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Birational Maps of Moduli Spaces of Vector Bundles on $K3$ Surfaces

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Abstract. In this note, we construct a birational map of a moduli space of stable sheaves on a $K3$ surface induced by a reflection functor.

0. Introduction

Let X be a $K3$ surface defined over $\mathbb{C}$ and H an ample line bundle on X. Let $(H^*(X,\mathbb{Z}), \langle \ , \ \rangle)$ be the Mukai lattice of X: for $x^i = (r^i, \xi^i, a^i) \in H^*(X,\mathbb{Z})$, $i = 1, 2$,

$$\langle x^1, x^2 \rangle := (\xi^1, \xi^2) - r^1 a^2 - a^1 r^2 \in \mathbb{Z}.$$

For a coherent sheaf E on X,

$$\begin{aligned} v(E) :=& \operatorname{ch}(E)\sqrt{\operatorname{td}_X} \\ =& (\operatorname{rk} E, c_1(E), \chi(E) - \operatorname{rk}(E)) \in H^*(X,\mathbb{Z}) \end{aligned}$$

is the Mukai vector of E, where td_X is the Todd class of X and we identify $H^4(X,\mathbb{Z})$ with $\mathbb{Z}$. We denote the moduli space of stable sheaves E of $v(E) = v$ by $M_H(v)$. If H is general and v is primitive, then $M_H(v)$ is a smooth projective scheme.

DEFINITION 0.1. For an object $\mathcal{E} \in \mathbf{D}(X \times X)$, we define an integral functor

$$\begin{array}{cccc} \Phi_{\mathcal{E}}: & \mathbf{D}(X) & \to & \mathbf{D}(X) \\ & x & \mapsto & \mathbf{R}p_{2*}(\mathcal{E} \otimes p_1^*(x)), \end{array} \tag{0.1}$$

where $p_1, p_2 : X \times X \to X$ are projections. The Fourier-Mukai transform of X is an equivalence $\mathbf{D}(X) \to \mathbf{D}(X)$ of this form $\Phi_{\mathcal{E}}$.

Let I_Δ be the ideal of the diagonal $\Delta \subset X \times X$. Then we have the Fourier-Mukai transform Φ_{I_Δ} whose inverse is given by $\Phi_{I_\Delta^*}[2] : \mathbf{D}(X) \to \mathbf{D}(X)$ with

$$\Phi_{I_\Delta^*}(x) := \mathbf{R}\operatorname{Hom}_{p_2}(I_\Delta, p_1^*(x)), x \in \mathbf{D}(X), \tag{0.2}$$

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where $I_\Delta^* = \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X\times X}}(I_\Delta, \mathcal{O}_{X\times X})$. The Fourier-Mukai transform $\Phi_{\mathcal{E}}$ induces an isometry $\Phi_{\mathcal{E}}^H$ of the Mukai lattice and we have a commutative diagram:

$$\begin{array}{ccc} \mathbf{D}(X) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathbf{D}(X) \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v} \\ H^*(X,\mathbb{Z}) & \xrightarrow{\Phi_{\mathcal{E}}^H} & H^*(X,\mathbb{Z}) \end{array} \tag{0.3}$$

If $\mathcal{E} = I_\Delta$, then $-\Phi_{\mathcal{E}}^H$ coincides with the reflection by the (-2)-vector $v(\mathcal{O}_X) = (1, 0, 1)$:

$$-\Phi_{\mathcal{E}}^H((r, \xi, a)) = (a, -\xi, r) = x + \langle x, v(\mathcal{O}_X)\rangle v(\mathcal{O}_X)\,, \tag{0.4}$$

where $x = (r, \xi, a)$.

Let E be a stable sheaf on X with $v(E) = v$. Assume that there is an integer i such that

(a) $H^i(\Phi_{I_\Delta^*}(E))$ is a stable sheaf.
(b) $H^j(\Phi_{I_\Delta^*}(E)) = 0$ for $j \neq i$.

Then we have a rational map $M_H(v) \cdots \to M_H(w)$ which becomes birational by the properties of the Fourier-Mukai transform, where $w = v(F)$. In this note, we give some conditions for E to satisfy (a) and (b).

THEOREM 0.1. *Let X be a $K3$ surface with $\operatorname{Pic}(X) = \mathbb{Z}H$. Let $v = (r, dH, a)$ be the Mukai vector of a coherent sheaf with $\langle v^2\rangle = d^2(H^2) - 2ra > 0$.*

(1) *Assume that $a \leq 0$.*
  (a) *If $r + a \geq 0$, then $\Phi_{I_\Delta[1]}$ induces a birational map*
$$M_H(r, dH, a) \cdots \to M_H(-a, dH, -r)\,.$$
  (b) *If $r + a \leq 0$, then $\Phi_{I_\Delta^*[1]}$ induces a birational map*
$$M_H(r, dH, a) \cdots \to M_H(-a, dH, -r)\,.$$
(2) *Assume that $a = 0, 1$, then $\mathcal{D} \circ \Phi_{I_\Delta}$ induces a birational map*
$$M_H(r, dH, a) \cdots \to M_H(a, dH, r)$$
unless $(H^2) = 2$ and $v = (2d-1, dH, 1)$, $d \geq 2$, where $\mathcal{D}(E) := \mathbf{R}\,\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$, $E \in \mathbf{D}(X)$.
(3) *If $(H^2) = 2$, then there is an auto-equivalence $\Phi : \mathbf{D}(X) \to \mathbf{D}(X)$ such that $\mathcal{D} \circ \Phi$ induces a birational map*
$$M_H(2d-1, dH, 1) \cdots \to M_H(1, dH, 2d-1),\ d \geq 2\,.$$

