 0, 1)$, $c(1) = (0, 1, 0)$ and c rotates clockwise around $\left(\frac{p}{p-2}, \frac{-1}{p-2}, \frac{-1}{p-2}\right)$, the slice section $\{p\theta_1 - \theta_2 - \theta_3 = 0\}$ over c gives the standard contact structure on $L(p, p-1)$. Namely, the link K is contactomorphic to $(L(p, p-1), \xi)$. Moreover, even if the curve c satisfies that $c([0, \frac{1}{3}]) = \{x_2 = 0, \varepsilon \leq x_3 \leq 1\}$ and $c([\frac{2}{3}, 1]) = \{\varepsilon \leq x_2 \leq 1, x_3 = 0\}$, the slice section $\{p\theta_1 - \theta_2 - \theta_3 = 0\}$ over c also gives the contact embedding of ξ by Gray stability. The slice sections over $c([0, \frac{1}{2}])$ and $c([\frac{1}{2}, 1])$ are the standard contact solid tori pasted by the coordinate linear transformation $\begin{pmatrix} p & -1 \\ 1 & 0 \end{pmatrix}$. This viewpoint is also useful for understanding the next example.

EXAMPLE 3.2 (Furukawa). The positive contact structure associated to the Anosov flow of T_A, where $A = \begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix}$.

Let $c : [0,1] \to \triangle$ be a C^∞-curve such that

$$\begin{cases} c([0, \frac{1}{6}]) = \{x_2 = 0, \varepsilon \le x_3 \le 1-\varepsilon\}\ , \\ c([\frac{1}{3}, \frac{1}{2}]) = \{x_3 = 0, \varepsilon \le x_1 \le 1-\varepsilon\}\ , \\ c([\frac{2}{3}, \frac{5}{6}]) = \{x_1 = 0, \varepsilon \le x_2 \le 1-\varepsilon\} \end{cases}$$

and c rotates clockwise around the point P, where

$$P = \begin{cases} (\frac{r-1}{r-3}, \frac{-1}{r-3}, \frac{-1}{r-3})\ \ t \in [\frac{1}{12}, \frac{5}{12})\,, \\ (\frac{-1}{q-3}, \frac{q-1}{q-3}, \frac{-1}{q-3})\ \ t \in [\frac{5}{12}, \frac{3}{4})\,, \\ (\frac{-1}{p-3}, \frac{-1}{p-3}, \frac{p-1}{p-3})\ \ t \in [0, \frac{1}{12}) \cup [\frac{3}{4}, 1]\,. \end{cases}$$

We define the submanifold X of S^5 as the slice section over c given by

$$\begin{cases} \{(r-1)\theta_1 - \theta_2 - \theta_3 = 0\}\ (t \in (\frac{1}{6}, \frac{1}{3}))\,, \\ \{(q-1)\theta_2 - \theta_3 - \theta_1 = 0\}\ (t \in (\frac{1}{2}, \frac{2}{3}))\,, \\ \{(p-1)\theta_3 - \theta_1 - \theta_2 = 0\}\ (t \in (\frac{5}{6}, 1))\,. \end{cases}$$

By the observation of Example 3.1, X is the resultant contact manifold of pasting the three pieces of $T^2 \times I$ by the linear maps $\begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix}$. Thus X is obtained by pasting the boundary tori of $(T^2 \times I, \ker(f(z)dx + g(z)dy))$ by the linear map A, where $\begin{pmatrix} f \\ g \end{pmatrix}$ is a curve rotating clockwise around $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ from $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $A\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore, it is contactomorphic to the positive contact structure associated to the suspension Anosov flow. Indeed, the kernel of the 1-form $a^{-z}dx \wedge dy(v_+, \cdot) - a^z dx \wedge dy(v_-, \cdot)$ restricted to the torus fiber of $T^2 \times \mathbf{R}$ rotates from v_+ to $-v_-$ and there exists z_0 such that the characteristic foliation on $\{z = z_0\}$ is given by $\ker(x + y)$. Hence, we can obtain $(T_A, \ker(\beta_+ + \beta_-))$ by pasting the boundary tori of $(T^2 \times I, \ker(h_1(z)dx + h_2(z)dy))$, where $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ satisfies the same condition of rotation as $\begin{pmatrix} f \\ g \end{pmatrix}$. We can isotope these two curves without breaking the condition. Namely, we can isotope X to $(T_A, \ker(\beta_+ + \beta_-))$ as a contact manifold. Therefore, X is contactomorphic to $(T_A, \ker(\beta_+ + \beta_-))$ by Gray stability. In the case of $(p, q, r) = (2, 3, 6), (2, 4, 4), (3, 3, 3)$, we can see that X is contactomorphic to $(T_A, \ker(dy + mz dx))(m = 1, 2, 3)$ by the same argument.

