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A Simple Proof of the Functional Relation for the Lerch Type Tornheim Double Zeta Function

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Abstract. In this paper, we give a simple proof of the functional relation for the Lerch type Tornheim double zeta function. By using it, we obtain simple proofs of some explicit evaluation formulas for double L-values.

1. Introduction and main results

We define the Lerch type Tornheim double zeta function by

$$T(s,t,u\,;x,y) := \lim_{R\to\infty} \sum_{m,n=1}^{m+n=R} \frac{e^{2\pi imx}e^{2\pi iny}}{m^s n^t (m+n)^u}\,, \tag{1.1}$$

where $x, y \in \mathbf{R}$, $\Re(s+t) > 1$, $\Re(t+u) > 1$ and $\Re(s+t+u) > 2$. By [6, Theorem 2.1], the function $T(s,t,u\,;x,y)$ can be continued meromorphically to $\mathbf{C}^3$. More precisely, all of its singularities are located on the subsets of $\mathbf{C}^3$ defined by the following equations;

$$\begin{aligned}
t = 1-k\,, \quad & k \in \mathbf{Z}_{\geq 0} \quad \text{if } x \not\equiv 1\,, \quad y \equiv 1 \mod 1\,,\\
s = 1-k\,, \quad & k \in \mathbf{Z}_{\geq 0} \quad \text{if } x \equiv 1\,, \quad y \not\equiv 1 \mod 1\,,\\
&\text{no singularity} \quad \text{if } x \not\equiv 1\,, \quad y \not\equiv 1 \mod 1\,.
\end{aligned}$$

We write $T(s,t,u) := T(s,t,u\,;1,1)$ and call this function the Tornheim double zeta function. The values $T(a,b,c)$ for $a,b,c \in \mathbf{N}$ were investigated by Tornheim in 1950 and later by Mordell in 1958, and some explicit formulas for them were obtained. Subbarao and Sitaramachandrarao, Huard, Williams and Zhang, and Tsumura studied explicit formulas of $T(a,b,c)$ for $a,b,c \in \mathbf{N}$. The value $T(0,a,b\,;x,y)$ and their multiple sum versions have been already defined by Arakawa and Kaneko [1], for the case $x, y \in \mathbf{Q}$, as special cases of their multiple L-values.

As a three-variable function, Matsumoto continued $T(s,t,u)$ meromorphically to the whole $\mathbf{C}^3$ plane in [3, Theorem 1]. Tsumura [9, Theorem 4.5], afterwards Nakamura [5,

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Theorem 1.2] found functional relations for the Tornheim double zeta function. Moreover, generalizations of these functional relations are proved by Matsumoto and Tsumura [4], and Nakamura [6].

In this paper, we show the following functional relation. This relation is essentially the same as [7, Theorem 3.1]. Therefore we can obtain all the results in [7] by this formula. Zhou gave a simple proof of [5, Theorem 1.2] in [11]. Recently, Li independently showed a proof similar to Zhou's one in [2]. By modifying their methods, we can prove the following theorem. Let $\zeta(s\,;x) := \sum_{n=1}^{\infty} e^{2\pi i n x} n^{-s}$.

THEOREM 1.1. *For $x, y \notin \mathbf{Z}$. $x \neq y$, $a, b \in \mathbf{N}$ and $s \in \mathbf{C}$, we have*

$$
\begin{aligned}
&T(a,b,s\,;x,y) + (-1)^b T(b,s,a\,;x-y,x) + (-1)^a T(s,a,b\,;y,y-x) \\
&= \sum_{j=1}^{a} \binom{a+b-j-1}{a-j} \zeta(a+b+s-j\,;y)\big(\zeta(j\,;x-y) + (-1)^j \zeta(j\,;y-x)\big) \\
&\quad + \sum_{j=1}^{b} \binom{a+b-j-1}{b-j} \zeta(a+b+s-j\,;x)\big(\zeta(j\,;y-x) + (-1)^j \zeta(j\,;x-y)\big) \\
&\quad - \binom{a+b-1}{a} \zeta(a+b+s\,;y) - \binom{a+b-1}{b} \zeta(a+b+s\,;x)\,.
\end{aligned}
\tag{1.2}
$$

Taking $x \to y$ in the above formula, we have [7, (3.1)] since

$$
\lim_{x \to y} \big(\zeta(a+b+s-1\,;y) - \zeta(a+b+s-1\,;x)\big)\big(\zeta(1\,;x-y) - \zeta(1\,;y-x)\big) = 0\,.
$$

The index of $T(0,a,b\,;x,y)$ is called admissible when $b \geq 2$ or $b = 1$ and $x \notin \mathbf{Z}$ since the series $T(0,a,b\,;x,y)$ converges (see [1, Proposition 1.1]). Define $K(a,b\,;x,y)$ by the right-hand side of (1.2) with $s = 0$. We obtain the following proposition by Theorem 1.1. It should be noted that $T(0,a,b\,;-y,x-y) = T(0,a,b\,;y,y-x)$ when $(x,y) = (1,1), (1,1/2), (1/2,1)$ or $(1/2,1/2)$. In these cases, the next proposition coincides with [7, Proposition 2.3] or [10, Proposition 1].