COROLLARY 0.2. *Let (X, H) be a pair of a $K3$ surface X and an ample divisor H on X. Let $v = (r, dH, a)$ be the Mukai vector of a coherent sheaf with $\langle v^2\rangle = d^2(H^2) - 2ra > 0$.*

If $a \le 1$ *and* $\gcd(r, d(H^2), a) = 1$, *then we have a birational map* $M_H(r, dH, a) \cdots \to M_H(a, dH, r)$.

PROOF. We first assume that $a \le 0$. We take a flat family $(\mathcal{X}, \mathcal{H}) \to S$ of polarized $K3$ surfaces over a smooth curve S such that $(\mathcal{X}, \mathcal{H})_{s_0} = (X, H)$, $s_0 \in S$ and $\mathrm{Pic}(\mathcal{X}_{s_1}) = \mathbb{Z}\mathcal{H}_{s_1}$, $s_1 \in S$. Then we have flat families $\mathcal{M}_i \to S$, $i = 1, 2$ of moduli spaces where $\mathcal{M}_1 := \mathcal{M}_{\mathcal{H}}(r, d\mathcal{H}, a)$ and $\mathcal{M}_2 := \mathcal{M}_{\mathcal{H}}(-a, d\mathcal{H}, -r)$. By our assumption, they are smooth and projective families. By the openness of the stability condition, the Fourier-Mukai transform induces a birational map $f : \mathcal{M}_1 \cdots \to \mathcal{M}_2$. Then [4, Theorem 4.3] implies the claim. □

REMARK 0.1. Related results are obtained by Zuo [17], Ballico and Chiantini [1], Nakashima [8] and Costa [2].

It is conjectured that an irreducible symplectic manifold M is birationally equivalent to an irreducible symplectic manifold with a Lagrangean fibration, if there is a line bundle which is isotropic with respect to Beauville bilinear form (cf. [3], [5], [9]). The following corollary supports this conjecture.

COROLLARY 0.3. *Let* (X, H) *be a pair of a* $K3$ *surface* X *and an ample divisor* H *on* X. *If* $\gcd(r, d(H^2)) = 1$, *then* $M_H(r, dH, 0)$, $d > 0$ *is birationally equivalent to a holomorphic symplectic manifold with a Lagrangean fibration.*

1. Preliminaries

Let $\mathcal{M}(v)$ be the moduli stack of coherent sheaves E on X with $v(E) = v$. Let $\mathcal{M}_H(v)^{ss}$ (resp. $\mathcal{M}_H(v)^s$) be the open substack of $\mathcal{M}(v)$ consisting of H-semi-stable sheaves (resp. H-stable sheaves). From now on, we assume that $\mathrm{Pic}(X) = \mathbb{Z}H$. Then, H is a general polarization, that is,

$$\frac{(c_1(F), H)}{\mathrm{rk}\, F} = \frac{(c_1(E), H)}{\mathrm{rk}\, E} \text{ if and only if } \frac{c_1(F)}{\mathrm{rk}\, F} = \frac{c_1(E)}{\mathrm{rk}\, E} \tag{1.1}$$

for any subsheaf F of a μ-semi-stable sheaf E with $v(E) = v$.

PROPOSITION 1.1. *Let* $\mathcal{M}$ *be an irreducible component of* $\mathcal{M}(v)$. *Then* $\dim \mathcal{M} \ge \langle v^2 \rangle + 1$.

PROOF. The claim is an easy consequence of the deformation theory of a coherent sheaf. For a proof, see the proof of [13, Prop. 3.4]. □

For the open substack $\mathcal{M}_H(v)^{ss}$, we have $\dim \mathcal{M}_H(v)^{ss} = \langle v^2 \rangle + 1$. Moreover we have the following claims.

THEOREM 1.2. [13, Thm. 0.1, Prop. 3.4], [15, Cor. 3.5] *Assume that* $\langle v^2 \rangle > 0$. *Then*

(1) $\mathcal{M}_H(v)^{ss}$ *is an irreducible normal stack of* $\dim \mathcal{M}_H(v)^{ss} = \langle v^2 \rangle + 1$.

(2) $\mathcal{M}_H(v)^s$ *is an open dense substack of* $\mathcal{M}_H(v)^{ss}$.

DEFINITION 1.1. For $v = (r, dH, a) \in \mathbb{Q} \oplus \mathbb{Q}H \oplus \mathbb{Q}$, we set $v \geq 0$, if (i) $r > 0$, or (ii) $r = 0$ and $d > 0$ or (iii) $r = d = 0$ and $a \geq 0$. If $v - w \geq 0$, then we write $v \geq w$.