3.2. The relation between $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ and T_{pqr}. In Arnold's classification list of hypersurface singularities, the simple elliptic singularities are

$$\tilde{E}_6 : z_1^3 + z_2^3 + z_3^3 + \lambda_1 z_1 z_2 z_3 = 0\ \ ({\lambda_1}^3 + 27 \neq 0)\,,$$

$$\tilde{E}_7 : z_1^2 + z_2^4 + z_3^4 + \lambda_2 z_1 z_2 z_3 = 0 \ \ ({\lambda_2}^4 - 64 \neq 0)\,,$$
$$\tilde{E}_8 : z_1^2 + z_2^3 + z_3^6 + \lambda_3 z_1 z_2 z_3 = 0 \ ({\lambda_3}^6 - 432 \neq 0)\,,$$

and the cusp singularities are

$$T_{pqr} : z_1^p + z_2^q + z_3^r + \lambda z_1 z_2 z_3 = 0 \ (\lambda \neq 0,\ p^{-1} + q^{-1} + r^{-1} < 1)\,.$$

As we stated in §2.3, the link of a simple elliptic singularity is a Nil-manifold, while that of a cusp singularity is a Sol-manifold. Thus they have different geometries. However, we can treat these singularities uniformly. Indeed, we can unite the former part of Theorem 2.13 and Theorem 2.15 meaningfully by the argument using the moment polytope of S^5, and also detect the contact structure on the link.

THEOREM 3.3. *The link of the surface singularity* $\{z_1^p + z_2^q + z_3^r + z_1 z_2 z_3 = 0\}$ *such that* $p^{-1} + q^{-1} + r^{-1} \leq 1$ *is the mapping torus* T_A *of* A, *where*

$$A = \begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, the canonical contact structure on the link is contactomorphic to the contact manifold X *of Example* 3.2.

PROOF. The strategy is roughly as follows. We consider the intersection $L_\lambda = S^5 \cap \{z_1^p + z_2^q + z_3^r - \lambda z_1 z_2 z_3 = 0\}$ for a sufficiently large positive real number λ. It is contactomorphic to the link of $\{z_1^p + z_2^q + z_3^r + z_1 z_2 z_3 = 0\}$. Since λ is sufficiently large, $|z_1 z_2 z_3|$ is small on L_λ. Indeed,

$$|z_1 z_2 z_3| < \frac{1}{\lambda}(|z_1|^p + |z_2|^q + |z_3|^r) < \frac{3}{\lambda}\,.$$

Thus L_λ is very close to L except on a neighborhood of the union of three circles $\{z_2 = z_3 = 0\} \cup \{z_3 = z_1 = 0\} \cup \{z_1 = z_2 = 0\}$. On the other hand, for sufficiently small ε, X of Example 3.2 is also very close to L except near the three circles. Moreover, L_λ is very close to X even on a neighborhood of the three circles. We can isotope L_λ to X as a contact submanifold, and by Gray stability, they are contactomorphic. Let us prove the existence of a contact isotopy from L_λ to X. Let $\phi : \mathbf{R}_{\geq 0} \to \mathbf{R}$ be a bump function supported on $\{s \in \mathbf{R} \mid 1 - 2\delta \leq s\}$ and $\phi \equiv 1$ on $\{s \in \mathbf{R} \mid 1 - \delta \leq s\}$ with $0 < \delta < \frac{1}{5}$. We define $F_\lambda = z_1 z_2 z_3 - \frac{1}{\lambda}(z_1^p + z_2^q + z_3^r)$ and $G_\lambda = z_1 z_2 z_3 - \frac{1}{\lambda}(\phi(r_1^2) z_1^p + \phi(r_2^2) z_2^q + \phi(r_3^2) z_3^r)$. Note that $G_\lambda^{-1}(0)$ satisfies the condition of X of Example 3.2. Hence it is enough to find a contact isotopy between $F_\lambda^{-1}(0)$ and $G_\lambda^{-1}(0)$. We define $H_t = (1-t)F_\lambda + tG_\lambda$. For sufficiently large λ, $H_t^{-1}(0)$ defines a contact isotopy. On the open set $\{|z_i| > \sqrt{1-\delta}\} \subset S^5$, $H_t^{-1}(0)$ is a complex hypersurface singularity link for each $t \in [0, 1]$. Thus it is a contact submanifold on the open set. On the other hand, $H_t^{-1}(0)$ is close to L on $U = \{|z_1|, |z_2|, |z_3| < \sqrt{1 - \frac{1}{2}\delta}\}$.

Indeed, if $|z_1| < \sqrt{1-\frac{1}{2}\delta}$ and λ is sufficiently large, $H_t(z_1, z_2, z_3) = 0$ can be solved for $z_j (j = 2,$ or $3)$ by the implicit function theorem and the graph is close to $\{z_j = 0\}$ in the sense of C^∞ topology. Since $L \cap U$ is a contact submanifold of U and the contactness is an open condition, there exists λ such that $H_t^{-1}(0) \cap U$ is a contact submanifold of U for each $t \in [0, 1]$. For such a positive number λ, $H_t^{-1}(0)$ is a contact submanifold of the standard contact 5-sphere for each $t \in [0, 1]$. Hence it is a contact isotopy between $F_\lambda^{-1}(0)$ and $G_\lambda^{-1}(0)$. Therefore, $L_\lambda = F_\lambda^{-1}(0)$ is diffeomorphic to the mapping torus T_A of A, where

$$A = \begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix}$$

and the canonical contact structure is the positive contact structure associated to the suspension Anosov flow on it. The above argument also works for the case of $(p, q, r) = (2, 3, 6), (2, 4, 4), (3, 3, 3)$, namely, the simple elliptic singularities. $\square$

Thus we can uniformly treat the links of these singularities with the canonical contact structures. Let us interpret Theorem 3.3. Put

$$A_{m,k} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_m \\ 0 & 1 \end{pmatrix} \in SL(2; \mathbf{Z}),$$

where $m \in \mathbf{Z}_{>0}$, $k = (k_1, \ldots, k_m) \in (\mathbf{Z}_{\geq 0})^m$. We note that $A_{m,k}$ is a hyperbolic matrix if $k \neq 0$ and $A_{m,0}$ is a parabolic matrix.

