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# Asymptotic Expansions of Solutions to the Heat Equations with Initial Value in the Dual of Gel'fand-Shilov Spaces

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**Abstract.** We will derive the asymptotic expansions of the solutions $U(x,t)$ to the heat equation with $(\mathcal{S}_r^r)'(\mathbf{R}^d)$, $r \geq 1/2$, initial value, where $(\mathcal{S}_r^r)'(\mathbf{R}^d)$ is the dual space of the Gel'fand-Shilov space $\mathcal{S}_r^r(\mathbf{R}^d)$. Moreover, we show that, when $1/2 \leq r \leq 1$, these asymptotic expansions satisfy the strong asymptotic condition on some circle $D_R = \{t \in \mathbf{C} \mid \mathrm{Re}\, t^{-1} > R^{-1}\}$. Therefore, we find that these asymptotic series for $(\mathcal{S}_r^r)'(\mathbf{R}^d)$ initial value are Borel summable by means of A. D. Sokal's result on the Borel summability. As an application, we show the asymptotic expansions of the Weyl transform with Planck's constant $\hbar$ in some state, which are refinement of a classical limit of the quantum mechanical expectation values expressed by the Weyl transform.

## 1. Introduction

In this paper, we investigate the asymptotic expansions of solutions to the heat equation with generalized functions initial value and its reconstruction. As a result, we obtain the asymptotic expansions of solutions to the heat equation with initial value in the dual spaces $(\mathcal{S}_r^r)'(\mathbf{R}^d)$ of the Gel'fand-Shilov spaces, which are natural generalizations of the space $\mathcal{S}'(\mathbf{R}^d)$ of Schwartz's tempered distributions and we obtain that when $1/2 \leq r \leq 1$, solutions to the heat equation are reconstructed by means of Borel summability.

Let $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ and

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\,.$$

Then solutions $U(x,t)$ of Cauchy's problem to the heat equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \Delta\right) U(x,t) = 0\,, \quad x \in \mathbf{R}^d\,, \quad t > 0\,, \\ \\ U(x,0) = u(x) \end{cases} \tag{1.1}$$

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are expressed by the following formal power series

$$U(x,t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\Delta^k u(x) \tag{1.2}$$

for suitable initial values $u$. Let us define the heat kernel $E(x,t)$ by

$$E(x,t)=\left(\frac{1}{\sqrt{4\pi t}}\right)^d e^{-\frac{x^2}{4t}}$$

for $x\in\mathbf{R}^d$ and $t>0$. Then we can also express the solutions of (1.1) by

$$U(x,t)=(u*E)(x,t)\,.$$

If an initial value $u\in C^\infty(\mathbf{R})$ satisfies the following condition:

$$|u^{(2k)}(x)|<\frac{Mk!}{r^k}\quad\text{and}\quad|u^{(2k+1)}(x)|<\frac{Mk!}{r^k}\,,\quad k=0,1,2,\dots,$$

for some constants $M>0$ and $r>0$, the solutions $U=u*E$ satisfies (1.2) uniformly for $(x,t)\in\mathbf{R}\times(0,r)$ (see Theorem 15.2 in [11]). If one considers $u\in C^l(\mathbf{R})$ as an initial function, one has the asymptotic expansions of solutions to the heat equation

$$U(x,t)\sim\sum_{k=0}^{l}\frac{t^k}{k!}\Delta^k u(x)$$

for $(x,t)\in\mathbf{R}\times(0,\infty)$ (See: Proposition 2.13. in [1]).

Let $u\in\mathcal{S}'(\mathbf{R}^d)$ and $U(x,t)=(u*E)(x,t)$. Then we have

$$U(x,t)\longrightarrow u\text{ as }t\longrightarrow 0+\ \text{ in }\mathcal{S}'(\mathbf{R}^d)\,, \tag{1.3}$$

(see [6]).

In [13], K. Yoshino obtained the asymptotic expansions of $U(x,t)$ with tempered distributions initial value to analyze (1.3) more precisely below:

THEOREM 1. *Let $U(x,t)\in C^\infty(\mathbf{R}^d\times(0,\infty))$. Suppose that it satisfies*

$$\left(\frac{\partial}{\partial t}-\Delta\right)U(x,t)=0\,,\quad \textit{in } \mathbf{R}^d\times(0,\infty)$$

*and that the estimate*

$$|U(x,t)|\le Ct^{-\nu}(1+|x|)^k\,,\quad x\in\mathbf{R}^d,\quad 0<t<1$$

*holds for some $C>0$, $\nu\ge 0$ and $k\ge 0$. Then there exists $u\in\mathcal{S}'(\mathbf{R}^d)$ such that $U=u*E$ and*

$$U(x,t)\sim\sum_{k=0}^{\infty}\frac{t^k}{k!}\Delta^k u(x)\,.$$

*Namely, for any even $N$ and $\varphi \in \mathcal{S}(\mathbf{R}^d)$,*

$$\lim_{t\to 0+} \left| \langle U(\cdot,t),\varphi\rangle - \sum_{k=0}^{\frac{N}{2}} \frac{t^k}{k!}\left\langle \Delta^k u,\varphi\right\rangle \right| t^{-\frac{N}{2}} = 0\,.$$

In [3], C. Dong and T. Matsuzawa established a characterization of $\left(\mathcal{S}_r^r\right)'(\mathbf{R}^d)$, where $\left(\mathcal{S}_r^r\right)'(\mathbf{R}^d)$ is the dual space of the Gel'fand-Shilov space $\mathcal{S}_r^r(\mathbf{R}^d)$. By using their result, we can obtain the extension of Theorem 1 below.

THEOREM 2 ([3]). *Assume $u \in \left(\mathcal{S}_r^r\right)'(\mathbf{R}^d)$, $r \ge 1/2$. Then the function $U(x,t) = (u * E)(x,t)$ is well defined as a $C^\infty$ function on $\mathbf{R}^d \times (0,\infty)$ and satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(x,t) = 0\,, \quad x \in \mathbf{R}^d\,, \quad t > 0\,, \tag{1.4}$$

$$\int_{\mathbf{R}^d} U(x,t)\varphi(x)dx \longrightarrow \langle u,\varphi\rangle\,,\ \textit{as}\ t \longrightarrow 0+ \quad \textit{for any}\ \varphi \in \mathcal{S}_r^r(\mathbf{R}^d)\,.$$

*Moreover, we have the following result*: (1) *in the case $r > 1/2$, for every $T > 0$ and $\varepsilon > 0$, there is a positive constant $C_\varepsilon$ such that*

$$|U(x,t)| \le C_\varepsilon \exp[\varepsilon(|x|^{\frac{1}{r}} + (1/t)^{1/(2r-1)})]\,, \quad x \in \mathbf{R}^d\,, \quad 0 < t < T\,, \tag{1.5}$$

*and* (2) *in the case $r = 1/2$, for every $t > 0$ and $\varepsilon > 0$, there is a positive constant $C_{\varepsilon,t}$ such that*

$$|U(x,t)| \le C_{\varepsilon,t} e^{\varepsilon|x|^2}\,, \quad x \in \mathbf{R}^d\,. \tag{1.6}$$

*Conversely every $C^\infty$-function $U(x,t)$ defined in $\mathbf{R}^d \times (0,\infty)$ satisfying the conditions* (1.4) *and* (1.5) *or* (1.6) *can be expressed in the form $U(x,t) = (u * E)(x,t)$ with an unique element $u \in \left(\mathcal{S}_r^r\right)'(\mathbf{R}^d)$.*

The aim of our investigation is to show that Yoshino's asymptotic expansion holds for these extended classes of $U = u * E$ and to show the condition which the solutions to the heat equation is reconstructed by Borel summability.

The following result is the main theorem of this paper:

THEOREM 3 (Main Theorem). (I) *Each solution $U = u * E$, $u \in \left(\mathcal{S}_r^r\right)'(\mathbf{R}^d)$ with $r \ge 1/2$, has the asymptotic expansion $U(x,t) \sim \sum_{k=0}^\infty \frac{t^k}{k!}\Delta^k u(x)$.*

(II) *Let $1/2 \le r \le 1$. For any $\varphi \in \mathcal{S}_r^r(\mathbf{R}^d)$, the asymptotic expansions*

$$\langle U(\cdot,t),\varphi\rangle \sim \sum_{k=0}^\infty \frac{t^k}{k!}\left\langle \Delta^k u,\varphi\right\rangle,\ u \in \left(\mathcal{S}_r^r\right)'(\mathbf{R}^d)\,,$$

*are Borel summable on* $D_R = \{t \in \mathbf{C} \mid \operatorname{Re} t^{-1} > R^{-1}\}$ *for some* $R > 0$ *and* $\langle U(\cdot, t), \varphi\rangle$ *are expressed by*

$$\langle U(\cdot, t), \varphi\rangle = \frac{1}{t}\int_0^\infty e^{-\zeta/t} f_B(\zeta)d\zeta$$

*in* $D_R$*, where* $f_B$ *is the Borel transform of the series* $\sum_{k=0}^\infty \frac{t^k}{k!}\langle \Delta^k u, \varphi\rangle$.