DEFINITION 1.2. For $v_i := (r_i, d_i H, a_i)$, $1 \leq i \leq s$ with $v_1/r_1 \geq v_2/r_2 \geq \cdots \geq v_s/r_s$, let $\mathcal{F}^{HN}(v_1, v_2, \dots, v_s)$ be the substack of $\mathcal{M}(v)$ whose element F has the Harder-Narasimhan filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F \tag{1.2}$$

such that $v_i = v(F_i/F_{i-1})$, $i = 1, 2, \dots, s$.

By the properties of Harder-Narasimhan filtration and the Serre duality,

$$\operatorname{Ext}^2(F_j/F_{j-1}, F_i/F_{i-1}) = \operatorname{Hom}(F_i/F_{i-1}, F_j/F_{j-1})^\vee = 0,\ i < j\,. \tag{1.3}$$

Then the following lemma holds (cf. [16, Lemma, 5.3]).

LEMMA 1.3.

$$\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) = \sum_{i<j} \langle v_j, v_i \rangle + \sum_{i \geq 1} \dim \mathcal{M}_H(v_i)^{ss}\,. \tag{1.4}$$

LEMMA 1.4. *Let $v = lv'$ be a Mukai vector such that $l > 0$ and v' is primitive. Then*

$$\dim \mathcal{M}_H(v)^{ss} \leq \langle v^2 \rangle + l^2\,. \tag{1.5}$$

PROOF. We note that

$$\dim \operatorname{Ext}^2(E, E) = \dim \operatorname{Hom}(E, E) \leq l^2 \tag{1.6}$$

for $E \in \mathcal{M}_H(v)^{ss}$. Hence $\dim \mathcal{M}_H(v)^{ss} \leq \langle v^2 \rangle + l^2$ by the deformation theory of a coherent sheaf. □

1.1. Brill-Noether locus. We set $v := (r, dH, a)$, $r \geq 0$, $d > 0$, $a \leq 0$. Let E be a stable sheaf with $v(E) = v$. Then $\chi(E) = r + a$. By the stability of E and $d > 0$, Serre duality implies that $H^2(X, E) = \operatorname{Hom}(E, \mathcal{O}_X)^\vee = 0$.

DEFINITION 1.3. We set

$$\mathcal{M}_H(v)_0^s := \{E \in \mathcal{M}_H(v)^s | H^0(X, E) = 0\}\,. \tag{1.7}$$

By the Brill-Noether theory, it is expected that $\mathcal{M}_H(v)_0^s \neq \emptyset$ if $r + a \leq 0$. In this subsection, we shall prove this expectation is true.

PROPOSITION 1.5. *Let $v = (r, dH, a)$ be a Mukai vector such that $r \geq 0$, $d > 0$ and $r + a \leq 0$. Then $\mathcal{M}_H(v)_0^s \neq \emptyset$.*

Before proving this proposition, we shall explain that $\Phi_{I_\Delta^*}(E)[1]$ is a coherent sheaf defined as the universal extension of E by $\mathcal{O}_X$ for $E \in \mathcal{M}_H(v)_0^s$. Assume that $r + a \leq 0$.

For $E \in \mathcal{M}_H(v)_0^s$, we consider the Fourier-Mukai transform $\Phi_{I_\Delta^*}(E)$. By using the exact sequence

$$0 \to I_\Delta \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0\,, \tag{1.8}$$

we have an exact sequence

(1.9)

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_{p_2}(\mathcal{O}_\Delta, p_1^*(E)) \longrightarrow H^0(X,E)\otimes\mathcal{O}_X \longrightarrow \mathrm{Hom}_{p_2}(I_\Delta, p_1^*(E)) \\ \longrightarrow \mathrm{Ext}^1_{p_2}(\mathcal{O}_\Delta, p_1^*(E)) \longrightarrow H^1(X,E)\otimes\mathcal{O}_X \longrightarrow \mathrm{Ext}^1_{p_2}(I_\Delta, p_1^*(E)) \\ \longrightarrow \mathrm{Ext}^2_{p_2}(\mathcal{O}_\Delta, p_1^*(E)) \longrightarrow H^2(X,E)\otimes\mathcal{O}_X \longrightarrow \mathrm{Ext}^2_{p_2}(I_\Delta, p_1^*(E)) \longrightarrow 0\,. \end{array}$$

Since $\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_{X\times X}) = \mathcal{O}_\Delta[-2]$, we have

$$\mathrm{Ext}^i_{p_2}(\mathcal{O}_\Delta, p_1^*(E)) = Rp_{2*}^{i-2}(\mathcal{O}_\Delta \otimes p_1^*(E)) = \begin{cases} E, & i=2 \\ 0, & i \neq 2\,. \end{cases} \tag{1.10}$$