THEOREM 3.4. *The link of the isolated surface singularity* $\{f_{m,k}(z_1, z_2, z_3) = 0\}$ *is contactomorphic to* $(T_{A_{m,k}}, \ker(\beta_+ + \beta_-))$ *if* $k \neq 0$, *and to* $(T_{A_{m,0}}, \ker(dy + mzdx))$ *if* $k = 0$, *where*

$$\begin{aligned} f_{1,(k_1)}(z_1, z_2, z_3) &= z_1^2 + z_2^3 + z_3^{6+k_1} + z_1 z_2 z_3 \\ f_{2,(k_1,k_2)}(z_1, z_2, z_3) &= z_1^2 + z_2^{4+k_1} + z_3^{4+k_2} + z_1 z_2 z_3 \\ f_{3,(k_1,k_2,k_3)}(z_1, z_2, z_3) &= z_1^{3+k_1} + z_2^{3+k_2} + z_3^{3+k_3} + z_1 z_2 z_3 \,. \end{aligned}$$

It is an extension of Theorem 3.5 of [14] and is the answer for Problem 4.6 of [14]. Moreover there is no hypersurface singularity for the case where $m > 3$ because of the latter part of Theorem 2.13 and Theorem 2.16. For the Euler characteristic of the Milnor fiber $P_{m,k}$, the Dynkin diagram is useful. Gabriélov showed that the Dynkin diagrams of these singularities are in the form of T_{pqr} ([3]). In particular, the Milnor number μ is equal to $p + q + r - 1$. The Milnor fiber is homotopic to the wedge of 2-spheres, where the number of 2-spheres are equal to μ. Therefore, the Euler characteristic is

$$\chi(P_{m,k}) = 1 + \mu = p + q + r = \begin{cases} 11 + k_1 \ (m = 1) \\ 10 + k_1 + k_2 \ (m = 2) \\ 9 + k_1 + k_2 + k_3 \ (m = 3)\,. \end{cases}$$

4. Hilbert modular cusps

4.1. Hilbert modular cusps. Let K be a totally real algebraic field of degree n over $\mathbf{Q}$. Then we have n distinct embeddings $x \mapsto x^{(i)}$ $(1 \leq i \leq n)$ of K in $\mathbf{R}$. Let H be an additive subgroup of K of rank n, and let V be a multiplicative subgroup of U_H^+ of rank $n-1$, where U_H^+ is the group of totally positive units e with $eH = H$. Let

$$G(H, V) = \left\{ \begin{pmatrix} e & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in H, e \in V \right\} .$$

Then $G(H, V)$ acts properly discontinuously and without fixed points on the product $\mathbf{H}^n$ of n copies of the upper half plane $\mathbf{H}$ by

$$(z_1, \ldots, z_n) \mapsto (e^{(1)}z_1 + b^{(1)}, \ldots, e^{(n)}z_n + b^{(n)}) .$$

We consider $\overline{\mathbf{H}^n/G(H, V)}$ which is the completion of $\mathbf{H}^n/G(H, V)$ by adding the one point ∞. The complete system of open neighborhoods of ∞ is given by the sets

$$\big(\mathrm{int}\ W(d)/G(H, V)\big) \cup \{\infty\}$$

where, for any positive d,

$$W(d) = \left\{(z_1, \ldots, z_n) \in \mathbf{H}^n \mid \Pi_{i=1}^n \mathrm{Im} z_i \geq d\right\} .$$

Then $\overline{\mathbf{H}^n/G(H, V)}$ is a normal complex space. The singularity ∞ is called a Hilbert modular cusp. Let us check that $\partial W(d)/G(H, V)$ is the link of ∞.

LEMMA 4.1. *The function*

$$\varphi(z_1, \ldots, z_n) = \frac{1}{\Pi_{j=1}^n \mathrm{Im} z_j}$$

on $\mathbf{H}^n$ *is spsh, and induces* $\tilde{\varphi}$ *which is spsh on* $\mathbf{H}^n/G(H, V)$.

PROOF. For the former part, we have to check that Levi matrix L_φ is positive definite. Put $z_j = x_j + iy_j$, then

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \frac{1}{i}\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \frac{1}{i}\frac{\partial}{\partial y_j}\right).$$

Since $\varphi = (y_1 \ldots y_n)^{-1}$,

$$\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} = \begin{cases} \frac{\varphi}{2y_i^2} & (i = j) , \\ \frac{\varphi}{4y_i y_j} & (i \neq j) . \end{cases}$$

Then it is easy to check that the matrix $L_\varphi = \left(\frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right)_{1\le i,j\le n}$ is positive definite. For the latter part, we have to check that φ is $G(H,V)$-invariant. The action of $G(H,V)$ is

$$\begin{pmatrix} e & b \\ 0 & 1 \end{pmatrix} \cdot (z_1,\dots,z_n) = (e^{(1)}z_1 + b^{(1)},\dots,e^{(n)}z_n + b^{(n)}),$$

and using the coordinates $(x_1, y_1, \dots, x_n, y_n)$, since $b^{(i)} \in \mathbf{R}$,

$$\begin{pmatrix} e & b \\ 0 & 1 \end{pmatrix} \cdot (x_1, y_1, \dots, x_n, y_n) = (e^{(1)}x_1 + b^{(1)}, e^{(1)}y_1, \dots, e^{(n)}x_n + b^{(n)}, e^{(n)}y_n).$$