Especially, we also show that the asymptotic expansions

$$U(x,t) \sim \sum_{k=0}^\infty \frac{t^k}{k!}\Delta^k u(x)$$

are convergent series when $r = 1/2$.

As an antecedent result about this investigation, K. Yoshino and Y. Oka obtained the similar asymptotic expansions of $U = u * E$ in the cases that $u$ is in $(\mathcal{S}_1)'(\mathbf{R}^d)$, the space of the distributions with the exponential growth [13], $u$ is in $(\mathcal{S}_1^1)'(\mathbf{R}^d)$, the space of Fourier-hyperfunctions and $u$ is in $A'(K)$, the space of hyperfunctions with a compact support [14].

As an application on the asymptotic expansions of the solutions to the heat equation on phase space, we give the asymptotic expansions on the classical limit of the quantum mechanical expectation values by the Weyl transform with symbol in $\left(\mathcal{S}_r^r\right)'(\mathbf{R}^{2d})$. The Weyl transform was introduced by H. Weyl for quantization from classical mechanics to quantum mechanics (see [7], [8], [10], [12] and etc).

The plan of the paper is as follows. In the next section, we introduce the definition and some properties for the spaces $(\mathcal{S}_r^r)'(\mathbf{R}^d)$ of the dual of Gel'fand-Shilov spaces. In section 3, we show that we can obtain the asymptotic expansions of solutions to the heat equation with these spaces initial value. In section 4, at first we define the Borel summable and the strong asymptotic condition and introduce A. D. Sokal's result on Watson's theorem. As a result, we give the condition for $(\mathcal{S}_r^r)'(\mathbf{R}^d)$ which the solutions to the heat equation are reconstructed by the Borel summability. We also show that when $r = 1/2$, the asymptotic expansions of solutions to the heat equation are convergent series. In section 5, we introduce the definition and some properties of the Weyl transform and give the asymptotic expansions on the classical limit of the quantum mechanical expectation values by the Weyl transform with symbol in $\left(\mathcal{S}_r^r\right)'(\mathbf{R}^{2d})$.

## 2. The Gel'fand-Shilov space $\mathcal{S}_r^r$ and its dual space $(\mathcal{S}_r^r)'$

First of all, we give some notations. We use a multi-index $\alpha \in \mathbf{Z}_+^d$, namely, $\alpha = (\alpha_1, \ldots, \alpha_d)$, where $\alpha_i \in \mathbf{Z}$ and $\alpha_i \geq 0$. So, for $x \in \mathbf{R}^d$, $x^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}$, where $\partial_{x_j}^{\alpha_j} = \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j}$.

The Gel'fand-Shilov space $\mathcal{S}_r^r(\mathbf{R}^d)$ is defined as follows (see [4]):

DEFINITION 1. Let $A, B \in (0,\infty)^d$. For $r = (r_1, \dots, r_d)$ and $r_i \geq 0,\ 1 \leq i \leq d$, we define the space $\mathcal{S}_{r,A}^{r,B}(\mathbf{R}^d)$ by

$\mathcal{S}_{r,A}^{r,B}(\mathbf{R}^d) = \{\varphi \in C^\infty(\mathbf{R}^d) \mid$ For any $\delta > 1$ and $\rho > 1$, there exists some constant $C_{\delta,\rho} > 0$ such that $|x^k \partial_x^q \varphi(x)| \leq C_{\delta,\rho}(\delta A)^k (\rho B)^q k^{kr} q^{qr}\ ,\ x \in \mathbf{R}^d,\ k, q \in \mathbf{Z}_+^d\}$,

where

$$(\delta A)^k = (\delta A_1)^{k_1} \cdots (\delta A_d)^{k_d}\ ,$$

$$(\rho B)^q = (\rho B_1)^{q_1} \cdots (\rho B_d)^{q_d}\ .$$

The space $\mathcal{S}_{r,A}^{r,B}(\mathbf{R}^d)$ is a Fréchet space with the semi-norms

$$\|\varphi\|^{\delta,\rho} = \sup_{\substack{x \in \mathbf{R}^d \\ k,q \in \mathbf{Z}_+^d}} \frac{|x^k \partial_x^q \varphi(x)|}{(\delta A)^k (\rho B)^q k^{kr} q^{qr}}\ .$$

The space $\mathcal{S}_r^r(\mathbf{R}^d)$ is given by the inductive limit

$$\mathcal{S}_r^r(\mathbf{R}^d) = \varinjlim_{A,B \to \infty} \mathcal{S}_{r,A}^{r,B}(\mathbf{R}^d)\ .$$

J. Chung, S. -Y. Chung and D. Kim characterized the Gel'fand-Shilov space via the Fourier transform as follows (see [2]):

PROPOSITION 1. *Let $r \geq 1/2$. The following statements are equivalent.*

(i) $\varphi \in \mathcal{S}_r^r(\mathbf{R}^d)$,

(ii) $\sup_{x \in \mathbf{R}^d} |\varphi(x)| e^{\varepsilon |x|^{\frac{1}{r}}} < \infty$ *and* $\sup_{\xi \in \mathbf{R}^d} |\hat{\varphi}(\xi)| e^{\eta |\xi|^{\frac{1}{r}}} < \infty$ , *for some* $\varepsilon, \eta > 0$.

We can easily give the example of the element of $\mathcal{S}_r^r(\mathbf{R})$ by Proposition 1 as follows:

EXAMPLE 1. (i) The Gaussian function $e^{-x^2}$ is in $\mathcal{S}_{1/2}^{1/2}(\mathbf{R})$, since its Fourier transform is

$$\mathcal{F}(e^{-x^2})(\xi) = e^{-\frac{1}{4}\xi^2}\ .$$

(ii) The function $\dfrac{1}{\cosh x}$ is in $\mathcal{S}_1^1(\mathbf{R})$, since its Fourier transform is

$$\mathcal{F}\left(\frac{1}{\cosh x}\right)(\xi) = \frac{\pi}{\cosh \frac{\pi}{2}\xi}\ .$$

(iii) We define the Hermite functions $\{h_n(x)\}_{n=0,1,2,\dots}$ on $\mathbf{R}$ by

$$h_n(x) = (2^n n!)^{-\frac{1}{2}} \pi^{-\frac{1}{4}} (-1)^n e^{\frac{x^2}{2}} \partial_x^n e^{-x^2}\ .$$

The Fourier transform of the Hermite function $h_n(x)$ is $(-i)^n h_n(\xi)$. Hence we find that

$$|h_n(x)| \le Ce^{-1/2x^2} \text{ and } |\widehat{h_n}(\xi)| \le C'e^{-1/2\xi^2}.$$

Therefore the Hermite function is a element of the $\mathcal{S}^{1/2}_{1/2}(\mathbf{R})$.

Let $a \in (0,\infty)^d$ be $a = r\big(eA^{\frac{1}{r}}\big)^{-1} = r_1\big(eA_1^{\frac{1}{r_1}}\big)^{-1}\cdots r_d\big(eA_d^{\frac{1}{r_d}}\big)^{-1}$. For any $a, B \in (0,\infty)^d$, we define the space $\mathcal{S}^{r,B}_{r,a}(\mathbf{R}^d)$ by
$\mathcal{S}^{r,B}_{r,a}(\mathbf{R}^d) = \{\varphi \in C^\infty(\mathbf{R}^d) \mid$ For any $\delta > 1$ and $\rho > 1$, there exists some constsnt $C_{\delta,\rho} > 0$ such that$|\partial_x^q \varphi(x)| \le C_{\delta,\rho}(\rho B)^q q^{qr} e^{-a_\delta |x|^{\frac{1}{r}}},\ x \in \mathbf{R}^d,\ q \in \mathbf{Z}^d_+\}$,

where $a_\delta = r\big(e(\delta A)^{\frac{1}{r}}\big)^{-1}$ and

$$\|\varphi\|^r_{\delta,\rho} = \sup_{\substack{x\in\mathbf{R}^d\\ \beta\in\mathbf{Z}^d_+}} \frac{|\partial_x^\beta \varphi(x)|}{(\rho B)^\beta \beta^{\beta r} e^{-a_\delta |x|^{\frac{1}{r}}}}.$$

Since $\mathcal{S}^{r,B}_{r,A}(\mathbf{R}^d)$ is isomorphic to $\mathcal{S}^{r,B}_{r,a}(\mathbf{R}^d)$ (see [4]), we mainly deal with $\mathcal{S}^{r,B}_{r,a}(\mathbf{R}^d)$.