Since $H^i(X,E) = 0$ for $i \neq 1$, we see that $H^i(\Phi_{I_\Delta^*}(E)) = 0$ for $i \neq 1$ and $F := H^1\,(\Phi_{I_\Delta^*}(E))$ fits in an exact sequence

$$0 \to H^1(X,E)\otimes\mathcal{O}_X \to F \to E \to 0\,. \tag{1.11}$$

Since $\Phi_{I_\Delta^*}(\mathcal{O}_X) = \mathcal{O}_X$, we have

$$\mathrm{Hom}(F,\mathcal{O}_X) = \mathrm{Hom}(\Phi_{I_\Delta^*}(E)[1], \Phi_{I_\Delta^*}(\mathcal{O}_X)) = \mathrm{Hom}(E[1],\mathcal{O}_X) = 0\,. \tag{1.12}$$

By Lemma 3.1, (1.11) is the universal extension of E by $\mathcal{O}_X$. In the next section, we shall prove that F is stable for a general E. Then we have a rational map $M_H(v)\cdots\to M_H(w)$ which becomes birational by the properties of the Fourier-Mukai transform, where $w = v(F)$. Thus we get Theorem 0.1 (1) for $r + a \leq 0$.

PROOF OF PROPOSITION 1.5. We first treat the case where $r = 0$. In this case, we can take a smooth curve $C \in |dH|$. Then it is easy to find a line bundle L on C with $H^0(C,L) = 0$ and $\dim H^1(C,L) = a$. Since C is reduced and irreducible, L is stable. Thus the claim holds.

We next treat the case where $r > 0$. We start with a special case.

LEMMA 1.6. *Let $v = (r, dH, a)$ be a Mukai vector such that $r > 0$, $d > 0$, $(r,d) = 1$ and $r + a \leq 0$. Then $\mathcal{M}_H(v)_0^s \neq \emptyset$.*

PROOF. We shall prove our claim by induction on r. (I) Assume that $r = 1$. Then $\mathcal{M}_H(v)^s$ consists of $I_Z(dH)$, where I_Z is the ideal sheaf of a 0-dimensional subscheme of length $\langle v^2\rangle/2+1$, that is, I_Z belongs to $\mathrm{Hilb}_X^{\langle v^2\rangle/2+1}$. Since $\chi(I_Z(dH)) = 1+a \leq 0$, we have $H^0(X, I_Z(dH)) = 0$ for a general I_Z. Moreover the same assertion also holds for $d = 0$.

(II) Let (r_1, d_1) be a pair of integers such that $d_1 r - d r_1 = 1$ and $0 < r_1 < r$. We set $(r_2, d_2) := (r - r_1, d - d_1)$. Then $d_1 > 0$ and $d - d_1 \geq 0$. Moreover if $d - d_1 = 0$, then

$r - r_1 = 1$. We shall choose Mukai vectors $v_i := (r_i, d_i H, a_i)$, $i = 1, 2$ such that $r_i + a_i \leq 0$, $i = 1, 2$. We shall choose $E_i \in \mathcal{M}_H(v_i)_0$, $i = 1, 2$. Then $H^0(X, E_1 \oplus E_2) = 0$. We shall prove that $E_1 \oplus E_2$ deforms to a stable sheaf. We set

$$\mathcal{M}(v)' := \mathcal{M}_H(v)^{ss} \cup \cup_{b \geq 0} \mathcal{F}^{HN}(v_1 - b\omega, v_2 + b\omega)\,,$$

where $\omega = (0, 0, 1)$. We first prove that $\mathcal{M}(v)'$ is an open substack of $\mathcal{M}(v)$.

Proof of the claim: If $E \in \mathcal{F}^{HN}(v_1 - b\omega, v_2 + b\omega)$ belongs to the closure of $\mathcal{F}^{HN}(u_1, u_2, \dots, u_s)$, then the Harder-Narasimhan polygon of $u_1, u_2, \dots, u_s$ is contained in the Harder-Narasimhan polygon of $v_1 - b\omega, v_2 + b\omega$. Then we see that $s = 2$ and $u_1 = v_1 - b'\omega$, $b' \geq b$. Therefore the claim holds.

We shall prove that

$$\dim \mathcal{F}^{HN}(v_1 - b\omega, v_2 + b\omega) < \langle v^2 \rangle + 1\,. \tag{1.13}$$

Since every irreducible component of $\mathcal{M}(v)$ is at least of dimension $\langle v^2 \rangle + 1$ (Prop. 1.1) and $\mathcal{M}_H(v)^{ss}$ is irreducible, (1.13) implies that $\mathcal{M}(v)'$ is also irreducible. Since $E_1 \oplus E_2$ belongs to $\mathcal{M}(v)'$, we get our claim $\mathcal{M}_H(v)_0^s \neq \emptyset$.