PROPOSITION 1.2. *For any admissible index, it holds that*

$$
\begin{aligned}
&T(0,a,b\,;-y,x-y) - (-1)^{a+b} T(0,a,b\,;y,y-x) \\
&\quad = (-1)^b \zeta(a\,;x)\zeta(b\,;y) - (-1)^b K(a,b\,;x,y) + \zeta(a\,;-y)\zeta(b\,;x) \\
&\qquad - \zeta(a+b\,;x-y)\,.
\end{aligned}
\tag{1.3}
$$

Let ϕ, χ and ψ be Dirichlet characters of conductor h, k, and q, respectively. For any admissible index, namely $b \geq 2$ or $b = 1$ and ψ is non-principal, we define $L(0,a,b\,;\phi,\chi,\psi)$

by

$$L(0, a, b\,; \phi, \chi, \psi) := \lim_{R\to\infty} \sum_{m,n=1}^{m+n=R} \frac{\phi(m)\chi(n)\psi(m+n)}{n^a(m+n)^b}\,. \tag{1.4}$$

Terhune [8] showed that if $\chi\psi(-1) = (-1)^{a+b+1}$ then $L(0, a, b\,; 1, \chi, \psi)$ can be expressed as a polynomial in the values of polylogarithms at certain roots of unity with coefficients in a cyclotomic field. In [7, Proposition 4.5], the author obtained explicit evaluation formulas of $L(0, a, b\,; 1, \chi, \psi)$ when $\chi\psi(-1) = (-1)^{a+b+1}$. The following proposition gives simpler ones. Denote the Gauss sum by $\tau(\chi) := \sum_{l=1}^{k-1} \chi(l)e^{2\pi i l/k}$.

PROPOSITION 1.3. *Let* $\phi\chi\psi(-1) = (-1)^{a+b+1}$. *For any admissible index, define* $2U(a, b\,; x, y)$ *by the right-hand side of* (1.3). *Then, we have*

$$\begin{aligned} &L(0, a, b\,; \phi, \chi, \psi) \\ &\quad = \frac{1}{\tau(\overline{\phi})\tau(\overline{\chi})\tau(\overline{\psi})} \sum_{j=1}^{h-1}\sum_{l=1}^{k-1}\sum_{r=1}^{q-1} \overline{\phi}(j)\overline{\chi}(l)\overline{\psi}(r)U(a, b\,; j/h + r/q, l/k + r/q)\,. \end{aligned} \tag{1.5}$$

2. Proof of results

PROOF OF THEOREM 1.1. Suppose $\Re(s) > 1$. We define $S(a, b, s\,; x, y)$ by

$$S(a, b, s\,; x, y) := T(a, b, s\,; x, y) + (-1)^b T(b, s, a\,; x-y, x) + (-1)^a T(s, a, b\,; y, y-x).$$

It is easy to see that $S(a, b, s\,; x, y) = S(a-1, b, s+1\,; x, y) + S(a, b-1, s+1\,; x, y)$ by $T(a, b, s\,; x, y) = T(a-1, b, s+1\,; x, y) + T(a, b-1, s+1\,; x, y)$. Hence we have

$$\begin{aligned} S(a, b, s\,; x, y) = &\sum_{j=1}^{a} \binom{a+b-j-1}{a-j} S(j, 0, a+b+s-j\,; x, y) \\ &+ \sum_{j=1}^{b} \binom{a+b-j-1}{b-j} S(0, j, a+b+s-j\,; x, y)\,. \end{aligned}$$

Now we consider the function $S(j, 0, a+b+s-j\,; x, y)$ in the formula above. By the definition of $S(a, b, s\,; x, y)$, it holds that

$$\begin{aligned} S(j, 0, a+b+s-j\,; x, y) &= T(j, 0, a+b+s-j\,; x, y) \\ &\quad + T(0, a+b+s-j, j\,; x-y, x) + (-1)^j T(a+b+s-j, j, 0\,; y, y-x) \\ &=: I_1 + I_2 + I_3\,. \end{aligned}$$

According to the harmonic product formula (see for example [1, p. 972, l. 13]), $I_1 + I_2$ coincides with $\zeta(a+b+s-j\,; y)\zeta(j\,; x-y) - \zeta(a+b+s\,; x)$. Obviously, we have

$I_3 = (-1)^j \zeta(a+b+s-j\,;y)\zeta(j\,;y-x)$. Hence we obtain

$$S(j,0,a+b+s-j\,;x,y) = \zeta(a+b+s-j\,;y)\big(\zeta(j\,;x-y) + (-1)^j\zeta(j\,;y-x)\big) - \zeta(a+b+s\,;x)\,.$$

