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A Characterization of a Multiple Weights Class

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Abstract. Moen introduced a multiple weights class. Moen proved that the multiple weights condition of vector of weights implies each weight function satisfies a certain A_p condition. In 2010, Chen and Xue improved sufficency under an unnatural condition. However we can remove this condition and prove necessity. We also give a typical example of the multiple weights class.

1. Main result

In the present paper, we obtain the following result.

THEOREM 1. *Let* $0 \le \alpha < mn$, $1 < p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Suppose that we are given a collection of m weights*

$$\begin{cases} (w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbf{R}^n)\,, \\ w_j^{-p_j'} \in A_{1+p_j'\left(\frac{1}{q}+m-\frac{1}{p}-\frac{1}{p_j'}\right)}(\mathbf{R}^n) \qquad (j = 1, \dots, m)\,. \end{cases}$$

Then the multisublinear fractional maximal function $\mathcal{M}_{\alpha,m}$, *which is given by*

$$\mathcal{M}_{\alpha,m}\left(\vec{f}\right)(x) := \sup_{B \ni x} |B|^{\frac{\alpha}{n}} \prod_{j=1}^{m} \frac{1}{|B|} \int_B \left| f_j(y_j) \right| dy_j$$

for $\vec{f} := (f_1, \dots, f_m)$, *satisfies the following inequality*:

$$\left\| \mathcal{M}_{\alpha,m}\left(\vec{f}\right)(w_1 \cdots w_m) \right\|_{L^q(\mathbf{R}^n)} \le C \prod_{j=1}^{m} \left\| f_j w_j \right\|_{L^{p_j}(\mathbf{R}^n)}$$

for some constant $C > 0$ *independent of the collection of positive measurable functions* $\vec{f} = (f_1, \dots, f_m)$. *Here and below* B *in* sup *runs over all balls in* $\mathbf{R}^n$.

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REMARK 1. When $\alpha = 0$, Lerner et al [4] proved Theorem 1 (p. 1233, Theorem 3.7). The remaining part of this paper is structured as follows: In Section 2, we give related definitions necessary later. In Section 3, we recall recent key results. We explain in Section 4 what is the advantage of our new results, where we transform Theorem 1 into Theorem 2 which we actually prove here. Section 4 is devoted also to the proof of Theorem 1 and Theorem 2 (to follow). Finally we give examples and corollaries in Section 5.

2. Some notations, terminologies and definitions

We shall below describe the terminologies. By a weight we mean a nonnegative locally integrable function w on $\mathbf{R}^n$. We recall the class $A_p(\mathbf{R}^n)$ with $1 \le p < \infty$.

DEFINITION 2.1 (cf. [2, 3, 5]).
1. When $1 < p < \infty$, one says that a weight w is in the weight class $A_p(\mathbf{R}^n)$ if

$$[w]_{A_p} := \sup_{B \subset \mathbf{R}^n:\text{ball}} \left(\frac{1}{|B|} \int_B w(x)dx \right) \cdot \left(\frac{1}{|B|} \int_B w(x)^{\frac{1}{1-p}} dx \right)^{p-1} < \infty \,,$$

where $|B|$ is Lebesgue measure of B.

2. When $p = 1$, one says that a weight w is in the weight class $A_1(\mathbf{R}^n)$ if there exists $C > 0$, for every ball $B \subset \mathbf{R}^n$ such that

$$\frac{1}{|B|} \int_B w(x)dx \le C \operatorname*{ess.inf}_{x \in B} w(x) \,. \tag{1}$$

Denote by $[w]_{A_1}$ by the infimum constant appearing in (1).

REMARK 2. By a long standing research, the theory of A_p weights has been well investigated. For example, the reverse Hölder inequality is a very important property of A_p weights (see [2, 3, 5] for the reverse Hölder inequality). Hence we are tempted to be oriented to a problem whether a given weight condition is characterized in terms of A_p weights.