PROPOSITION 2 ([4]). *The following properties are known*:

(i) $\mathcal{S}^r_r(\mathbf{R}^d) \equiv \{0\}$ *for* $0 \le r < 1/2$,

(ii) $\widehat{\mathcal{S}^{r,B}_{r,A}}(\mathbf{R}^d) = \mathcal{S}^{r,A}_{r,B}(\mathbf{R}^d)$, *where we denote the image of Fourier transform of* $\mathcal{S}^{r,B}_{r,A}(\mathbf{R}^d)$ *by* $\widehat{\mathcal{S}^{r,B}_{r,A}}(\mathbf{R}^d)$.

DEFINITION 2. We denote by $(\mathcal{S}^r_r)'(\mathbf{R}^d)$ the dual space of the Gel'fand-Shilov space $\mathcal{S}^r_r(\mathbf{R}^d)$.

As a remark, it is known that $U(x,t)$ has the following properties (see for example [3]).

PROPOSITION 3. *Let* $r \ge 1/2$. *If* $\varphi \in \mathcal{S}^{r,B}_{r,a}(\mathbf{R}^d)$, *then we have*

$$U(x,t) \equiv \int_{\mathbf{R}^d} E(x-y,t)\varphi(y)dy \in \mathcal{S}^r_r(\mathbf{R}^d_x), \quad t>0, \quad (r>1/2)$$

*or*

$$U(x,t) \equiv \int_{\mathbf{R}^d} E(x-y,t)\varphi(y)dy \in \mathcal{S}^{1/2}_{1/2}(\mathbf{R}^d_x), \quad 0<t<\frac{1}{8a_i}, \quad 1 \le i \le d.$$

*Moreover we have*

$$U(x,t) \to \varphi \textit{ in } \mathcal{S}^r_r(\mathbf{R}^d) \textit{ as } t \to 0+.$$

PROPOSITION 4. *Let* $u \in \left(\mathcal{S}_r^r\right)'(\mathbf{R}^d)$, $r \geq 1/2$ *and* $U(x,t) = (u * E)(x,t) = \left\langle u_y, E(x-y,t)\right\rangle$. *Then for any* $\varphi \in \mathcal{S}_r^r(\mathbf{R}^d)$, *we have*

$$\lim_{t\to 0+}\int_{\mathbf{R}^d} U(x,t)\varphi(x)dx = \langle u,\varphi\rangle\,.$$

### 3. The proof of (I) of Theorem 3

At first, we show the following proposition:

PROPOSITION 5. *Let* $\varphi \in \mathcal{S}_{r,a}^{r,B}(\mathbf{R}^d)$, $r \geq 1/2$. *Then we have for any* $\delta, \rho > 1$ *there exists* $C_{\delta,\rho} > 0$ *such that*

$$\|\partial^\alpha \varphi\|_{\delta, e^r\rho}^r \leq C_{\delta,\rho}(\rho\tilde{B}e^{2r}/r^r)^{|\alpha|}(\alpha!)^r$$

*for any* $\alpha \in \mathbf{Z}_+^d$, *where* $\tilde{B} = \max\{B_1,\dots,B_d\}$.

PROOF. We only show our assertion as $d = 1$ to avoid the confusion of the notation. Let $\varphi \in \mathcal{S}_{r,a}^{r,B}(\mathbf{R})$. Then by the definition of $\mathcal{S}_{r,a}^{r,B}(\mathbf{R})$, for any $q \in \mathbf{Z}_+$,

$$|\partial_x^q\varphi(x)| \leq C_{\delta,\rho}(\rho B)^q q^{qr} e^{-a_\delta|x|^{1/r}}\,.$$

Put $\psi(x) = \partial_x^n\varphi(x)$. then we have

$$\begin{aligned}|\partial_x^q\psi(x)| = |\partial_x^{q+n}\varphi(x)| &\leq C_{\delta,\rho}(\rho B)^{q+n}(q+n)^{(q+n)r}e^{-a_\delta|x|^{1/r}}\\ &= C_{\delta,\rho}(\rho B)^{q+n}q^{qr}e^{nr}(q+n)^{nr}e^{-a_\delta|x|^{1/r}}\,. \qquad (3.1)\end{aligned}$$

If we put

$$f(x) = \frac{(x+n)^{nr}}{\lambda^{x+n}}\,, \quad x > 0\,, \quad n = 0,1,2,\dots,$$

for any $\lambda > 1$, since $f(x) \leq \left(\frac{nr}{e}\right)^{nr}(\log\lambda)^{-nr}$, we obtain the following estimate:

$$(q+n)^{nr} \leq \left(\frac{nr}{e}\right)^{nr}(\log\lambda)^{-nr}\lambda^{q+n}\,.$$

Then we have

$$(3.1) \leq C_{\delta,\rho}(\lambda\rho B)^q q^{qr}(\rho B)^n(nr)^{nr}\left(\frac{\lambda}{(\log\lambda)^r}\right)^n\,.$$

Let us take the lower bound for $\lambda$ on $\dfrac{\lambda}{(\log\lambda)^r}$; we obtain that $\dfrac{e^r}{r^r} = \inf_{\lambda>1}\dfrac{\lambda}{(\log\lambda)^r}$. By Stirling's formula, we have

$$\frac{|\partial_x^{q+n}\varphi(x)|e^{a_\delta|x|^{1/r}}}{(e^r\rho B)^q q^{qr}} \leq C_{\delta,\rho}(\rho B)^n(e^{2nr}/r^{rn})(n!)^r\,.$$

Therefore we obtain

$$\|\partial^n \varphi\|^r_{\delta, e^r\rho} \le C_{\delta,\rho}(\rho B e^{2r}/r^r)^n (n!)^r. \qquad \square$$

We need the following Lemma 1 for the proof of (I) of Theorem 3.

LEMMA 1. Let $\varphi$ be in $\mathcal{S}^{r,B}_{r,a}(\mathbf{R}^d)$ for $r \ge 1/2$ and $a$, $B \in (0,\infty)^d$. Moreover $t$ satisfies $0 < t < 1/(8a_{\delta i})$, $i = 1, \dots, d$. Then for $N = 0, 1, 2, \dots,$

$$\int_{\mathbf{R}^d} e^{-z^2} z^\alpha \left\{ \int_0^1 (1-\theta)^N \partial_y^\alpha \varphi(y + \sqrt{4t} z\theta) d\theta \right\} dz, \ \alpha \in \mathbf{Z}^d_+$$

is in $\mathcal{S}^{r,B}_{r,c_r a}(\mathbf{R}^d)$, where $c_r = (c_{r_1}, \dots, c_{r_d})$ and $c_{r_i} = (2^{\max\{1/r_i - 1, 0\}})^{-1}$.

PROOF. Let $\varphi \in \mathcal{S}^{r,B}_{r,a}(\mathbf{R}^d)$, $r \ge 1/2$. Then for any $\delta, \rho > 1$ and $N = 0, 1, 2, \dots,$

$$\begin{aligned} &\frac{\left| \partial_y^\beta \int_{\mathbf{R}^d} e^{-z^2} z^\alpha \left\{ \int_0^1 (1-\theta)^N \partial_y^\alpha \varphi(y + \sqrt{4t} z\theta) d\theta \right\} dz \right| e^{c_r a_\delta |y|^{1/r}}}{(e^r \rho B)^\beta \beta^{\beta r}} \\ &\le \int_{\mathbf{R}^d} e^{-z^2} |z|^\alpha \left\{ \int_0^1 |1-\theta|^N \frac{|\partial_y^{\beta+\alpha} \varphi(y + \sqrt{4t} z\theta)|}{(e^r \rho B)^\beta \beta^{\beta r}} d\theta \right\} dz e^{c_r a_\delta |y|^{1/r}} \end{aligned} \tag{3.2}$$

for any $\alpha, \beta \in \mathbf{Z}^d_+$. By Proposition 5, we have

$$\frac{|\partial_y^{\beta+\alpha} \varphi(y + \sqrt{4t} z\theta)|}{(e^r \rho B)^\beta \beta^{\beta r}} \le C_{\delta,\rho} (\rho \tilde{B} e^{2r}/r^r)^{|\alpha|} (\alpha!)^r e^{-a_\delta |y + \sqrt{4t} z\theta|^{1/r}}.$$

So we have

$$\begin{aligned} (3.2) \le\ & C_{\delta,\rho} (\rho \tilde{B} e^{2r}/r^r)^{|\alpha|} (\alpha!)^r \int_{\mathbf{R}^d} e^{-z^2} |z|^\alpha \\ & \times \left\{ \int_0^1 |1-\theta|^N e^{-a_\delta |y + \sqrt{4t} z\theta|^{1/r}} d\theta \right\} e^{c_r a_\delta |y|^{1/r}} dz. \end{aligned} \tag{3.3}$$

Let $c_{r_i} = (2^{\max\{1/r_i - 1, 0\}})^{-1}$. For $1 \le i \le d$, we have

$$c_{r_i} |y_i|^{1/r_i} - |\sqrt{4t} z_i \theta|^{1/r_i} \le |y_i + \sqrt{4t} z_i \theta|^{1/r_i}.$$