Proof of (1.13):

We shall first estimate $\dim \mathcal{F}^{HN}(v_1 - b\omega, v_2 + b\omega)$.

$$\begin{aligned} &\dim \mathcal{F}^{HN}(v_1 - b\omega, v_2 + b\omega) \\ =&\dim \mathcal{M}_H(v_1 - b\omega)^{ss} + \dim \mathcal{M}_H(v_2 + b\omega)^{ss} + \langle v_1 - b\omega, v_2 + b\omega \rangle \\ =&\langle (v_1 - b\omega)^2 \rangle + \langle (v_2 + b\omega)^2 \rangle + \langle v_1 - b\omega, v_2 + b\omega \rangle + 2 \end{aligned} \tag{1.14}$$

Hence

(1.15)
$$\begin{aligned} (\langle v^2 \rangle + 1) - \dim \mathcal{F}^{HN}(v_1 - b\omega, v_2 + b\omega) =&\langle v_1 - b\omega, v_2 + b\omega \rangle - 1 \\ =&d_1 d_2 (H^2) - r_2 a_1 - r_1 a_2 + (r_2 - r_1) b - 1. \end{aligned}$$

We note that $a_1 + a_2 \leq -r_1 - r_2 = -r$ and $a_2 + b \leq (d_2^2 (H^2) + 2)/2r_2$. If $r_1 \geq r_2$, then we see that

(1.16)
$$\begin{aligned} \langle v^2 \rangle + 1 - \dim \mathcal{F}^{HN}(v_1 - b\omega, v_2 + b\omega) &= d_1 d_2 (H^2) - r_2 (a_1 + a_2) \\ &\quad - (r_1 - r_2)(a_2 + b) - 1 \\ &\geq d_1 d_2 (H^2) + r_2 r - (r_1 - r_2) \frac{d_2^2 (H^2) + 2}{2r_2} - 1 \\ &= d_1 d_2 (H^2) \left(1 - \frac{(r_1 - r_2)}{2r_2} \frac{d_2}{d_1} \right) + r_2 r - \frac{r_1}{r_2} \\ &> d_1 d_2 \frac{r}{2r_1} (H^2) + r_2 r - \frac{r_1}{r_2} > 0, \end{aligned}$$

where we used the inequality $d_1/r_1 > d_2/r_2$. If $r_1 \le r_2$, then since $a_1 \le -r_1$, we see that

$$\begin{aligned}\langle v^2\rangle + 1 - \dim \mathcal{F}^{HN}(v_1 - b\omega, v_2 + b\omega) &= d_1 d_2 (H^2) - (r_2 - r_1)(a_1 - b) \\ &\quad - r_1(a_1 + a_2) - 1 \\ &\ge d_1 d_2 (H^2) + (r_2 - r_1) r_1 + r_1 r - 1 > 0 . \quad \square\end{aligned}$$

By using Lemma 1.6, we treat the general case. We set $v := (lr', ld'H, a)$, where $l := \gcd(r, d)$. We choose integers $a_1, a_2, \ldots, a_l$ such that $\sum_{i=1}^{l} a_i = a$ and $r' + a_i \le 0$ for $1 \le i \le l$. We set $v_i := (r', d'H, a_i)$. By Lemma 1.6, $\mathcal{M}_H(v_i)_0^s \ne \emptyset$, $1 \le i \le l$. We choose elements $E_i \in \mathcal{M}_H(v_i)_0^s$, $1 \le i \le l$ and set $E := \oplus_{i=1}^{l} E_i$. Then E is μ-semi-stable and $H^0(X, E) = 0$. Since $\langle v^2\rangle \ge 2l^2$, [11, Lem. 4.4] implies that our proposition holds. $\square$

2. Proof of Theorem 0.1

2.1. Estimates on the Mukai pairing. In order to estimate the dimension of the loci of unstable sheaves, we prepare some estimates of the Mukai pairing.

LEMMA 2.1. *Let $v_1 := (r_1, d_1 H, a_1)$, $r_1 > 0$, and $v_2 := (r_2, d_2 H, a_2)$, $r_2 > 0$ be Mukai vectors such that*

$$d_1/r_1 \ge d_2/r_2 > 0 . \tag{2.1}$$

We set $l := \gcd(r_2, d_2, a_2)$. Assume that $a_1 \le 0$, $a_1 + a_2 \le 0$ and $\langle v_2^2\rangle \ge -2l^2$. Then

$$\langle v_1, v_2\rangle - 1 > 0 . \tag{2.2}$$

Moreover, if $\langle v_2^2\rangle \le 0$, then

$$\langle v_1, v_2\rangle - l^2 > 0 . \tag{2.3}$$

PROOF. Assume that $\langle v_2^2\rangle > 0$. Then $a_2 < r_2 d_2^2 (H^2)/2$. By our assumption, we have $d_1 \ge r_1 d_2/r_2$. If $r_1 \ge r_2$, then we see that

$$\begin{aligned}\langle v_1, v_2\rangle - 1 &= d_1 d_2 (H^2) - (r_1 - r_2) a_2 - r_2(a_1 + a_2) - 1 \\ &\ge d_1 d_2 (H^2) - \frac{(r_1 - r_2) d_2^2}{r_2} \frac{(H^2)}{2} - 1 \\ &\ge d_2^2 \frac{r_1 + r_2}{2 r_2} (H^2) - 1 \\ &\ge d_2^2 (H^2) - 1 > 0 .\end{aligned} \tag{2.4}$$

If $r_1 < r_2$, then

$$\begin{aligned}\langle v_1, v_2\rangle - 1 &= d_1 d_2 (H^2) - (r_2 - r_1) a_1 - r_1(a_1 + a_2) - 1 \\ &\ge d_1 d_2 (H^2) - 1 > 0 .\end{aligned} \tag{2.5}$$