Next, we define the multiple weights classes $A_{\vec{P}}(\mathbf{R}^n)$ and $A_{\vec{P},q}(\mathbf{R}^n)$. For each $1 \le p \le \infty$, p' will denote the dual exponent of p, i.e., $p' = p/(p-1)$ with the usual modifications $1' = \infty$ and $\infty' = 1$.

DEFINITION 2.2.
1 ([4]). Let $1 < p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{P} := (p_1, \ldots, p_m)$. One says that a vector of weights $\vec{w} = (w_1, \ldots, w_m)$ is in the multiple weights class $A_{\vec{P}}(\mathbf{R}^n)$ if

$$\left[\vec{w}\right]_{A_{\vec{P}}} := \sup_{B \subset \mathbf{R}^n} \left(\frac{1}{|B|} \int_B v_{\vec{w}}(x)dx \right)^{\frac{1}{p}} \prod_{j=1}^m \left(\frac{1}{|B|} \int_B w_j(y_j)^{\frac{1}{1-p_j}} dy_j \right)^{\frac{1}{p'_j}} < \infty \,,$$

where $v_{\vec{w}} := w_1^{\frac{p}{p_1}} \cdots w_m^{\frac{p}{p_m}}$.

2 ([6]). Let $1 \le p_1, \ldots, p_m \le \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\vec{P} := (p_1, \ldots, p_m)$ and $0 < q \le \infty$. One says that a vector of weights $\vec{w}$ is in the multiple weights class $A_{\vec{P},q}(\mathbf{R}^n)$ if

$$[\vec{w}]_{A_{\vec{P},q}} := \sup_{B \subset \mathbf{R}^n} \left(\frac{1}{|B|} \int_B \prod_{j=1}^m w_j(x)^q dx \right)^{\frac{1}{q}} \prod_{j=1}^m \left(\frac{1}{|B|} \int_B w_j(y_j)^{-p'_j} dy_j \right)^{\frac{1}{p'_j}} < \infty .$$

When $q = \infty$, $(\frac{1}{|B|} \int_B (w_1 \cdots w_m)(x)^q dx)^{\frac{1}{q}}$ is understood as $\operatorname{ess.sup}_{x \in B}(w_1 \cdots w_m)(x)$. Moreover when $p_i = 1$, $(\frac{1}{|B|} \int_B w_i(y_i)^{-p'_i} dy_i)^{\frac{1}{p'_i}}$ is understood as $\left(\operatorname{ess.inf}_{y_i \in B} w_i(y_i)\right)^{-1}$. When $m = 1$, one abbreviates $A_{\vec{P},q}(\mathbf{R}^n)$ to $A_{p,q}(\mathbf{R}^n)$.

REMARK 3.

1. In [6], the restriction $\frac{1}{m} < q < \infty$ is imposed. However, in the definition we are still interested in the case $0 < q \le \frac{1}{m}$ and $q = \infty$. Consequently, we shall incorporate this case.

2. If $0 < q_1 \le q_2$, then we have

$$A_{\vec{P},\infty}(\mathbf{R}^n) \subset A_{\vec{P},q_2}(\mathbf{R}^n) \subset A_{\vec{P},q_1}(\mathbf{R}^n)$$

by virtue of the Hölder inequality.

3. Recent results

Lerner et al [4] also proved that the multiple weights class $A_{\vec{P}}(\mathbf{R}^n)$ is characterized in terms of A_p classes.