Hence we obtain

$$\begin{aligned} e^{-a_{\delta i} |y_i + \sqrt{4t} z_i \theta|^{1/r_i}} e^{c_{r_i} a_{\delta i} |y_i|^{1/r_i}} &\le e^{-a_{\delta i} (c_{r_i} |y_i|^{1/r_i} - |\sqrt{4t} z_i \theta|^{1/r_i})} e^{c_{r_i} a_{\delta i} |y_i|^{1/r_i}} \\ &\le e^{a_{\delta i} |\sqrt{4t} z_i|^{1/r_i}}. \end{aligned}$$

So we have

$$(3.3) \le C'_{\delta,\rho} (\rho \tilde{B} e^{2r}/r^r)^{|\alpha|} (\alpha!)^r \int_{\mathbf{R}^d} e^{-z^2} |z|^\alpha e^{a_\delta |\sqrt{4t} z|^{\frac{1}{r}}} dz. \tag{3.4}$$

Since $0 < t < 1/(8a_{\delta i}),\ i = 1, \ldots, d$, from the assumption of Lemma 1, the above integral converges for any $r \geq 1/2$. Therefore as $r > 1/2$, we obtain for a sufficient small $t$,

$$\begin{aligned}(3.4) &\leq C'_{\delta,\rho}(\rho\tilde{B}e^{2r}/r^r)^{|\alpha|}(\alpha!)^r \int_{\mathbf{R}^d} e^{-z^2}|z|^\alpha e^{a_\delta|z|^{\frac{1}{r}}}dz \\ &\leq C''_{\delta,\rho,a,r}(\rho\tilde{B}e^{2r}/r^r)^{|\alpha|}(\alpha!)^r \int_{\mathbf{R}^d} e^{-\frac{z^2}{2}}|z|^\alpha dz \\ &\leq C'''_{\delta,\rho,a,r,d}(\sqrt{2}\rho\tilde{B}e^{2r}/r^r)^{|\alpha|}(\alpha!)^r \Gamma\left(1/2(\alpha+1)\right).\end{aligned} \tag{3.5}$$

Since $\Gamma\left(1/2(\alpha+1)\right) \leq \pi^{d/2}(\alpha!)^{1/2}$, we have

$$(3.5) \leq C''''_{\delta,\rho,a,r,d}(\sqrt{2}\rho\tilde{B}e^{2r}/r^r)^{|\alpha|}(\alpha!)^{r+1/2}.$$

On the other hand, as $r = 1/2$, since $0 < t < 1/(8a_{\delta i}),\ i = 1, \ldots, d$, we obtain

$$\begin{aligned}(3.4) &\leq C'_{\delta,\rho}(\sqrt{2}\rho\tilde{B}e)^{|\alpha|}(\alpha!)^r \int_{\mathbf{R}^d} e^{-z^2}|z|^\alpha e^{a_\delta|\sqrt{4t}z|^2}dz \\ &\leq C'_{\delta,\rho}(\sqrt{2}\rho\tilde{B}e)^{|\alpha|}(\alpha!)^r \int_{\mathbf{R}^d} e^{-1/2z^2}|z|^\alpha dz \\ &\leq C''_{\delta,\rho,a,d}(2\rho\tilde{B}e)^{|\alpha|}(\alpha!)^{1/2}\Gamma\left(1/2(\alpha+1)\right).\end{aligned}$$

This completes the proof of Lemma 1. □

THE PROOF OF (I) OF THEOREM 3. From the assumption of Theorem 2, there exists $u \in (\mathcal{S}_r^r)'(\mathbf{R}^d)$, $r \geq 1/2$, such that $U(x,t) = (u * E)(x,t)$. For any $\varphi \in \mathcal{S}_r^r(\mathbf{R}^d)$, $r \geq 1/2$,

$$\begin{aligned}\langle U(\cdot,t), \varphi\rangle &= \left\langle u_y, \left(\frac{1}{\sqrt{4\pi t}}\right)^d \int_{\mathbf{R}^d} e^{-\frac{|x-y|^2}{4t}}\varphi(x)dx\right\rangle \\ &= \left\langle u_y, \int_{\mathbf{R}^d} \pi^{-\frac{d}{2}}e^{-z^2}\varphi(y+\sqrt{4t}z)dz\right\rangle,\end{aligned}$$

where $z = \dfrac{x-y}{\sqrt{4t}}$. Since

$$\int_{\mathbf{R}^d} e^{-z^2}z^\alpha dz = \prod_{i=1}^d \int_{\mathbf{R}} e^{-z_i^2}z_i^{\alpha_i}dz_i = \begin{cases}\prod_{i=1}^d \Gamma\left(1/2(\alpha_i+1)\right), & \alpha_i : \text{even}, \\ 0, & \alpha_i : \text{odd},\end{cases}$$

we have by Taylor's formula,

$$\begin{aligned}&\int_{\mathbf{R}^d} \pi^{-\frac{d}{2}}e^{-z^2}\varphi(y+\sqrt{4t}z)dz \\ &= \int_{\mathbf{R}^d} \pi^{-\frac{d}{2}}e^{-z^2}\left\{\sum_{|\alpha|\leq N} \frac{(\sqrt{4t})^{|\alpha|}}{\alpha!}z^\alpha\partial_y^\alpha\varphi(y)\right.\end{aligned}$$

$$+ (\sqrt{4t})^{N+1} \sum_{|\alpha|=N+1} \frac{(N+1)z^\alpha}{\alpha!} \int_0^1 (1-\theta)^N \partial_y^\alpha \varphi(y+\sqrt{4t}z\theta)d\theta \Big\} dz$$

$$= \pi^{-\frac{d}{2}} \sum_{|\alpha|\le N} \frac{(\sqrt{4t})^{|\alpha|}}{\alpha!} \partial_y^\alpha \varphi(y) \int_{\mathbf{R}^d} e^{-z^2} z^\alpha dz$$

$$+ \pi^{-\frac{d}{2}} (\sqrt{4t})^{N+1} \sum_{|\alpha|=N+1} \frac{(N+1)}{\alpha!} \int_{\mathbf{R}^d} e^{-z^2} z^\alpha \left\{ \int_0^1 (1-\theta)^N \partial_y^\alpha \varphi(y+\sqrt{4t}z\theta)d\theta \right\} dz$$

$$= \sum_{k=0}^{\frac{N}{2}} \frac{t^k}{k!} \Delta^k \varphi(y) + \pi^{-\frac{d}{2}} (\sqrt{4t})^{N+1} \sum_{|\alpha|=N+1} \frac{(N+1)}{\alpha!} \int_{\mathbf{R}^d} e^{-z^2} z^\alpha$$

$$\times \left\{ \int_0^1 (1-\theta)^N \partial_y^\alpha \varphi(y+\sqrt{4t}z\theta)d\theta \right\} dz\,,$$

where $\Delta = \partial_{y_1}^2 + \cdots + \partial_{y_d}^2$. So we have the following equality:

$$\left| \langle U(\cdot,t), \varphi\rangle - \left\langle \sum_{k=0}^{\frac{N}{2}} \frac{t^k}{k!} \Delta^k u_y, \varphi \right\rangle \right| t^{-\frac{N}{2}}$$

$$= \left| \left\langle u_y\ ,\ \pi^{-\frac{d}{2}} (\sqrt{4t})^{N+1} \sum_{|\alpha|=N+1} \frac{(N+1)}{\alpha!} \int_{\mathbf{R}^d} e^{-z^2} z^\alpha \right.\right. \tag{3.6}$$

$$\left.\left. \times \left\{ \int_0^1 (1-\theta)^N \partial_y^\alpha \varphi(y+\sqrt{4t}z\theta)d\theta \right\} dz \right\rangle \right| t^{-\frac{N}{2}}\,.$$

We obtain the following estimate by the continuity of $u$, Lemma 1 and (3.6): For any $a, B \in (0,\infty)^d$,

$$(3.6) \le C_{a,B,\delta,\rho,d} Q^{N+1} \{(N+1)!\}^{(r-1/2)} t^{\frac{N+1}{2}} t^{-\frac{N}{2}}$$

$$= C_{a,B,\delta,\rho,d} Q^{N+1} \{(N+1)!\}^{(r-1/2)} t^{\frac{1}{2}} \to 0\,,$$

for some constant $Q > 0$ as $t \to 0+$. This completes the proof of (I) of Theorem 3.