By exchanging the parameters x and y, we have

$$S(0,j,a+b+s-j\,;x,y) = \zeta(a+b+s-j\,;x)\big(\zeta(j\,;y-x) + (-1)^j\zeta(j\,;x-y)\big) - \zeta(a+b+s\,;y)\,.$$

Therefore we obtain this theorem when $\Re(s) > 1$ by the well-known formula $\sum_{j=1}^{a}\binom{a+b-j-1}{a-j} = \binom{a+b-1}{b}$. Hence we obtain Theorem 1.1 by the analytic continuation. □

PROOF OF PROPOSITION 1.2. By putting $s=0$ in (1.2), we have

$$\zeta(a\,;x)\zeta(b\,;y) + (-1)^b T(b,0,a\,;x-y,x) + (-1)^a T(0,a,b\,;y,y-x) = K(a,b\,;x,y)$$

According to the harmonic product formula, one has

$$T(0,a,b\,;-y,x-y) + T(b,0,a\,;x-y,x) + \zeta(a+b\,;x-y) = \zeta(a\,;-y)\zeta(b\,;x)\,.$$

Therefore we have Proposition 1.2 by removing the term $T(b,0,a\,;x-y,x)$. □

PROOF OF PROPOSITION 1.3. Recall the well-known formula

$$\chi(n) = \frac{1}{\tau(\overline{\chi})}\sum_{l=1}^{k-1}\overline{\chi}(l)e^{2\pi i l n/k} = \frac{\chi(-1)}{\tau(\overline{\chi})}\sum_{l=1}^{k-1}\overline{\chi}(l)e^{-2\pi i l n/k}.$$

By using the above formula and $\phi\chi\psi(-1) = (-1)^{a+b+1}$, we have

$$\begin{aligned}
&\tau(\overline{\phi})\tau(\overline{\chi})\tau(\overline{\psi})L(0,a,b\,;\phi,\chi,\psi)\\
&= \sum_{j=1}^{h-1}\sum_{l=1}^{k-1}\sum_{r=1}^{q-1}\overline{\phi}(j)\overline{\chi}(l)\overline{\psi}(r)T(0,a,b\,;j/h+r/q,l/k+r/q) \qquad (\#)\\
&= (-1)^{a+b+1}\sum_{j=1}^{h-1}\sum_{l=1}^{k-1}\sum_{r=1}^{q-1}\overline{\phi}(j)\overline{\chi}(l)\overline{\psi}(r)T(0,a,b\,;-j/h-r/q,-l/k-r/q). \qquad (\$)
\end{aligned}$$

Therefore, one has $2\tau(\overline{\phi})\tau(\overline{\chi})\tau(\overline{\psi})L(0,a,b\,;\phi,\chi,\psi) = (\#)+(\$)$. Hence we obtain Proposition 1.3 by Proposition 1.2 which is proved by the functional relation (1.2). □

References

[1] T. ARAKAWA and M. KANEKO, On multiple L-values, J. Math. Soc. Japan **56**, no. 4 (2004), 967–991.

[2] Z. LI, On functional relations for the alternating analogues of Tornheim's double zeta function, arXiv:1011.2897v1.

[3] K. MATSUMOTO, On Mordell-Tornheim and other multiple zeta-functions, Proceedings of the Session in Analytic Number Theory and Diophantine Equations, Bonner Math. Schriften **360**, Univ. Bonn, Bonn, 2003, 17 pp.
[4] K. MATSUMOTO and H. TSUMURA, Functional relations among certain double polylogarithms and their character analogues, Šiauliai Math. Semin. **11** (2008), 189–205.
[5] T. NAKAMURA, A functional relation for the Tornheim double zeta function, Acta Arith. **125**, no. 3 (2006), 257–263.
[6] T. NAKAMURA, Double Lerch series and their functional relations, Aequationes Mathematicae **75**, no. 3 (2008), 251–259.
[7] T. NAKAMURA, Double Lerch value relations and functional relations for Witten zeta functions, Tokyo J. Math. **31**, no. 2 (2008), 551–574.
[8] D. TERHUNE, Evaluation of double L-values, J. Number Theory **105**, no. 2 (2004), 275–301.
[9] H. TSUMURA, On functional relations between the Mordell-Tornheim double zeta functions and the Riemann zeta function, Math. Proc. Camb. Phil. Soc. **142** (2007), 395–405.
[10] X. ZHOU, T. CAI and D. M. BRADLEY, Signed q-analogs of Tornheim's double series, Proc. Amer. Math. Soc. **136**, no. 8 (2008), 2689–2698.
[11] X. ZHOU, A functional relation for the Tornheim double zeta function, preprint.

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