THEOREM A. *Let $\vec{w}$ be a vector weight, $1 < p_1, \ldots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then $\vec{w} \in A_{\vec{P}}(\mathbf{R}^n)$ if and only if*

$$\begin{cases} v_{\vec{w}} \in A_{mp}(\mathbf{R}^n), \\ w_j^{\frac{1}{1-p_j}} \in A_{mp'_j}(\mathbf{R}^n) \qquad (j = 1, \ldots, m). \end{cases}$$

Precisely, we have the following inequalities; for $j = 1, \ldots, m$,

$$\left[w_j^{\frac{1}{1-p_j}} \right]_{A_{mp'_j}}^{\frac{1}{p'_j}}, [v_{\vec{w}}]_{A_{mp}}^{\frac{1}{p}} \le [\vec{w}]_{A_{\vec{P}}} \le [v_{\vec{w}}]_{A_{mp}}^{\frac{1}{p}} \prod_{j=1}^m \left[w_j^{\frac{1}{1-p_j}} \right]_{A_{mp'_j}}^{\frac{1}{p'_j}} .$$

On the other hand, Moen proved the following theorem:

THEOREM B ([6]). *Let $0 \le \alpha < mn$, $1 < p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that we are given a collection of m weights $\vec{w} \in A_{\vec{P},q}(\mathbf{R}^n)$. Then the multisublinear fractional maximal function $\mathcal{M}_{\alpha,m}$ satisfies the following inequality:*

$$\left\| \mathcal{M}_{\alpha,m}\left(\vec{f}\right)(w_1 \cdots w_m) \right\|_{L^q(\mathbf{R}^n)} \le C \prod_{j=1}^m \| f_j w_j \|_{L^{p_j}(\mathbf{R}^n)}$$

for some constant $C > 0$ *independent of the collection of positive measurable functions* $\vec{f} = (f_1, \ldots, f_m)$.

To prove Theorem 1, we use Theorem B. On the other hand, it seems that the multiple weights class $A_{\vec{P},q}(\mathbf{R}^n)$ is not completely characterized in terms of A_p classes. So we will characterize $A_{\vec{P},q}(\mathbf{R}^n)$ in terms of $A_p(\mathbf{R}^n)$ classes.

4. The characterization

In 2010, Chen and Xue [1] proved the following theorem:

THEOREM C. *Let* $0 < \alpha < mn$, $1 \le p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Suppose that* $\vec{w} \in A_{\vec{P},q}(\mathbf{R}^n)$ *and* $\frac{\alpha}{n} < m - 2 + \frac{1}{p_i} + \frac{1}{p_j}$ *for any* $1 \le i, j \le m$, *then*

$$\begin{cases} (w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbf{R}^n)\,, \\ w_i^{-p_i'} \in A_{1+p_i'\left(\frac{1}{q}+m-\frac{1}{p}-\frac{1}{p_i'}\right)}(\mathbf{R}^n) \qquad (i = 1, \ldots, m)\,. \end{cases}$$

When $p_i = 1$, *we regard the condition* $w_i^{-p_i'} \in A_{1+p_i'\cdot(\frac{1}{q}+m-\frac{1}{p}-\frac{1}{p_i'})}(\mathbf{R}^n)$ *as* $w_i^{(\frac{1}{q}+m-\frac{1}{p})^{-1}} \in A_1(\mathbf{R}^n)$.

In the following we always assume that $1 \le p_1, \ldots, p_m \le \infty$ and $0 < q \le \infty$. We denote that $\frac{1}{p} := \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and

$$s_j := \frac{1}{q} + m - \frac{1}{p} - \frac{1}{p_j'} \quad (j = 1, \ldots, m). \tag{2}$$

REMARK 4.
1. In [1] the authors postulated a superfluous assumption; actually $\frac{\alpha}{n} < m - 2 + \frac{1}{p_i} + \frac{1}{p_j}$ for all $i, j = 1, \ldots, m$ is unnecessary. What counts about Theorem 2 below is the fact that $\frac{\alpha}{n} < m - 2 + \frac{1}{p_i} + \frac{1}{p_j}$ for all $i, j = 1, \ldots, m$ is unnecessary and that we can incorporate the case when $\alpha = 0$.