By the proof of (I) of Theorem 3, we obtain the following results: □

THEOREM 4. *Let $T > 0$ and $U(x,t) = (u * E)(x,t)$, $u \in \left(\mathcal{S}_r^r\right)'(\mathbf{R}^d)$ with $r \ge 1/2$. Then we obtain the following estimate for $0 < t < T$ :*

(i) *For any $\varphi \in \mathcal{S}_{r,a}^{r,B}(\mathbf{R}^d)$, $r > 1/2$, we have for any $a, B \in (0,\infty)^d$ and $\delta, \rho > 1$,* *there exists* $C_{a,B,d} > 0$ *and* $\|\cdot\|^r_{\delta,\rho}$ *such that*

$$\left|\langle U(\cdot,t),\varphi\rangle - \sum_{k=0}^{N/2}\frac{t^k}{k!}\big\langle \Delta^k u,\varphi\big\rangle\right|$$
$$\le C_{a,B,d}\|\varphi\|^r_{\delta,\rho}(2\sqrt{2}\rho\tilde{B}e^{2r+1}/r^r)^{N+1}\{(N+1)!\}^{(r-1/2)}t^{\frac{N+1}{2}}\,.$$

(ii) *For any* $\varphi \in \mathcal{S}^{1/2,B}_{1/2,a}(\mathbf{R}^d)$, *we have for any* $a, B \in (0,\infty)^d$ *and* $\delta, \rho > 1$, *there exists* $C_{a,B,d} > 0$ *and* $\|\cdot\|^{1/2}_{\delta,\rho}$ *such that*

$$\left|\langle U(\cdot,t),\varphi\rangle - \sum_{k=0}^{N/2}\frac{t^k}{k!}\big\langle \Delta^k u,\varphi\big\rangle\right| \le C_{a,B,d}\|\varphi\|^{1/2}_{\delta,\rho}(4\rho\tilde{B}e^2)^{N+1}t^{\frac{N+1}{2}}\,.$$

Especially if we take $\varphi$ in a bounded set $\mathcal{B}$ of $\mathcal{S}^r_r(\mathbf{R}^d)$, $r \ge 1/2$, then we have for $0 < t < T$,

$$\left|\langle U(\cdot,t),\varphi\rangle - \sum_{k=0}^{N/2}\frac{t^k}{k!}\big\langle \Delta^k u,\varphi\big\rangle\right| \le Ch^{N+1}\{(N+1)!\}^{(r-1/2)}t^{\frac{N+1}{2}}$$

for some constant $C > 0$ and $h > 0$, independent of $\varphi$. In fact, $\mathcal{B}$ is included in some inductive space $\mathcal{S}^{r,B}_{r,a}(\mathbf{R}^d)$ and the differential operators are bounded in the space $\mathcal{S}^{r,B}_{r,a}(\mathbf{R}^d)$.

## 4. Borel summability

At first, we define the Borel summable as follows (see [9]):

DEFINITION 3. We say that the formal power series $\sum_{n=0}^{\infty} a_n t^n$ is Borel summable if

(i) $f_B(\zeta) = \sum_{n=0}^{\infty}\frac{a_n\zeta^n}{n!}$ converges in some circle $\{\zeta \in \mathbf{C} \mid |\zeta| < \delta\}$,

(ii) $f_B(\zeta)$ has an analytic continuation to a neighborhood of the positive real axis and

(iii) $\frac{1}{t}\int_0^{\infty} e^{-\frac{\zeta}{t}} f_B(\zeta)d\zeta$ converges for some $t \ne 0$.

$f_B(\zeta)$ is called the Borel transform of the series $\sum_{n=0}^{\infty} a_n t^n$.

DEFINITION 4. Let $f$ be analytic in the circle $D_R = \{t \in \mathbf{C} \mid \mathrm{Re}\, t^{-1} > R^{-1}\}$ for some $R > 0$ and satisfy the following condition:

$$f(t) = \sum_{n=0}^{N} a_n t^n + R_N(t)\,,$$

$$|R_N(t)| \le C\sigma^{N+1}(N+1)!|t|^{N+1} \tag{4.1}$$

for some constants $C > 0$ and $\sigma > 0$ uniformly in $N$ and in $t \in D_R$. Then we say that $f(t)$ satisfies the strong asymptotic condition (4.1) on $D_R$.

On the Borel summability, Sokal's result is known as follows (see [9]):

THEOREM 5. *Suppose that $f(t)$ satisfies the strong asymptotic condition* (4.1) *on $D_R$. Then the Borel transform $f_B(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n$ converges for $|\zeta| < 1/\sigma$ and has an analytic continuation to the striplike region $S_\sigma = \{\zeta \in \mathbf{C} \mid dist\ (t, \mathbf{R}_+) < 1/\sigma\}$, satisfying the bound*

$$|f_B(\zeta)| \leq Ke^{|\zeta|/R} \tag{4.2}$$

*uniformly in every $S_{\sigma'}$ with $\sigma' > \sigma$. Furthermore, $f$ can be represented by absolutely convergent integral*

$$f(t) = \frac{1}{t}\int_0^\infty e^{-\zeta/t} f_B(\zeta)d\zeta \tag{4.3}$$

*for any $t \in D_R$.*

*Conversely, if $f_B(\zeta)$ is a holomorphic in $S_{\sigma''}$ $(\sigma'' \leq \sigma)$ and satisfies* (4.2), *then the function $f(t)$ defined by* (4.3) *is a holomorphic in $D_R$ and satisfies a strong asymptotic condition* (4.1) *uniformly in every $D_{R'}$ with $R' < R$.*

In the previous section, we obtain the asymptotic expansions of $U(x,t)$ with $u \in (\mathcal{S}_r^r)'(\mathbf{R}^d)$, $r \geq 1/2$, initial value as follows:

$$\langle U(\cdot,t), \varphi\rangle \sim \sum_{n=0}^{\infty} \frac{t^n}{n!}\left\langle \Delta^n u, \varphi\right\rangle$$

for any $\varphi \in \mathcal{S}_r^r(\mathbf{R}^d)$. Here we put

$$a_n = \frac{\langle \Delta^n u, \varphi\rangle}{n!}$$

and we consider the asymptotic series

$$\langle U(\cdot,t), \varphi\rangle \sim \sum_{n=0}^{\infty} a_n t^n\,. \tag{4.4}$$

At first, we obtain the following estimate for $\mathcal{S}_{r,a}^{r,B}(\mathbf{R}^d)$.

PROPOSITION 6. *Let $\varphi \in \mathcal{S}_{r,a}^{r,B}(\mathbf{R}^d)$, $r \geq 1/2$. Then for any $\delta,\ \rho > 1$, there exists $C_{\delta,\rho} > 0$ such that*

$$\|\Delta^n\varphi\|_{\delta,e^r\rho}^r \leq C_{\delta,\rho}(2d^{1/2}\rho\tilde{B}e^{2r}/r^r)^{2n}(n!)^{2r}\,,$$

*where $\tilde{B} = \max\{B_1,\ldots,B_d\}$.*

PROOF. When $d = 1$, by Proposition 5, for any $\delta, \rho > 1$, there exists $C_{\delta,\rho} > 0$ such that

$$\|\partial^{2n}\varphi(x)\|^r_{\delta,e^r\rho} \le C_{\delta,\rho}(\rho B e^{2r}/r^r)^{2n}\{(2n)!\}^r .$$

From the Stirling's formula, we have

$$\begin{aligned}(2n)! &\sim \sqrt{2\pi}(2n)^{2n+1/2}e^{-2n}\\ &= 2^{2n+1/2}\frac{1}{\sqrt{2\pi}}n^{-1/2}(\sqrt{2\pi}n^{n+1/2}e^{-n})^2\\ &\sim 2^{2n+1/2}\frac{1}{\sqrt{2\pi}}n^{-1/2}(n!)^2 .\end{aligned}$$

Hence we obtain

$$\|\partial^{2n}\varphi(x)\|^r_{\delta,e^r\rho} \le C'_{\delta,\rho}(2\rho B e^{2r}/r^r)^{2n}(n!)^{2r} . \tag{4.5}$$

For a higher dimension, by $\tilde{B} = \max\{B_1, \dots, B_d\}$ and (4.5),

$$\|\Delta^n\varphi\|^r_{\delta,e^r\rho} \le C''_{\delta,\rho}(2d^{1/2}\rho\tilde{B}e^{2r}/r^r)^{2n}(n!)^{2r} . \qquad \square$$

Now we find that the asymptotic expansions (4.4) satisfy the strong asymptotic condition for $1/2 \le r \le 1$ as follows:

THEOREM 6. *Let $1/2 \le r \le 1$. If $U(x,t)$ satisfies the assumption of Theorem 2, then for any $\varphi \in \mathcal{S}^r_r(\mathbf{R}^d)$, the asymptotic expansion*

$$\langle U(\cdot,t),\varphi\rangle \sim \sum_{k=0}^{\infty}\frac{t^k}{k!}\langle\Delta^k u,\varphi\rangle, \quad u \in (\mathcal{S}^r_r)'(\mathbf{R}^d)$$

*satisfies the strong asymptotic condition on $D_{\{\tilde{A}^2\theta/(4e^2)\}}$, $\tilde{A} = \min\{A_1, \dots, A_d\}$ and $0 < \theta < 1$.*

Now we prepare three lemmas for the proof of Theorem 6. To avoid the confusion of the notation, we deal with only one dimensional case.