If $\langle v_2^2\rangle \leq 0$, then we set $v_2 = l(r_2', d_2'H, a_2')$. Then a_2' satisfies the inequality

$$\frac{(d_2')^2(H^2)}{2r_2'} \leq a_2' \leq \frac{(d_2')^2(H^2)+2}{2r_2'} . \tag{2.6}$$

Since

$$\langle v_1, v_2\rangle - l^2 = l(d_1d_2'(H^2) - (r_2'a_1 + r_1a_2') - l) , \tag{2.7}$$

we shall prove that

$$d_1d_2'(H^2) - (r_2'a_1 + r_1a_2') > l . \tag{2.8}$$

$$\begin{aligned} d_1d_2'(H^2) - (r_2'a_1 + r_1a_2') &\geq d_1d_2'(H^2) - (-r_2'la_2' + r_1a_2') \\ &\geq d_1d_2'(H^2) - r_1a_2' + r_2'a_2'l \\ &= d_1d_2'\left((H^2) - \frac{r_1}{d_1d_2'}a_2'\right) + r_2'a_2'l \\ &= d_1d_2'\left((H^2) - \frac{r_1}{d_1d_2'}a_2'\right) + d_2'^2\frac{(H^2)}{2}l . \end{aligned} \tag{2.9}$$

By using (2.1) and the inequality (2.6), we see that

$$\begin{aligned} (H^2) - \frac{r_1}{d_1d_2'}a_2' &\geq (H^2) - \frac{r_2'a_2'}{(d_2')^2} \\ &= \frac{1}{(d_2')^2}\left((d_2')^2(H^2) - r_2'a_2'\right) \\ &= \frac{1}{(d_2')^2}\left(\frac{(d_2')^2(H^2)}{2} + \frac{1}{2}((d_2')^2(H^2) - 2r_2'a_2')\right) \\ &\geq \frac{1}{(d_2')^2}\left(\frac{(d_2')^2(H^2)}{2} - 1\right) \geq 0 . \end{aligned} \tag{2.10}$$

If $d_1d_2'(H^2) - (r_2'a_1 + r_1a_2') = l$, then we have $r_2'a_2' = (d_2')^2(H^2)/2 = 1$. Thus $r_2' = a_2' = d_2' = (H^2)/2 = 1$. Since $d_1/r_1 \geq d-2'/r_2' = 1$, $d_1d_2'(H^2) - (r_2'a_1 + r_1a_2') = 2d_1 - r_1 + l > l$, which is a contradiction. Therefore we get (2.8). □

LEMMA 2.2. *Let $v_1 := (r_1, d_1H, a_1), r_1 > 0$ and $v_2 := (r_2, d_2H, a_2), r_2 > 0$ be Mukai vectors. Assume that $a_1 \leq 0$, $a_1 + a_2 = 1$ and $d_1/r_1 > d_2/r_2 > 0$.*

(1) *If $\langle v_2^2\rangle \geq -2$, then $\langle v_1, v_2\rangle - 1 > 0$, unless $(H^2) = 2$, $v_1 = (2d_1 - 1, d_1H, 0)$ and $v_2 = (2, H, 1)$.*

(2) *If $l := \gcd(r_2, d_2, a_2) \geq 2$ and $-2l^2 \leq \langle v_2^2\rangle \leq 0$, then $\langle v_1, v_2\rangle - l^2 > 0$.*

PROOF. (1) (i) We first assume that $a_2 \geq 2$. If $r_1 \geq r_2$, then

$$
\begin{aligned}
\langle v_1, v_2\rangle - 1 &= d_1 d_2 (H^2) - r_2 a_1 r_1 a_2 \\
&= d_1 d_2 (H^2) - (r_2 - r_1) a_2 - r_2 (a_1 + a_2) - 1 \\
&\geq d_1 d_2 (H^2) \left(1 - \frac{(r_1 - r_2)}{2 r_2} \frac{d_2}{d_1}\right) - r_2 - \frac{r_1}{r_2} \\
&\geq d_1 d_2 \frac{r_1 + r_2}{2 r_1} (H^2) - r_2 - \frac{r_1}{r_2} \\
&= \left(d_1 d_2 \frac{(H^2)}{2} - \frac{d_1}{d_2}\right) + \left(\frac{d_1 d_2 r_2 (H^2)}{2 r_1} - r_2\right) \\
&> \frac{d_2^2 (H^2)}{2} - r_2 \\
&\geq (a_2 - 1) r_2 - 1 \geq 0 .
\end{aligned} \tag{2.11}
$$

If $r_1 < r_2$, then

$$
\begin{aligned}
\langle v_1, v_2\rangle - 1 &= d_1 d_2 (H^2) - (r_1 - r_2) a_2 - r_2 (a_1 + a_2) - 1 \\
&\geq d_1 d_2 (H^2) - 2 r_1 + r_2 - 1 \\
&> \frac{r_1}{r_2} d_2^2 (H^2) + r_2 - 2 r_1 - 1 \\
&\geq \frac{r_1}{r_2} (4 r_2 - 2) + r_2 - 2 r_1 - 1 \\
&= \frac{2 (r_2 - 1) r_1 + r_2 (r_2 - 1)}{r_2} \geq 0 .
\end{aligned} \tag{2.12}
$$