2. When $1 < p < \infty$, we know $w \in A_p(\mathbf{R}^n)$ if and only if $w^{\frac{1}{1-p}} \in A_{p'}(\mathbf{R}^n)$ (cf. [2, 3, 5]). Hence if $1 < p_j < \infty$, then we have $w_j^{-p_j'} \in A_{1+p_j's_j}(\mathbf{R}^n)$ if and only if $w_j^{s_j^{-1}} \in A_{1+\frac{1}{p_j's_j}}(\mathbf{R}^n)$. Therefore, it is natural to regard the case of $p_i = 1$ such as stated above.

We will completely characterize the multiple weights class $A_{\vec{P},q}(\mathbf{R}^n)$ in terms of A_p classes (cf. [2, 3, 5]).

THEOREM 2. *Let* $1 \le p_1, \dots, p_m \le \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $0 < q \le \infty$. *A vector* $\vec{w}$ *of weights satisfies* $\vec{w} \in A_{\vec{P},q}(\mathbf{R}^n)$ *if and only if*

$$\begin{cases} (w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbf{R}^n), \\ w_j^{-p'_j} \in A_{1+p'_j \cdot s_j}(\mathbf{R}^n) \qquad (j = 1, \dots, m), \end{cases}$$

where $s_j = \frac{1}{q} + m - \frac{1}{p} - \frac{1}{p'_j}$ $(j = 1, \dots, m)$. *An analogy is available for* $q = \infty$, *if we regard the condition* $(w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbf{R}^n)$ *as the condition* $(w_1 \cdots w_m)^{-\frac{1}{m-\frac{1}{p}}} \in A_1(\mathbf{R}^n)$. *Also a tacit understanding as Theorem C is made when* $p_i = 1$.

REMARK 5. When $0 < q < \infty$, we have $(w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbf{R}^n)$ if and only if $(w_1 \cdots w_m)^{-\frac{1}{m-\frac{1}{p}}} \in A_{1+\frac{1}{q}\left(m-\frac{1}{p}\right)^{-1}}(\mathbf{R}^n)$, which justifies the above understanding that $(w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbf{R}^n)$.

Reexaming the proof of Theorem 2, we obtain the following precise inequalities:

COROLLARY 2.1. *Under the condition of Theorem* 2, *one has the following inequalities: for* $1 < p_1, \dots, p_m < \infty$, $0 < q < \infty$, *one has*

$$\left[(w_1 \cdots w_m)^q\right]_{A_{1+q\left(m-\frac{1}{p}\right)}}^{\frac{1}{q}}, \left[w_j^{-p'_j}\right]_{A_{1+p'_j \cdot s_j}}^{\frac{1}{p'_j}} \le [\vec{w}]_{A_{\vec{P},q}}, \tag{3}$$

for $j = 1, \dots, m$. *Conversely, one has*

$$[\vec{w}]_{A_{\vec{P},q}} \le \left[(w_1 \cdots w_m)^q\right]_{A_{1+q\left(m-\frac{1}{p}\right)}}^{\frac{1}{q}} \cdot \prod_{j=1}^{m} \left[w_j^{-p'_j}\right]_{A_{1+p'_j \cdot s_j}}^{\frac{1}{p'_j}}. \tag{4}$$

In the remaining cases in Theorem 2, *modify inequalities* (3) *and* (4) *following Remarks* 4 *and* 5.

By Hölder's inequality, we will prove Theorem 2:

PROOF OF THEOREM 2. When $m = 1$, Theorem 2 obviously holds and well-known [5]. So we assume that $m > 1$. Moreover, we assume that $1 < p_1, \dots, p_m < \infty$ and that $0 < q < \infty$, the remaining cases being dealt similarly. Let us first assume that $\vec{w} \in A_{\vec{P},q}(\mathbf{R}^n)$. Chen and Xue [1] proved that $(w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbf{R}^n)$. The same argument does work under our weaker assumption and we omit the proof. So, we prove the condition

$$w_j^{-p'_j} \in A_{1+p'_j \cdot s_j}(\mathbf{R}^n).$$

We put indices

$$q_i := \begin{cases} q \cdot s_j & (i = j)\,, \\ p_i' \cdot s_j & (i \neq j) \end{cases}$$

for $1 \le i \le m$, where $s_j = \frac{1}{q} + m - \frac{1}{p} - \frac{1}{p_j'}$ $(j = 1, \ldots, m)$. By (2), a calculation shows that $\sum_{i=1}^m \frac{1}{q_i} = 1$. Let B be a ball chosen arbitrarily. By Hölder's inequality