Hence

$$\begin{pmatrix} e & b \\ 0 & 1 \end{pmatrix} \cdot \varphi = \frac{1}{(e^{(1)}y_1)\cdots(e^{(n)}y_n)} = \frac{1}{y_1\cdots y_n} = \varphi.$$

Thus φ is $G(H,V)$-invariant and $\tilde{\varphi}$ is induced. □

We define $\tilde{\varphi}(\infty) = 0$. Then $\tilde{\varphi}$ is a real analytic function on $\overline{\mathbf{H}^n/G(H,V)}$. Thus $\partial W(d)/G(H,V) = \tilde{\varphi}^{-1}(d)$ is the link of the Hilbert modular cusp ∞. Let J be the standard complex structure on $\mathbf{H}^n$. For the spsh function φ of Lemma 4.1, we put

$$\lambda = -J^* d\varphi, \quad \omega = -dJ^* d\varphi, \quad g(u,v) = \omega(u, Jv).$$

Then ω is a symplectic form on $\mathbf{H}^n$ compatible with J, and g is a J-invariant Riemannian metric. Moreover, $\alpha = \lambda \mid \partial W(d)$ is a contact form on $\partial W(d)$. Since φ is $G(H,V)$-invariant, the contact structure $(\partial W(d)/G(H,V), \ker \tilde{\alpha})$ is induced. It is the canonical contact structure on the link of the Hilbert modular cusp ∞. By the embedding $M \ni b \mapsto (b^{(1)}, b^{(2)}, \dots, b^{(n)}) \in \mathbf{R}^n$, $H \cong \mathbf{Z}^n$ is a lattice in $\mathbf{R}^n$. Since $eH = H$ and e is totally positive for $e \in V$, V is a subgroup of $SL(H)$. Thus the link $\partial W(d)/G(H,V)$ is a T^n bundle over T^{n-1} with hyperbolic monodromies. We note that the contact form $\tilde{\alpha}$ on the link $\partial W(1)/G(H,V)$ is given by

$$\alpha = \frac{dx_1}{y_1} + \cdots + \frac{dx_n}{y_n}.$$

4.2. The 2-dimensional case. Hirzebruch constructed the minimal resolution of a 2-dimensional Hilbert modular cusp ∞. As we will see, the exceptional set is a cycle of non-singular rational curves. Thus it is a cusp singularity in the sense of Definition 2.14.

Let us assume that $\mathbf{Z} \ni k \mapsto b_k \in \mathbf{N}$ is a function satisfying $b_k \ge 2$ for all k, and $b_k \ge 3$ for some k. For each integer k, take a copy R_k of $\mathbf{C}^2$ with coordinates (u_k, v_k). We define $R'_k = R_k \setminus \{u_k = 0\}$ and $R''_k = R_k \setminus \{v_k = 0\}$. The equations

$$u_{k+1} = u_k^{b_k} v_k, \tag{1}$$

$$v_{k+1} = \frac{1}{u_k} \tag{2}$$

give a biholomorphic map $\varphi_k : R'_k \to R''_{k+1}$. In the disjoint union $\bigcup R_k$ we make all the identifications (1) and (2). We get a set Y. The axes $\{v_k = 0\}$ and $\{u_{k+1} = 0\}$ are pasted together by the equation (2) and form $\mathbf{C}P^1$. The self-intersection number is $-b_k$ by the equation (1). We put this rational curve S_k, then the intersection numbers are

$$S_i \cdot S_i = -b_i\,,\quad S_i \cdot S_{i+1} = 1\,,\quad S_i \cdot S_j = 0 \ \ (|\, i - j\,| \geq 2)\,.$$

Let us assume that the number series $\{b_k\}$ above is periodic and l is the period: $b_{k+l} = b_k (k \in \mathbf{Z})$. We denote the infinite continued fraction

$$b_k - \cfrac{1}{b_{k+1} - \cfrac{1}{b_{k+2} - \cdots}}$$

by w_k. It is a quadratic irrational number which is greater than 1 and belongs to the real quadratic field $\mathbf{Q}[w_0]$. We consider the $\mathbf{Z}$-module $H = \mathbf{Z} \cdot w_0 + \mathbf{Z} \cdot 1$. We define the action on $\mathbf{C}^2$ as follows. For $a \in H$, $\bar{a}$ denotes the conjugate irrational number of the quadratic irrational number a.

$$a : (z_1, z_2) \mapsto (z_1 + a, z_2 + \bar{a})\,.$$

Since $a \in \mathbf{R}$, $\mathbf{C}^2/H$ is diffeomorphic to $T^2 \times \mathbf{R}^2$. We define the map

$$\Phi : Y - \bigcup_{j \in \mathbf{Z}} S_j \to \mathbf{C}^2/H$$

given by $\Phi(u_0, v_0) = (z_1, z_2)$, where

$$2\pi i z_1 = w_0 \log u_0 + \log v_0,\quad 2\pi i z_2 = \bar{w}_0 \log u_0 + \log v_0\,.$$

It is well-defined and biholomorphic. We put

$$Y^+ = \Phi^{-1}(\mathbf{H}^2/H) \cup \bigcup_{j \in \mathbf{Z}} S_j\,.$$

We define $\{\mu_k\}_{k \in \mathbf{Z}}$ inductively by $\mu_0 = 1$ and $\mu_{k+1} = w_{k+1}^{-1} \mu_k$.