LEMMA 2. *Let $t = p + iq$, $p > 0$ and $0 < \theta < 1$. Then we have the following estimate;*

$$\left|\partial^k_\xi e^{-\theta t\xi^2}\right| \le C(2e)^k\left(\frac{|t|^2}{\theta p}\right)^{k/2}k^{k/2}$$

*for some constant $C > 0$.*

PROOF. For $t > 0$, we have

$$\partial^k_\xi e^{-\theta t\xi^2} = \int_{\mathbf{R}}\frac{1}{\sqrt{4\pi\theta t}}e^{-\frac{x^2}{4\theta t}}(-ix)^k e^{-ix\xi}dx$$

$$= \frac{1}{\sqrt{\pi}}(-i(\sqrt{4\theta t}))^k \int_{\mathbf{R}} e^{-z^2} z^k e^{-i(\sqrt{4\theta t} z)\xi} dz$$
$$= \frac{1}{\sqrt{\pi}}(-i(\sqrt{4\theta t}))^k \int_{\mathbf{R}} e^{-(x+iy)^2}(x+iy)^k e^{-i(\sqrt{4\theta t}(x+iy))\xi} dz$$
$$= \frac{1}{\sqrt{\pi}}(-i(\sqrt{4\theta t}))^k e^{y^2} e^{\sqrt{4\theta t}\xi y} \sum_{l=0}^{k} \binom{k}{l} (iy)^{k-l} \int_{\mathbf{R}} e^{-x^2-2ixy} x^l e^{-i\sqrt{4\theta t} x\xi} dx$$
$$= \frac{1}{\sqrt{\pi}}(-i(\sqrt{4\theta t}))^k e^{(y+\sqrt{\theta t}\xi)^2-\theta t\xi^2} \sum_{l=0}^{k} \binom{k}{l} (iy)^{k-l} \int_{\mathbf{R}} e^{-x^2-2ixy} x^l e^{-i\sqrt{4\theta t} x\xi} dx\,,$$

where we put $z = \dfrac{x}{\sqrt{4\theta t}}$ and shift the integral line $\mathbf{R}$ to $\mathbf{R} + iy$ $y > 0$. Therefore, letting $y = -\sqrt{\theta t}\xi$, we obtain

$$\partial_\xi^k e^{-\theta t\xi^2} = \frac{1}{\sqrt{\pi}}(-i(\sqrt{4\theta t}))^k e^{-\theta t\xi^2} \sum_{l=0}^{k} \binom{k}{l} (-i\sqrt{\theta t}\xi)^{k-l} \int_{\mathbf{R}} e^{-x^2} x^l dx\,. \tag{4.6}$$

The both hand side of this equality (4.6) are holomorphic functions on $\{\mathrm{Re}\, t > 0\}$. Hence (4.6) holds on $\{\mathrm{Re}\, t > 0\}$. If we put $t = p + iq$, $p > 0$, we have

$$\left|\partial_\xi^k e^{-\theta t\xi^2}\right| \le \frac{1}{\sqrt{\pi}} \left|\sqrt{4\theta t}\right|^k e^{-\theta p\xi^2} \sum_{l=0}^{k} \binom{k}{l} \left|\sqrt{\theta t}\xi\right|^{k-l} \int_{\mathbf{R}} e^{-x^2}|x|^l dx\,. \tag{4.7}$$

Since $\xi^{k-l} e^{-\theta p\xi^2} \le \left(\frac{k-l}{2\theta p}\right)^{(k-l)/2}$ $(k \ge l)$ and $\int_{\mathbf{R}} e^{-x^2}|x|^l dx = \Gamma\left(\frac{l+1}{2}\right) \le C l^{l/2} e^l (1/2)^{l/2}\sqrt{\pi}$ for some constant $C > 0$, we obtain

$$(4.7) \le C e^k |t|^{k/2} k^{k/2} \sum_{l=0}^{k} \binom{k}{l} \left(\frac{|t|}{\theta p}\right)^{(k-l)/2}$$
$$= C e^k |t|^{k/2} k^{k/2} \left(1 + \sqrt{\frac{|t|}{\theta p}}\right)^k$$
$$= C e^k |t|^{k/2} k^{k/2} \left(\frac{|t|}{\theta p}\right)^{k/2} \left(\sqrt{\frac{\theta p}{|t|}} + 1\right)^k$$
$$\le C(2e)^k \left(\frac{|t|^2}{\theta p}\right)^{k/2} k^{k/2}\,.$$

Therefore we obtain the following estimate:

$$\left|\partial_\xi^k e^{-\theta t\xi^2}\right| \le C(2e)^k \left(\frac{|t|^2}{\theta p}\right)^{k/2} k^{k/2}$$

for some constant $C > 0$. □

LEMMA 3. *Let $r \geq 1/2$. If $\varphi \in \mathcal{S}^{r,B}_{r,A}(\mathbf{R})$, then we have the following estimate. For any $\delta, \rho > 1$, there exists $C_{\delta,\rho} > 0$ such that*

$$|\partial_\xi^\beta \mathcal{F}^{-1}(\Delta^m \varphi)(\xi)| \leq C_{\delta,\rho}(e^r\delta A)^\beta \beta^{\beta r} e^{-b_{e^r\rho}|\xi|^{1/r}} Q^{m+1}\{(m+1)!\}^{2r}, \ \beta \in \mathbf{Z}_+$$

*for some $Q > 0$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform.*

PROOF. Let $\varphi \in \mathcal{S}^{r,B}_{r,A}(\mathbf{R})$. Then we have for any $\delta, \rho > 1$,

$$\begin{aligned}
|\xi^\alpha \partial_\xi^\beta \mathcal{F}^{-1}(\Delta^{m+1}\varphi)(\xi)| &\leq \left| \frac{1}{2\pi}\int_{\mathbf{R}} (\Delta^{m+1}\varphi)(x)(ix)^\beta \left(\frac{1}{i}\partial_x\right)^\alpha e^{ix\cdot\xi}dx \right| \\
&\leq \left| \frac{(-1)^\alpha i^{\beta-\alpha}}{2\pi}\int_{\mathbf{R}} x^\beta \partial_x^\alpha(\Delta^{m+1}\varphi)(x)e^{ix\cdot\xi}dx \right| \\
&\leq \int_{\mathbf{R}} \left| x^\beta \partial_x^{\alpha+2(m+1)}\varphi(x)\right| dx \\
&= \int_{\mathbf{R}} (1+|x|^2)\frac{1}{1+|x|^2}\left| x^\beta \partial_x^{\alpha+2(m+1)}\varphi(x)\right| dx \\
&= \int_{\mathbf{R}} \left\{ \left| x^\beta \partial_x^{\alpha+2(m+1)}\varphi(x)\right| + \left| x^{\beta+2} \partial_x^{\alpha+2(m+1)}\varphi(x)\right| \right\} \frac{1}{1+|x|^2}dx \\
&\leq C_{\delta,\rho}(\delta A)^\beta(\rho B)^{(\alpha+2(m+1))}\beta^{\beta r}(\alpha+2(m+1))^{(\alpha+2(m+1))r} \\
&\quad + C'_{\delta,\rho}(\delta A)^{\beta+2}(\rho B)^{\alpha+2(m+1)}(\beta+2)^{(\beta+2)r}(\alpha+2(m+1))^{(\alpha+2(m+1))r}
\end{aligned} \tag{4.8}$$

for any $\alpha, \beta \in \mathbf{Z}_+$. Since

$$(\beta+2)^{(\beta+2)r} \leq (2e)^{2r}(e^r)^\beta \beta^{\beta r}$$

and

$$(\alpha+2(m+1))^{(\alpha+2(m+1))r} \leq (2e)^{2(m+1)r}(e^r)^\alpha \alpha^{\alpha r}(m+1)^{2(m+1)r}$$

(in detail, see the proof of Proposition 5), we obtain by Stirling's formula,

$$\begin{aligned}
(4.8) &\leq C''_{\delta,\rho}(e^r\delta A)^\beta(e^r\rho B)^\alpha \beta^{\beta r}\alpha^{\alpha r}\{(\rho B)^2(2e)^{2r}\}^{m+1}(m+1)^{2(m+1)r} \\
&\leq C''_{\delta,\rho}(e^r\delta A)^\beta(e^r\rho B)^\alpha \beta^{\beta r}\alpha^{\alpha r}\{(\rho B)^2(2e^2)^{2r}\}^{m+1}\{(m+1)!\}^{2r}.
\end{aligned}$$