(ii) We next treat the case of $a_2 = 1$. In this case, $a_1 = 0$. (a) If $r_1, r_2 \geq 3$, then

$$
\begin{aligned}
\langle v_1, v_2\rangle - 1 &= d_1 d_2 (H^2) - r_1 - 1 \\
&> r_1 \left(\frac{d_2^2}{r_2} (H^2) - 1\right) - 1 \\
&\geq r_1 \left(1 - \frac{2}{r_2}\right) - 1 \geq 0 .
\end{aligned} \tag{2.13}
$$

(b) If $r_2 \geq 3$ and $r_1 \leq 2$, then $d_2^2 (H^2) \geq 4$, and hence $d_2 (H^2) \geq 4$. Then we see that

$$
\langle v_1, v_2\rangle - 1 = d_1 d_2 (H^2) - r_1 - 1 \geq 4 d_1 - 3 > 0 . \tag{2.14}
$$

(c) If $r_2 = 1$, then $d_1 > r_1 d_2$. Hence we see that

$$
\langle v_1, v_2\rangle - 1 = d_1 d_2 (H^2) - r_1 - 1 > r_1 d_2^2 (H^2) - r_1 - 1 \geq r_1 - 1 \geq 0 . \tag{2.15}
$$

(d) If $r_2 = 2$, then $d_1 > r_1 d_2 / 2$. (d-1) If $d_2^2 (H^2) \geq 4$, then same computation as in (c) implies our claim. (d-2) If $d_2^2(H^2) = 2$, that is, $d_2 = 1$ and $(H^2) = 2$, then

$$\langle v_1, v_2\rangle - 1 = d_1 d_2 (H^2) - r_1 - 1 = 2d_1 - r_1 - 1 \geq 0\,. \tag{2.16}$$

If $\langle v_1, v_2\rangle - 1 = 0$, then $a_1 = 0$ and $2d_1 - r_1 - 1 = 0$. Thus $v_1 = (2d_1 - 1, d_1 H, 0)$ and $v_2 = (2, H, 1)$.

(2) Since

$$\langle v_1, v_2\rangle - l^2 = l(d_1 d_2'(H^2) - (r_2' a_1 + r_1 a_2') - l)\,, \tag{2.17}$$

we shall prove that

$$d_1 d_2'(H^2) - (r_2' a_1 + r_1 a_2') > l\,. \tag{2.18}$$

$$\begin{aligned} d_1 d_2'(H^2) - (r_2' a_1 + r_1 a_2') &= d_1 d_2'(H^2) - (r_2'(1 - l a_2') + r_1 a_2') \\ &= d_1 d_2'(H^2) - r_1 a_2' + r_2'(-1 + l a_2') \\ &= d_1 d_2' \left((H^2) - \frac{r_1}{d_1 d_2'} a_2' \right) + r_2'(-1 + l a_2')\,. \end{aligned} \tag{2.19}$$

(i) If $a_2' \geq 2$ or $r_2' \geq 1$, then $r_2'(-1 + l a_2') \geq l$. On the other hand, we see that

$$\begin{aligned} (H^2) - \frac{r_1}{d_1 d_2'} a_2' &> (H^2) - \frac{r_2' a_2'}{(d_2')^2} \\ &= \frac{1}{(d_2')^2}\left((d_2')^2 (H^2) - r_2' a_2' \right) \\ &\geq \frac{1}{(d_2')^2}\left(\frac{(d_2')^2 (H^2)}{2} - 1 \right) \geq 0\,. \end{aligned} \tag{2.20}$$

Hence we get (2.18). (ii) If $a_2' = 1$ and $r_2 = 1$, then $(d_2')^2(H^2) \leq 2r_2' = 2$. Hence $d_2' = 1$ and $(H^2) = 2$. Since $d_1/r_1 > 1$, we see that

$$d_1 d_2'(H^2) - (r_2' a_1 + r_1 a_2') - l = 2d_1 - r_1 - 1 > r_1 - 1 \geq 0\,. \tag{2.21}$$

□

2.2. Proof of Theorem 0.1 (1). (I) We shall first prove (b). So we assume that $r + a \leq 0$. By Proposition 1.5, $\mathcal{M}_H(v)_0^s \neq \emptyset$. For $E \in \mathcal{M}_H(v)_0^s$, we shall consider the universal extension

$$0 \to \mathcal{O}_X^{\oplus n} \to F \to E \to 0, \tag{2.22}$$

where $n = \dim \operatorname{Ext}^1(E, \mathcal{O}_X) = \langle v, v(\mathcal{O}_X)\rangle$. We shall prove that F is a semi-stable sheaf for a general $E \in \mathcal{M}_H(v)_0^s$.