PROPOSITION 4.2. *The numbers μ_k satisfy the following conditions.*
(1) $\mu_{k+1} = b_k \mu_k - \mu_{k-1}$.
(2) $\mu_{k+l} = \mu_l \mu_k,\ (\mu_l)^n = \mu_{nl}$.

PROPOSITION 4.3. *The number $\mu_l = (w_1 w_2 \dots w_l)^{-1}$ is a unit in H and $\mu_l < 1$.*

PROOF. Since $\begin{pmatrix} \mu_{k+1} \\ \mu_k \end{pmatrix} = \begin{pmatrix} b_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_k \\ \mu_{k-1} \end{pmatrix}$,

$$\begin{pmatrix} \mu_{k+1} \\ \mu_k \end{pmatrix} = \begin{pmatrix} b_k & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_{-1} \end{pmatrix}.$$

Put

$$P = \begin{pmatrix} b_{l-1} & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since $\mu_{k+l} = \mu_l \mu_k$,

$$P \begin{pmatrix} \mu_0 \\ \mu_{-1} \end{pmatrix} = \begin{pmatrix} \mu_l \\ \mu_{l-1} \end{pmatrix} = \mu_l \begin{pmatrix} \mu_0 \\ \mu_{-1} \end{pmatrix}.$$

That is,

$$P \begin{pmatrix} 1 \\ w_0 \end{pmatrix} = \mu_l \begin{pmatrix} 1 \\ w_0 \end{pmatrix}.$$

Since $\{1, w_0\}$ is a basis of H and $P \in SL(2; \mathbf{Z})$, $\{\mu_l, \mu_l w_0\}$ is also a basis. Hence $\mu_l H = H$ and μ_l is a unit in H. Since $w_k > 1$ for any k, $\mu_l < 1$ is obvious. □

We consider the group $V = \{(\mu_l)^n \mid n \in \mathbf{Z}\}$. The action of V on Y^+ is given by

$$(\mu_l)^n : R_k \to R_{k+nl} ,$$

where $(\mu_l)^n(u_k, v_k) = (u_k, v_k)$ for $(\mu_l)^n \in V$. The action of V on $\mathbf{H}^2/H$ is given by

$$(\mu_l)^n \cdot (z_1, z_2) = ((\mu_l)^n z_1, (\bar{\mu}_l)^n z_2) \quad \text{for } (\mu_l)^n \in V .$$

PROPOSITION 4.4. *These two actions of V on Y^+ and on $\mathbf{H}^2/H$ are equivariant with respect to Φ, and the action of V on Y^+ is free and properly discontinuous.*

PROOF. See the lemma of section 2.3 of [4]. □

Hence, the two quotient spaces Y^+/V and $(\mathbf{H}^2/H)/V$ are complex manifolds. We note that $(\mathbf{H}^2/H)/V \cong \mathbf{H}^2/G(H, V)$, since H is a normal subgroup of $G(H, V)$. Since $Y(b_0, \dots, b_{l-1}) = Y^+/V$ is the minimal resolution of $\left(\overline{\mathbf{H}^2/G(H, V)}, \infty\right)$ and the exceptional set $\bigcup_{j=0}^{l-1} S_j$ is a cycle of non-singular rational curves, ∞ is a cusp singularity.

THEOREM 4.5. *The link $(\partial W/G, \ker\tilde{\alpha})$ is contactomorphic to the Sol-manifold $(T_A, \ker(\beta_+ + \beta_-))$, where*

$$A = \left(\begin{pmatrix} b_{l-1} & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_0 & -1 \\ 1 & 0 \end{pmatrix}\right)^{-1}.$$

PROOF. We have $A \begin{pmatrix} 1 \\ w_0 \end{pmatrix} = \mu_l^{-1} \begin{pmatrix} 1 \\ w_0 \end{pmatrix}$ and $A \begin{pmatrix} 1 \\ \bar{w}_0 \end{pmatrix} = \mu_l \begin{pmatrix} 1 \\ \bar{w}_0 \end{pmatrix}$. Now we put $\begin{pmatrix} 1 \\ w_0 \end{pmatrix} = \sqrt{w_0 - \bar{w}_0} v_+$, $\begin{pmatrix} 1 \\ \bar{w}_0 \end{pmatrix} = -\sqrt{w_0 - \bar{w}_0} v_-$ and $\mu_l^{-1} = a$. Then the map

$$F : T_A \to \partial W(1)/G \ ; \quad ((x, y), z) \mapsto (y - w_0 x, \ \ a^z, \ \ y - \bar{w}_0 x, \ \ a^{-z})$$

is the diffeomorphism. Since

$$\begin{aligned}
F\left(A\begin{pmatrix}1\\ w_0\end{pmatrix}, 0\right) = (0, 1, a(w_0-\bar{w}_0), 1) \sim F\left(\begin{pmatrix}1\\ w_0\end{pmatrix}, 1\right) &= (0, a, (w_0-\bar{w}_0), a^{-1}),\\
F\left(A\begin{pmatrix}1\\ \bar{w}_0\end{pmatrix}, 0\right) = (a^{-1}(\bar{w}_0-w_0), 1, 0, 1) \sim F\left(\begin{pmatrix}1\\ \bar{w}_0\end{pmatrix}, 1\right) &= (\bar{w}_0-w_0, a, 0, a^{-1}),
\end{aligned}$$