Therefore for any $\delta, \rho > 1$, there exists $C'''_{\delta,\rho} > 0$ such that

$$|\partial_\xi^\beta \mathcal{F}^{-1}(\Delta^m \varphi)(\xi)| \leq C'''_{\delta,\rho}(e^r\delta A)^\beta \beta^{\beta r} e^{-b_{e^r\rho}|\xi|^{1/r}} Q^{m+1}\{(m+1)!\}^{2r},$$

where $b_{e^r\rho} = r/(e(e^r\rho B)^{1/r})$ and $Q = (\rho B)^2(2e^2)^{2r}$. □

LEMMA 4. *Let* $\varphi \in \mathcal{S}^{r,B}_{r,A}(\mathbf{R})$, $r \geq 1/2$ *and* $t$ *in* $D_{\{A^2\theta/(4e^2)\}} = \{\mathrm{Re}\ t^{-1} > (A^2\theta/(4e^2))^{-1}\}$, $0 < \theta < 1$. *Then we have*

$$e^{-\theta t\xi^2}\mathcal{F}^{-1}(\Delta^{m+1}\varphi)(\xi)$$

*in* $\mathcal{S}^{r,A}_{r,b}(\mathbf{R})$, $r \geq 1/2$, *where* $b$ *appears in Lemma* 3. *Especially, we obtain the following estimate*;

$$\|e^{-\theta t\xi^2}\mathcal{F}^{-1}(\Delta^{m+1}\varphi)\|^r_{2e^r\delta, e^r\rho} \leq C_{\delta,\rho}Q^{m+1}\{(m+1)!\}^{2r}$$

*for some constant* $C_{\delta,\rho} > 0$ *and* $Q > 0$. $Q$ *is independent of* $t$ *and* $m$.

PROOF. For any $\varphi \in \mathcal{S}^{r,B}_{r,A}(\mathbf{R}_\xi)$, $r \geq 1/2$ and $t > 0$, we find that

$$e^{-\theta t\xi^2}\mathcal{F}^{-1}(\Delta^{m+1}\varphi)(\xi)$$

is a holomorphic on $\{\mathrm{Re}\ t > 0\}$. Therefore by Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}\left|\partial^\beta_\xi e^{-\theta t\xi^2}\mathcal{F}^{-1}(\Delta^{m+1}\varphi)(\xi)\right| &\leq \sum_{k=0}^{\beta}\binom{\beta}{k}\left|\partial^k_\xi e^{-\theta t\xi^2}\right|\left|\partial^{\beta-k}_\xi\mathcal{F}^{-1}(\Delta^{m+1}\varphi)(\xi)\right| \\ &\leq C_{\delta,\rho}\sum_{k=0}^{\beta}\binom{\beta}{k}(2e)^k k^{k/2}\left(\frac{|t|^2}{\theta p}\right)^{k/2}(e^r\delta A)^{\beta-k} \\ &\quad\times(\beta-k)^{(\beta-k)r}e^{-b_{e^r\rho}|\xi|^{1/r}}Q^{m+1}\{(m+1)!\}^{2r}\end{aligned} \tag{4.9}$$

for some constant $Q > 0$.

Since $\left(\frac{4e^2|t|^2}{\theta p}\right)^{1/2} < A$ for $t = p + iq$, we have

$$(4.9) \leq C_{\delta,\rho}(2e^r\delta A)^\beta\beta^{\beta r}e^{-b_{e^r\rho}|\xi|^{1/r}}Q^{m+1}\{(m+1)!\}^{2r}. \qquad \square$$

THE PROOF OF THEOREM 6. From the assumption, there exists $u \in (\mathcal{S}^r_r)'(\mathbf{R})$, $1/2 \leq r \leq 1$, such that $U(x,t) = (u * E)(x,t)$. For any $\varphi \in \mathcal{S}^r_r(\mathbf{R})$, $1/2 \leq r \leq 1$, we have for $t > 0$,

$$\begin{aligned}\langle U(x,t),\varphi\rangle &= \left\langle\mathcal{F}^{-1}\mathcal{F}U(x,t),\varphi\right\rangle \\ &= \left\langle\hat{U}(\xi,t),\mathcal{F}^{-1}\varphi(\xi)\right\rangle \\ &= \left\langle\hat{u}(\xi)e^{-t\xi^2},\mathcal{F}^{-1}\varphi(\xi)\right\rangle \\ &= \left\langle\hat{u}(\xi),e^{-t\xi^2}\mathcal{F}^{-1}\varphi(\xi)\right\rangle.\end{aligned}$$

By the Taylor's formula, we obtain that for $t > 0$,

$$\begin{aligned}\langle U(x,t),\varphi\rangle &- \left\langle u(x), \sum_{n=0}^{m}\frac{t^n}{n!}\Delta^n\varphi(x)\right\rangle \\ &= \left\langle \hat{u}(\xi), \frac{t^{m+1}}{m!}\int_0^1 (1-\theta)^m(-\theta\xi^2)^{m+1}e^{-\theta t\xi^2}d\theta\mathcal{F}^{-1}\varphi(\xi)\right\rangle \\ &= \left\langle \hat{u}(\xi), \frac{t^{m+1}}{m!}\int_0^1 (1-\theta)^m\theta^{m+1}e^{-\theta t\xi^2}d\theta\mathcal{F}^{-1}(\Delta^{m+1}\varphi)(\xi)\right\rangle \\ &= R_m(t)\,.\end{aligned} \tag{4.10}$$

Since the both hand side of (4.10) is holomorphic on $\{\mathrm{Re}\ t > 0\}$, the equality (4.10) holds on $\{\mathrm{Re}\ t > 0\}$.

Therefore by the continuity of $\hat{u}$ and Lemma 4, we obtain the following estimate for $R_m(t)$ on $D_{\{A^2\theta/(4e^2)\}}$;

$$\begin{aligned}|R_m(t)| &\le C\left\|\frac{t^{m+1}}{m!}\int_0^1(1-\theta)^m e^{-\theta t\xi^2}d\theta\mathcal{F}^{-1}(\Delta^{m+1}\varphi)(\xi)\right\|^r_{2e^r\delta,e^r\rho} \\ &\le C'Q^{m+1}|t|^{m+1}\{(m+1)!\}^{(2r-1)}\end{aligned}$$

for some constants $C' > 0$ and $Q > 0$ uniformly on $m$ and $t$. □

As a remark, the last inequality is also formed as $r > 1$ by Lemma 4.

By Theorem 5 and Theorem 6, we obtain the following result ((II) of Main Theorem):

COROLLARY 1. *Let $1/2 \le r \le 1$. For any $\varphi \in \mathcal{S}^r_r(\mathbf{R}^d)$, the asymptotic expansion*

$$\langle U(\cdot,t),\varphi\rangle \sim \sum_{k=0}^{\infty}\frac{t^k}{k!}\langle\Delta^k u,\varphi\rangle,\ u \in \left(\mathcal{S}^r_r\right)'(\mathbf{R}^d)\,,$$

*is Borel summable on $D_{\{\tilde{A}^2\theta/(4e^2)\}}$ and $\langle U(\cdot,t),\varphi\rangle$ is expressed by*

$$\langle U(\cdot,t),\varphi\rangle = \frac{1}{t}\int_0^\infty e^{-\zeta/t}f_B(\zeta)d\zeta$$

*in $D_{\{\tilde{A}^2\theta/(4e^2)\}}$, where $f_B$ is the Borel transform of the series $\sum_{k=0}^{\infty}\frac{t^k}{k!}\langle\Delta^k u,\varphi\rangle$.*

On the other hand, by Proposition 6, we also find that the asymptotic series converges for $r = 1/2$ as follows:

PROPOSITION 7. *Let $u \in \left(\mathcal{S}^{1/2}_{1/2}\right)'(\mathbf{R}^d)$. For any $\varphi \in \mathcal{S}^{1/2}_{1/2}(\mathbf{R}^d)$, the asymptotic series $\langle U(\cdot,t),\varphi\rangle \sim \sum_{k=0}^{\infty}\frac{t^k}{k!}\langle\Delta^k u,\varphi\rangle$ converges if $|t| < \dfrac{1}{(2d^{1/2}e\rho\tilde{B})^2}$.*

For example, let $u$ be the Dirac delta function and $\varphi = e^{-x^2}$. Then by replacing $\theta t$ in (4.6) to 1, we have

$$\begin{aligned} a_k &= \left\langle \Delta^k \delta, e^{-x^2} \right\rangle \\ &= \left\langle \delta, \Delta^k e^{-x^2} \right\rangle \\ &= \left\langle \delta, \frac{1}{\sqrt{\pi}} 2^{2k} e^{-x^2} \sum_{l=0}^{2k} x^l \int_{\mathbf{R}} e^{-y^2} y^{2k-l} dy \right\rangle \\ &= \frac{(-1)^k}{\sqrt{\pi}} \Gamma\left(k + \frac{1}{2}\right). \\ &= (-1)^k 2^k (2k-1)!!\,. \end{aligned}$$

So we have

$$a_k/k! = (-4)^k \frac{(2k-1)!!}{(2k)!!}$$

Therefore we find that $\sum_{k=0}^{\infty} \frac{t^k}{k!} \langle \Delta^k \delta, e^{-x^2} \rangle$ converges and

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \left\langle \Delta^k \delta, e^{-x^2} \right\rangle = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} (-4t)^k = \frac{1}{\sqrt{1+4t}}$$

if $|t| < 1/4$.