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B w_j(x)^{s_j{}^{-1}}dx\right)^{s_j} \\
&= \left(\frac{1}{|B|}\int_B \prod_{i=1}^m w_i(x)^{s_j^{-1}} \cdot \prod_{i\neq j} w_i(x)^{-s_j^{-1}} dx\right)^{s_j} \\
&\le \left(\frac{1}{|B|}\int_B (w_1 \cdots w_m)(x)^q dx\right)^{\frac{1}{q}} \prod_{i \neq j}\left(\frac{1}{|B|}\int_B w_i(x)^{-p_i'}dx\right)^{\frac{1}{p_i'}} .
\end{aligned}$$

Therefore we obtain the following inequality:

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B w_j(x)^{-p_j'}dx\right)\left(\frac{1}{|B|}\int_B w_j(x)^{s_j^{-1}}dx\right)^{p_j'\cdot s_j} \\
&\le \left(\frac{1}{|B|}\int_B w_j(x)^{-p_j'}dx\right) \\
&\quad \left[\left(\frac{1}{|B|}\int_B (w_1 \cdots w_m)(x)^q dx\right)^{\frac{1}{q}} \prod_{i \neq j}\left(\frac{1}{|B|}\int_B w_i(x)^{-p_i'}dx\right)^{\frac{1}{p_i'}}\right]^{p_j'} \\
&= \left[\left(\frac{1}{|B|}\int_B (w_1 \cdots w_m)(x)^q dx\right)^{\frac{1}{q}} \prod_{i=1}^{m}\left(\frac{1}{|B|}\int_B w_i(x)^{-p_i'}dx\right)^{\frac{1}{p_i'}}\right]^{p_j'} \\
&\le [\vec{w}]_{A_{\vec{P},q}}^{p_j'} .
\end{aligned}$$

Hence we obtain the desired inequality (3). In the remaining cases, it is easy to obtain the desired results with some minor modifications.

Conversely, we prove necessity. We assume that

$$\begin{cases} (w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbf{R}^n)\,, \\ w_j^{-p_j'} \in A_{1+p_j'\cdot s_j}(\mathbf{R}^n) & (j = 1, \ldots, m)\,. \end{cases}$$

We check that $\vec{w}$ satisfies the inequality (4). Set indices

$$\begin{cases} l := \left[q \cdot \sum_{k=1}^{m} s_k\right]^{-1} > 0\,, \\ l_j := \frac{1}{s_j} \cdot \sum_{k=1}^{m} s_k > 0 \qquad (j = 1, \dots, m)\,. \end{cases} \tag{5}$$

These indices satisfy the following conditions; $\sum_{j=1}^{m} \frac{1}{l_j} = 1$ and $\frac{1}{ql \cdot l_j} = s_j$ $(j = 1, \dots, m)$. By the definition of $A_{1+q\left(m-\frac{1}{p}\right)}(\mathbf{R}^n)$ and $A_{1+p_j' \cdot s_j}(\mathbf{R}^n)$ for $j = 1, \dots, m$ and Hölder's inequality for $l_1, \dots, l_m$, we obtain