F is well-defined. Moreover the map

$$H : \partial W(1)/G \to T_A \ ; \ (x_1, y_1, x_2, y_2) \mapsto \left(\left(\frac{x_2-x_1}{w_0-\bar{w}_0}, \frac{w_0x_2-\bar{w}_0x_1}{w_0-\bar{w}_0}\right), \log_a y_1\right)$$

is the inverse of F. Hence $\partial W/G$ is diffeomorphic to T_A. Next we show that F is the contactomorphism $(T_A, \ker(\beta_+ + \beta_-)) \cong (\partial W(1)/G, \ker\tilde{\alpha})$, where the 1-form $\tilde{\alpha}$ is induced by

$$\alpha = \lambda \mid \partial W(1) = \frac{dx_1}{y_1} + \frac{dx_2}{y_2}.$$

Let us check that $F^*\tilde{\alpha} = \sqrt{w_0-\bar{w}_0}(\beta_+ + \beta_-)$.

$$\begin{aligned}
F^*\tilde{\alpha} = F^*\left(\frac{dx_1}{y_1} + \frac{dx_2}{y_2}\right) &= a^{-z}d(y-w_0x) + a^z d(y-\bar{w}_0x),\\
d(y-w_0x) &= \sqrt{w_0-\bar{w}_0}\cdot dx\wedge dy(v_+, \ \cdot\ ),\\
d(y-\bar{w}_0x) &= -\sqrt{w_0-\bar{w}_0}\cdot dx\wedge dy(v_-, \ \cdot\ ).
\end{aligned}$$

Therefore,

$$\begin{aligned}
F^*\tilde{\alpha} &= \sqrt{w_0-\bar{w}_0}(a^{-z}dx\wedge dy(v_+, \ \cdot\ ) - a^z dx\wedge dy(v_-, \ \cdot\ ))\\
&= \sqrt{w_0-\bar{w}_0}(\beta_+ + \beta_-).
\end{aligned}$$

$\square$

If the two matrices

$$\begin{pmatrix}0 & 1\\ -1 & b_0\end{pmatrix}\cdots\begin{pmatrix}0 & 1\\ -1 & b_{l-1}\end{pmatrix}$$

and

$$\begin{pmatrix}r-1 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}q-1 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}p-1 & -1\\ 1 & 0\end{pmatrix}$$

are conjugate in $SL(2;\mathbf{Z})$, the two links K and $\partial W/G$ are orientation preserving diffeomorphic. Such a series $b_0, \dots, b_{l-1}$ is uniquely determined by p, q, r up to cyclic permutation.

Concretely,

$$\begin{cases} b_0 = 3, b_1 = \cdots = b_{r-7} = 2 \ \ (p = 2, q = 3, r \geq 7)\,, \\ b_0 = 4, b_1 = \cdots = b_{r-5} = 2 \ \ (p = 2, q = 4, r \geq 5)\,, \\ b_0 = b_{q-4} = 3, b_1 = \cdots = b_{q-5} = b_{q-3} = \ldots b_{q+r-9} = 2 \ \ (p = 2, q \geq 5, r \geq 5), \\ b_0 = 5, b_1 = \cdots = b_{r-4} = 2 \ \ (p = q = 3, r \geq 4)\,, \\ b_0 = 4, b_1 = \cdots = b_{q-4} = b_{q-2} = \cdots = b_{q+r-7} = 2 \ \ (p = 3, q, r \geq 4)\,, \\ b_1 = \cdots = b_{p-4} = b_{p-2} = \cdots = b_{p+q-7} = b_{p+q-5} = \cdots = b_{p+q+r-10} = 2\,, \\ b_0 = b_{p-3} = b_{p+q-6} = 3 \ (p, q, r \geq 4)\,. \end{cases}$$

Since cusp singularities are taut, the T_{pqr} singularity is analytically equivalent to the cusp singularity ∞. Then the two links $(K, \xi_0|K)$ and $(\partial W/G, \ker\tilde{\alpha})$ are contactomorphic, because the contact isotopy type of the link of an isolated singularity is unique [2].

THEOREM 4.6. *$(K, \xi_0|K)$ is contactomorphic to $(T_A, \ker(\beta_+ + \beta_-))$.*

5. The Lutz-Mori tube

DEFINITION 5.1 (Liouville pairs). A Liouville pair on an oriented manifold N^{2n-1} is a pair of a positive contact form $\tilde{\alpha}_+$ and a negative contact form $\tilde{\alpha}_-$ on N^{2n-1} such that the 1-form $\beta = \exp(-s)\tilde{\alpha}_- + \exp s \cdot \tilde{\alpha}_+$ on $\mathbf{R} \times N^{2n-1}$ satisfies $d\beta^n > 0$.

DEFINITION 5.2 (The Lutz-Mori tube). Let $(\tilde{\alpha}_+, \tilde{\alpha}_-)$ be a Liouville pair on N^{2n-1}. The 1-form

$$\gamma = \frac{1 - \cos s}{2}\tilde{\alpha}_+ + \frac{1 + \cos s}{2}\tilde{\alpha}_- - s \sin s d\theta$$

defines the contact structure on $N^{2n-1} \times D^2$, where (s, θ) are the polar coordinates of the disk D^2 with the radius π. We call $(N^{2n-1} \times D^2, \ker\gamma)$ the Lutz-Mori tube.