## 5. The solutions to the heat equation on phase space and Weyl transform

On the quantization from classical mechanics, H. Weyl introduced the following operator (see [8], [10]): For any $F \in \mathcal{S}(\mathbf{R}^{2d})$,

$$[\mathcal{W}(F)\varphi](\xi) = \iint_{\mathbf{R}^{2d}} F(x, y)[\pi(x, y)\varphi](\xi) dx dy, \ \varphi \in L^2(\mathbf{R}^d)\,,$$

where $[\pi(x, y)\varphi](\xi) = e^{i(x \cdot \xi + \frac{1}{2} x \cdot y)} \varphi(\xi + y)$. We call this transform $\mathcal{W}(F)$ the Weyl transform with symbol $F$. It is known that if the symbol $F \in L^2$, then $\mathcal{W}(F)$ is $L^2$-bounded (see [8], [10]). The Weyl transform $\mathcal{W}(F)$ is also expressed by using the matrix element of $\pi$: For any $\varphi, \ \psi \in L^2(\mathbf{R}^d)$,

$$\begin{aligned} (\mathcal{W}(F)\varphi, \psi) &= \iint_{\mathbf{R}^{2d}} F(x, y)(\pi(x, y)\varphi, \psi) dx dy \\ &= (2\pi)^d \iint_{\mathbf{R}^{2d}} F(x, y) V(\varphi, \psi)(x, y) dx dy\,, \end{aligned}$$

where $V(\varphi,\psi)(x,y)$ is the Fourier-Wigner transform of $\varphi$ and $\psi$ defined by

$$V(\varphi,\psi)(x,y) = (2\pi)^{-d}\int_{\mathbf{R}^d} e^{ix\cdot p}\varphi\left(p+\frac{y}{2}\right)\overline{\psi\left(p-\frac{y}{2}\right)}dp\,.$$

The Fourier-Wigner transform has the following property (see for example [5], [7]):

PROPOSITION 8. *Let* $\varphi,\psi\in\mathcal{S}_r^r(\mathbf{R}^d)$, $r\geq\frac{1}{2}$. *Then* $V(\varphi,\psi)\in\mathcal{S}_r^r(\mathbf{R}^{2d})$.

We define the Weyl transform with symbol $T\in(\mathcal{S}_r^r)'(\mathbf{R}^{2d})$ by

$$\langle\mathcal{W}(T)\varphi,\psi\rangle = (2\pi)^d\left\langle T, V(\varphi,\overline{\psi})\right\rangle,\ \varphi,\psi\in\mathcal{S}_r^r(\mathbf{R}^d)\,.$$

It follows from Proposition 8 that this definition is well defined. The Weyl transform $\mathcal{W}(T)$ has the following property (see [7]):

PROPOSITION 9. *Let* $r\geq 1/2$. *Then the map* $\mathcal{W}$ *from* $\mathcal{S}(\mathbf{R}^{2d})$ *to the space of bounded operators on* $L^2(\mathbf{R}^d)$ *extends uniquely to a bijection from* $(\mathcal{S}_r^r)'(\mathbf{R}^{2d})$, $r\geq 1/2$, *to the space of continuous linear maps from* $\mathcal{S}_r^r(\mathbf{R}^d)$ *to* $(\mathcal{S}_r^r)'(\mathbf{R}^d)$.

We consider the heat equations on phase space:

$$\left(\frac{\partial}{\partial\hbar}-\Delta_{x,y}\right)U(x,y,\hbar)=0,\ x,y\in\mathbf{R}^d\,,$$

where $\hbar>0$ is the Plank's constant. In this setting the Fourier-Wigner transform and the Weyl transform are given by

$$V_\hbar(\varphi,\psi)(x,y)=\left(\frac{1}{2\pi\hbar}\right)^d\int_{\mathbf{R}^d}e^{\frac{ip\cdot x}{\hbar}}\varphi\left(p+\frac{y}{2}\right)\overline{\psi}\left(p-\frac{y}{2}\right)dp$$

and for any $F_\hbar\in\left(\mathcal{S}_r^r\right)'(\mathbf{R}^{2d})$, $r\geq 1/2$,

$$\left\langle\mathcal{W}_\hbar(F_\hbar)\varphi,\psi\right\rangle=(2\pi\hbar)^d\left\langle F_\hbar, V_\hbar(\varphi,\overline{\psi})\right\rangle,\ \varphi,\psi\in\mathcal{S}_r^r(\mathbf{R}^d)\,.$$

As a remark, as $\hbar=1$, the operator $\mathcal{W}_1(F_1)$ coincides with the normal Weyl transform $\mathcal{W}(F)$. Moreover for any $\hbar>0$, Proposition 8 and Proposition 9 hold.

Especially, we deal with the operator $\tilde{\mathcal{W}}_\hbar(F)$ defined by for any $F\in\left(\mathcal{S}_r^r\right)'(\mathbf{R}^{2d})$, $r\geq 1/2$,

$$\left\langle\tilde{\mathcal{W}}_\hbar(F)\varphi,\psi\right\rangle=\left\langle\mathcal{W}_\hbar(F_\hbar)\varphi,\psi\right\rangle=(2\pi\hbar)^d\left\langle(8\pi\hbar)^{-d}F, V_\hbar(\varphi,\overline{\psi})\right\rangle,\ \varphi,\psi\in\mathcal{S}_r^r(\mathbf{R}^d)\,.$$

PROPOSITION 10. *Let* $F(\xi,\eta)\in\left(\mathcal{S}_r^r\right)'(\mathbf{R}^{2d})$, $r\geq 1/2$ *and* $\varphi(p)=\psi(p)=(4\pi\hbar)^{-d/4}e^{-\frac{1}{2\hbar}p^2}$ $(p\in\mathbf{R}^d)$. *Then we have*

$$\left\langle\tilde{\mathcal{W}}_\hbar(F(x-\cdot,y-\cdot))\varphi,\psi\right\rangle\to 4^{-d}F\ in\ \left(\mathcal{S}_r^r\right)'(\mathbf{R}^{2d})\ as\ \hbar\to 0+\,. \tag{5.1}$$

PROOF. From the assumption, we have

$$\begin{aligned} V_\hbar(\varphi, \psi)(\xi, \eta) &= (4\pi\hbar)^{-d/2}(2\pi\hbar)^{-d}\int_{\mathbf{R}^d} e^{\frac{ip\cdot\xi}{\hbar}} e^{-1/2\hbar(2p^2+\frac{\eta^2}{2})}dp \\ &= (4\pi\hbar)^{-d/2}e^{-\frac{\eta^2}{4\hbar}}(2\pi)^{-d}\int_{\mathbf{R}^d} e^{i\xi\cdot s}e^{-\hbar s^2}ds \\ &= (4\pi\hbar)^{-d}e^{-\frac{\xi^2+\eta^2}{4\hbar}}\,. \end{aligned}$$

Therefore we obtain that

$$V_\hbar(\varphi, \psi)(\xi, \eta) = E(\xi, \eta, \hbar)\,.$$

By Proposition 4, we have for any $f \in \mathcal{S}_r^r(\mathbf{R}^{2d})$,

$$\begin{aligned} \big\langle\big\langle(\tilde{\mathcal{W}}_\hbar(F(x-\cdot, y-\cdot))\varphi, \psi\big\rangle, f\big\rangle &= \big\langle(2\pi\hbar)^d\big\langle(8\pi\hbar)^{-d}(F(x-\cdot, y-\cdot), V_\hbar(\varphi, \psi)\big\rangle, f\big\rangle \\ &= \big\langle 4^{-d}F * E, f\big\rangle \\ &\to 4^{-d}\big\langle F, f\big\rangle \text{ in } \big(\mathcal{S}_r^r\big)'(\mathbf{R}^{2d}) \end{aligned}$$

as $h \to 0+$. □

We have the following asymptotic expansions more strictly than (5.1):

THEOREM 7. *Let* $F(\xi, \eta) \in \big(\mathcal{S}_r^r\big)'(\mathbf{R}^{2d})$, $r \geq 1/2$ *and* $\varphi(p) = \psi(p) = (4\pi\hbar)^{-d/4}e^{-\frac{1}{2\hbar}p^2}$ $(p \in \mathbf{R}^d)$. *Then we have*

$$\big\langle\tilde{\mathcal{W}}_\hbar(F(x-\xi, y-\eta))\varphi, \psi\big\rangle \sim 4^{-d}\sum_{n=0}^{\infty}\frac{\hbar^n}{n!}\Delta^n F\,.$$

PROOF. We immediately obtain this result from (I) of Theorem 3. □

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