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \prod_{j=1}^{m} w_j(x)^q dx\right)^{\frac{1}{q}} \prod_{j=1}^{m} \left(\frac{1}{|B|}\int_B w_j(x)^{-p_j'} dx\right)^{\frac{1}{p_j'}} \\
&\le \left[(w_1 \cdots w_m)^q\right]_{A_{1+q\left(m-\frac{1}{p}\right)}}^{\frac{1}{q}} \cdot \prod_{j=1}^{m} \left[w_j^{-p_j'}\right]_{A_{1+p_j' \cdot s_j}}^{\frac{1}{p_j'}} \\
&\quad \left(\frac{1}{|B|}\int_B \prod_{j=1}^{m} w_j(x)^{-\frac{1}{m-\frac{1}{p}}} dx\right)^{-\left(m-\frac{1}{p}\right)} \prod_{j=1}^{m} \left(\frac{1}{|B|}\int_B w_j(x)^{ql \cdot l_j} dx\right)^{-\frac{1}{ql \cdot l_j}} \\
&\le \left[(w_1 \cdots w_m)^q\right]_{A_{1+q\left(m-\frac{1}{p}\right)}}^{\frac{1}{q}} \cdot \prod_{j=1}^{m} \left[w_j^{-p_j'}\right]_{A_{1+p_j' \cdot s_j}}^{\frac{1}{p_j'}} \\
&\quad \left(\frac{1}{|B|}\int_B \prod_{j=1}^{m} w_j(x)^{-\frac{1}{m-\frac{1}{p}}} dx\right)^{-\left(m-\frac{1}{p}\right)} \left(\frac{1}{|B|}\int_B \prod_{j=1}^{m} w_j(x)^{ql} dx\right)^{-\frac{1}{ql}}.
\end{aligned}$$

On the other hand, applying Hölder's inequality for $1 + ql\left(m - \frac{1}{p}\right) > 1$, we estimate 1 from above:

$$\begin{aligned}
1 &= \left(\frac{1}{|B|}\int_B \prod_{j=1}^{m} w_j(x)^{\frac{ql}{1+ql\left(m-\frac{1}{p}\right)}} \cdot \prod_{j=1}^{m} w_j(x)^{-\frac{ql}{1+ql\left(m-\frac{1}{p}\right)}} dx\right)^{\frac{1}{ql}+m-\frac{1}{p}} \\
&\le \left(\frac{1}{|B|}\int_B \prod_{j=1}^{m} w_j(x)^{ql} dx\right)^{\frac{1}{ql}} \cdot \left(\frac{1}{|B|}\int_B \prod_{j=1}^{m} w_j(x)^{-\frac{1}{m-\frac{1}{p}}} dx\right)^{m-\frac{1}{p}}.
\end{aligned}$$

Consequently, we have the desired inequality (4). Therefore we obtain $\vec{w} \in A_{\vec{P},q}(\mathbf{R}^n)$.

In the remaining cases, as we have announced in the beginning of the proof of this theorem, it is easy to obtain the desired results with some minor modifications. For the sake of convenience we provide a proof when $1 < p_1, \dots, p_m < q = \infty$. Firstly we assume that

$$\begin{cases} (w_1 \cdots w_m)^{-\frac{1}{m-\frac{1}{p}}} \in A_1(\mathbf{R}^n)\,, \\ w_j^{-p_j'} \in A_{1+p_j' \cdot s_j}(\mathbf{R}^n) \qquad (j = 1, \dots, m)\,. \end{cases}$$

We will check $\vec{w} \in A_{\vec{P},\infty}(\mathbf{R}^n)$. By the same method (i.e., by Hölder's inequality for $l_1 \dots, l_m$), we obtain

$$\begin{aligned}
&\operatorname*{ess.sup}_{x\in B}(w_1\cdots w_m)(x)\prod_{j=1}^m\left(\frac{1}{|B|}\int_B w_j(x)^{-p_j'}dx\right)^{\frac{1}{p_j'}}\\
&\le\left[(w_1\cdots w_m)^{-\frac{1}{m-\frac{1}{p}}}\right]_{A_1}^{m-\frac{1}{p}}\prod_{j=1}^m\left[w_j^{-p_j'}\right]_{A_{1+p_j'\cdot s_j}}^{\frac{1}{p_j'}}\\
&\quad\left(\frac{1}{|B|}\int_B(w_1\cdots w_m)(x)^{-\left(m-\frac{1}{p}\right)^{-1}}dx\right)^{\frac{1}{p}-m}\\
&\quad\left(\frac{1}{|B|}\int_B\prod_{j=1}^m w_j(x)^{\frac{1}{m-1}\cdot\left(m-\frac{1}{p}\right)^{-1}}dx\right)^{-(m-1)\left(m-\frac{1}{p}\right)}.
\end{aligned}$$