DEFINITION 5.3 (The Lutz-Mori twist). Suppose that $(N^{2n-1}, \ker\tilde{\alpha}_+)$ is a codimension two contact submanifold of a contact manifold $(M^{2n+1}, \ker\beta)$ with trivial normal bundle. By the tubular neighborhood theorem for contact submanifolds, there is a small tubular neighborhood of N^{2n-1} contactomorphic to $(N^{2n-1} \times D^2, \ker(\tilde{\alpha}_+ + r^2 d\theta))$ $(0 < r < \varepsilon)$. Putting $s = r + \pi$ and slightly deform the contact structure $\ker(\tilde{\alpha}_+ + r^2 d\theta)$ on $\{0 < r < \varepsilon\}$, we can insert the Lutz-Mori tube along the contact submanifold N^{2n-1} to get a possibly new contact structure on M^{2n+1}. We call this operation the Lutz-Mori twist.

Mori originally defined the Lutz-Mori tube by using a Geiges pair $(\tilde{\alpha}_+, \tilde{\alpha}_-)$, which is a special type of a Liouville pair satisfying the condition that

$$\tilde{\alpha}_+ \wedge d\tilde{\alpha}_+^k \wedge d\tilde{\alpha}_-^{n-k-1} = \tilde{\alpha}_- \wedge d\tilde{\alpha}_-^k \wedge d\tilde{\alpha}_+^{n-k-1} = 0$$

for all $0 \leq k \leq n-1$. Massot, Niederkrüger, and Wendl in [11] generalized the definition as above by using a Liouville pair. Moreover they constructed the example of the $(2n+1)$-dimensional Lutz-Mori tube by proving the following proposition. Let $N^{2n-1} = \partial W(1)/G(H, V)$ and $\tilde{\alpha}_\pm$ be the 1-forms on N^{2n-1} induced by the 1-forms $\alpha_\pm$ on $\partial W(1)$, where

$$\alpha_\pm = \pm\frac{dx_1}{y_1} + \cdots + \frac{dx_n}{y_n}.$$

PROPOSITION 5.4 (The section 7 of [11]). *The pair $(\tilde{\alpha}_+, \tilde{\alpha}_-)$ is a Liouville pair on N^{2n-1}. In particular, the 1-form*

$$\gamma = \frac{1-\cos s}{2}\tilde{\alpha}_+ + \frac{1+\cos s}{2}\tilde{\alpha}_- - s\sin s d\theta$$

defines the $(2n+1)$-dimensional Lutz-Mori tube $(N^{2n-1} \times D^2, \ker\gamma)$.

They also proved that the Lutz-Mori tube contains a bordered Legendrian open book (bLob), a generalization of a plastikstufe ([16]), which prevents the weak symplectic filling.

EXAMPLE 5.5. In the case where $N^1 = S^1$ with a coordinate z, $(\tilde{\alpha}_+, \tilde{\alpha}_-) = (dz, -dz)$ is a Liouville pair on S^1. The contact form

$$\gamma = -\cos s dz - s\sin s d\theta$$

defines the usual Lutz tube structure on the solid torus $S^1 \times D^2$.

EXAMPLE 5.6. In the case where $N^3 = T_A$, $(\tilde{\alpha}_+, \tilde{\alpha}_-) = (\beta_+ + \beta_-, \beta_+ - \beta_-)$ is a Liouville pair on T_A. The contact form

$$\gamma = \beta_+ - \cos s \cdot \beta_- - s\sin s \cdot d\theta$$

defines the 5-dimensional Lutz-Mori tube $(T_A \times D^2, \ker\gamma)$. This is the original example which Mori constructed in [14].

REMARK 5.7. The usual Lutz twist can be performed in a 3-dimensional Darboux chart. The resultant contact structure is overtwisted, thus it is not weakly fillable. If A is conjugate to $\begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix}$ in $SL(2; \mathbf{Z})$ for some p, q, r such that $p^{-1} + q^{-1} + r^{-1} < 1$, then the Lutz-Mori twist associated to T_A can be performed in a 5-dimensional Darboux chart, because the link of T_{pqr} singularity is a contact submanifold of (S^5, ξ_0) contactomorphic to $(T_A, \ker(\beta_+ + \beta_-))$. If we perform the Lutz-Mori twist along the link, the resultant contact structure on S^5 is not weakly fillable. Thus it is not contactomorphic to the standard one.

Next we show an example of the 7-dimensional Lutz-Mori tube whose core is the link of a 3-dimensional Hilbert modular cusp.

EXAMPLE 5.8. Let $\zeta = \exp\frac{2\pi i}{7}$ and $\rho_1 = \zeta + \zeta^{-1}$. The minimal polynomial of $\rho_1 = 2\cos\frac{2\pi}{7}$ is $x^3 + x^2 - 2x - 1 = 0$. The other solutions are $\rho_2 = \zeta^2 + \zeta^{-2} = 2\cos\frac{4\pi}{7}$ and $\rho_3 = \zeta^4 + \zeta^{-4} = 2\cos\frac{8\pi}{7}$. Thus ρ_1 is an irrational number of degree 3. Let $H = \mathbf{Z} \oplus \mathbf{Z}\rho_1 \oplus \mathbf{Z}\rho_2$. Then $r_1 = 2 + \rho_1$, $r_2 = 2 + \rho_2$ and $r_3 = 2 + \rho_3$ are totally positive units in the $\mathbf{Z}$-module H satisfying $r_1 r_2 r_3 = 1$. Thus the multiplicative group $V = \{r_1^n r_2^m \mid n, m \in \mathbf{Z}\}$ is a subgroup of U_H^+ of rank 2. The link of the Hilbert modular cusp $(\overline{\mathbf{H}^3/G(H,V)}, \infty)$ is a T^3 bundle over T^2 with monodromies A_1 and $A_2 \in SL(3;\mathbf{Z})$, which are representation matrices of the actions of r_1 and r_2 to the lattice H. Let us determine the matrices. We fix a basis of the lattice $H \subset \mathbf{R}^3$ as $u_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $u_2 = \begin{pmatrix}\rho_1\\\rho_2\\\rho_3\end{pmatrix}$, and $u_3 = \begin{pmatrix}\rho_2\\\rho_3\\\rho_1\end{pmatrix}$. Since

$$r_1 \cdot u_1 = \begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix} = \begin{pmatrix}2+\rho_1\\2+\rho_2\\2+\rho_3\end{pmatrix} = 2u_1 + u_2\,,$$

$$r_1 \cdot u_2 = \begin{pmatrix}r_1\rho_1\\r_2\rho_2\\r_3\rho_3\end{pmatrix} = \begin{pmatrix}2\rho_1+\rho_1^2\\2\rho_2+\rho_2^2\\2\rho_3+\rho_3^2\end{pmatrix} = 2u_1 + 2u_2 + u_3\,,$$

$$r_1 \cdot u_3 = \begin{pmatrix}r_1\rho_2\\r_2\rho_3\\r_3\rho_1\end{pmatrix} = \begin{pmatrix}\rho_1^2-3\\\rho_2^2-3\\\rho_3^2-3\end{pmatrix} = -u_1 + u_3\,,$$

we have $A_1 = \begin{pmatrix}2&2&-1\\1&2&0\\0&1&1\end{pmatrix}$. Similarly, $A_2 = \begin{pmatrix}2&-1&1\\0&2&-1\\1&-1&1\end{pmatrix}$. They have common eigen vectors $v_1 = \rho_1 u_1 + u_2 - \rho_2 u_3$, $v_2 = \rho_2 u_1 + u_2 - \rho_3 u_3$ and $v_3 = \rho_3 u_1 + u_2 - \rho_1 u_3$. Indeed, they satisfy that

$$A_1 v_1 = r_1 v_1\,, \quad A_1 v_2 = r_2 v_2\,, \quad A_1 v_3 = r_3 v_3\,,$$
$$A_2 v_1 = r_2 v_1\,, \quad A_2 v_2 = r_3 v_2\,, \quad A_2 v_3 = r_1 v_3\,.$$

Let T_{A_1,A_2} be the T^3 bundle over T^2 with monodromy A_1 and A_2. Then the 1-forms

$$\beta_1 = r_1^{-s} r_2^{-t} dx \wedge dy \wedge dz(\cdot, v_2, v_3)\,,$$
$$\beta_2 = r_2^{-s} r_3^{-t} dx \wedge dy \wedge dz(v_1, \cdot, v_3)\,,$$
$$\beta_3 = r_3^{-s} r_1^{-t} dx \wedge dy \wedge dz(v_1, v_2, \cdot)$$

define foliations on T_{A_1,A_2}, where (x, y, z) is a coordinate on T^3 and (s, t) is a coordinate on T^2. The pair $(\beta_1 + \beta_2 + \beta_3, -\beta_1 + \beta_2 + \beta_3)$ is a Liouville pair on T_{A_1,A_2}. Therefore the

1-form

$$\gamma = \beta_2 + \beta_3 - \cos s \cdot \beta_1 - s \sin s d\theta$$

defines the 7-dimensional Lutz tube $(T_{A_1,A_2} \times D^2, \ker\gamma)$.

Thomas and Vasquez gave the minimal resolutions of some 3-dimensional Hilbert modular cusps ([18]). Their examples contain the cusp of Example 5.8. The resolutions are toric manifolds and the weighted dual graph can be seen as a triangulation of the 2-torus. It is known that the 3-dimensional Hilbert modular cusps are not Gorenstein, thus the singularities cannot be embedded in $\mathbf{C}^4$. However, we can ask the following question.

PROBLEM 5.9. *Is it possible to embed the link of a* 3*-dimensional Hilbert modular cusp in* (S^7, ξ_0) *as a contact submanifold?*

If it is possible, the associated 7-dimensional Lutz-Mori twist can be performed in a Darboux chart. However, it may be a hard problem. For, there are not so many known results about the existence of codimension two contact embeddings in a Darboux chart. Up to now, the only found obstructions are the Chern classes of the contact manifold [7], which vanish for $(T_{A_1A_2}, \ker(\beta_1 + \beta_2 + \beta_3))$. What is worse is that we do not even know the existence of an embedding of T_{A_1,A_2} in S^7 as a submanifold. The author suspect that open book decompositions might be useful for solving these difficulties. The supporting open book decomposition of the 5-dimensional contact manifold $(\partial W(1)/G(H,V), \ker\tilde{\alpha})$ is given by a cusp form, which is a holomorphic function on $\overline{\mathbf{H}^3/G(H,V)}$ vanishing at ∞. Interpreting the information of the minimal resolution given by Thomas and Vasquez, we can see the structure of the open book, in principle. It will be the first step of solving the problem.

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