On the other hand, by Hölder's inequality for $m > 1$, we obtain

$$\begin{aligned}
1&=\left(\frac{1}{|B|}\int_B(w_1\cdots w_m)(x)^{-\frac{1}{m}\left(m-\frac{1}{p}\right)^{-1}}\cdot(w_1\cdots w_m)(x)^{\frac{1}{m}\left(m-\frac{1}{p}\right)^{-1}}dx\right)^{m\left(m-\frac{1}{p}\right)}\\
&\le\left(\frac{1}{|B|}\int_B(w_1\cdots w_m)(x)^{-\left(m-\frac{1}{p}\right)^{-1}}dx\right)^{m-\frac{1}{p}}\\
&\quad\left(\frac{1}{|B|}\int_B(w_1\cdots w_m)(x)^{\frac{1}{m-1}\cdot\left(m-\frac{1}{p}\right)^{-1}}dx\right)^{(m-1)\cdot\left(m-\frac{1}{p}\right)}.
\end{aligned}$$

Therefore we have the desired inequality. □

5. Some remarks

Theorem 2 and a typical example of A_p weights give us a typical example of the multiple weights class $A_{\vec{P},q}(\mathbf{R}^n)$. Recall that $|x|^\theta \in A_p(\mathbf{R}^n)$ if and only if $-n < \theta < n(p-1)$. For details we refer to [p. 141, 2].

EXAMPLE 2.1. Let $\theta_1,\dots,\theta_m \in \mathbf{R}$, $1 < p_1,\dots,p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $0 < q < \infty$. Then $\left(|x|^{\theta_1},\dots,|x|^{\theta_m}\right) \in A_{\vec{P},q}(\mathbf{R}^n)$ if and only if

$$\begin{cases}-\frac{n}{q} < \theta_1 + \cdots + \theta_m\,,\\ -n\left(\frac{1}{q} + m - \frac{1}{p} - \frac{1}{p_j'}\right) < \theta_j < \frac{n}{p_j'} \quad (j = 1,\dots,m)\,.\end{cases}$$

Theorem 2 and the reverse Hölder inequality give us the following two corollaries. The reverse Hölder inequality can be found in [Theorem 7.4, 2].

COROLLARY 2.2. *Under the condition of Theorem* 2, *if* $\vec{w} \in A_{\vec{P},q}(\mathbf{R}^n)$ *then there exists* $\varepsilon > 0$ *such that*

$$\left(w_1^{1+\varepsilon}, \dots, w_m^{1+\varepsilon}\right) \in A_{\vec{P},q}(\mathbf{R}^n).$$

COROLLARY 2.3. *Let* $0 \le \alpha < mn$ *and* $1 \le p_1, \dots, p_m \le \infty$. *Moreover suppose that* $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n} \ge 0$. *If* $\vec{w} \in A_{\vec{P},q}(\mathbf{R}^n)$ *then there exists* $\tilde{p}_j < p_j$ *for* $j = 1, \dots, m$ *such that*

$$\vec{w} \in A_{\tilde{\vec{P}},\tilde{q}}(\mathbf{R}^n),$$

where $\frac{1}{\tilde{q}} := \frac{1}{\tilde{p}_1} + \cdots + \frac{1}{\tilde{p}_m} - \frac{\alpha}{n} \left(> \frac{1}{q}\right)$ *and* $\tilde{\vec{P}} := (\tilde{p}_1, \dots, \tilde{p}_m)$.

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