(Step 1) Assume that F is not semi-stable. For the Harder-Narasimhan filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F \tag{2.23}$$

of F, we set

$$E_i := F_i/F_{i-1}, \quad v_i := v(E_i) = (r_i, d_i H, a_i)\,. \tag{2.24}$$

Then we get

$$\frac{d_1}{r_1} \geq \frac{d_2}{r_2} \geq \cdots \geq \frac{d_s}{r_s} > 0\,. \tag{2.25}$$

Proof of (2.25): By the property of the Harder-Narasimhan filtration, it is sufficient to prove $d_s/r_s > 0$. We shall consider the quotient $q : F \to E_s$ and the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X^{\oplus n} \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "q"] & E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & q(\mathcal{O}_X^{\oplus n}) \arrow[r] & E_s \arrow[r] & E_s/q(\mathcal{O}_X^{\oplus n}) \arrow[r] & 0 \end{tikzcd} \tag{2.26}$$

If $d_s/r_s < 0$, then $q(\mathcal{O}_X^{\oplus n}) = 0$. Thus q induces a surjective homomorphism $E \to E_s$. Since E is stable and $d > 0$, q must be 0, which is a contradiction. If $d_s/r_s = 0$, then $q(\mathcal{O}_X^{\oplus n})$ is a semi-stable sheaf of $c_1(q(\mathcal{O}_X^{\oplus n})) = 0$. By Lemma 3.2, $q(\mathcal{O}_X^{\oplus n}) = \mathcal{O}_X^{\oplus m}$ for some $m > 0$. Since $c_1(E_s/\mathcal{O}_X^{\oplus m}) = 0$ and $E_s/\mathcal{O}_X^{\oplus m}$ is a quotient of E, $E_s/\mathcal{O}_X^{\oplus m}$ is a torsion sheaf of dimension 0. Since E_s is torsion free and $\mathcal{O}_X^{\oplus m}$ is a locally free subsheaf of E_s, we get $E_s/\mathcal{O}_X^{\oplus m} = 0$. Then we get a splitting $F \cong \mathcal{O}_X^{\oplus m} \oplus F'$, which contradicts the choice of extension class. Therefore (2.25) holds.

(Step 2) We shall next prove that

$$\begin{aligned} a_1 &\leq 0\,, \\ a_1 + a_2 &\leq 0\,, \\ &\vdots \\ a_1 + a_2 + \cdots + a_s &\leq 0\,. \end{aligned} \tag{2.27}$$

In particular,

$$\begin{aligned} \langle v_1^2 \rangle &\geq d_1^2 (H^2) > 0\,, \\ \langle (v_1 + v_2)^2 \rangle &\geq (d_1 + d_2)^2 (H^2) > 0\,, \\ &\vdots \\ \langle (v_1 + v_2 + \cdots + v_s)^2 \rangle &\geq (d_1 + d_2 + \cdots + d_s)^2 (H^2) > 0\,. \end{aligned} \tag{2.28}$$

Proof of (2.27): We shall consider an exact sequence

$$0 \to \mathcal{O}_X^{\oplus n} \cap F_i \to F_i \to F_i/(\mathcal{O}_X^{\oplus n} \cap F_i) \to 0\,. \tag{2.29}$$

Since F_i is a filter of the Harder-Narasimhan filtration of F, $F_i/(\mathcal{O}_X^{\oplus n} \cap F_i) \neq 0$. Since $F_i/(\mathcal{O}_X^{\oplus n} \cap F_i)$ is a subsheaf of E and $H^0(X, E) = 0$, $H^0(X, F_i/(\mathcal{O}_X^{\oplus n} \cap F_i)) = 0$. Since $\mathcal{O}_X^{\oplus n} \cap F_i$ is a subsheaf of $\mathcal{O}_X^{\oplus n}$, $H^0(X, \mathcal{O}_X^{\oplus n} \cap F_i) \otimes \mathcal{O}_X$ is a subsheaf of $\mathcal{O}_X^{\oplus n} \cap F_i$. Therefore $\dim H^0(X, F_i) \leq \mathrm{rk}(\mathcal{O}_X^{\oplus n} \cap F_i) \leq \mathrm{rk}(F_i)$. Since $\chi(F_i) = \mathrm{rk}(F_i) + \sum_{j=1}^{i} a_j$, we get (2.27).

(Step 3) We shall prove that

$$\dim \mathcal{F}^{HN}(v_1, v_2, \ldots, v_s) \leq \langle v^2 \rangle . \tag{2.30}$$

Proof of (2.30): By Lemma 1.3, we have

$$\dim \mathcal{F}^{HN}(v_1, v_2, \ldots, v_s) = \sum_{i<j} \langle v_j, v_i \rangle + \sum_{i \geq 1} \dim \mathcal{M}_H(v_i)^{ss} . \tag{2.31}$$

Since $\langle v_1^2 \rangle > 0$, $\dim \mathcal{M}_H(v_1)^{ss} = \langle v_1^2 \rangle + 1$ by Theorem 1.2. Applying Lemma 2.1 and Lemma 1.4, we see that

$$(\langle v_1^2 \rangle + 1) + \dim \mathcal{M}_H(v_2)^{ss} + \langle v_2, v_1 \rangle < \langle (v_1 + v_2)^2 \rangle + 1 . \tag{2.32}$$

We set $v_2' := v_1 + v_2$ and $v_i' := v_i$, $i > 2$. Then we get that

$$\dim \mathcal{F}^{HN}(v_1, v_2, \ldots, v_s) < \sum_{2 \leq i < j} \langle v_j', v_i' \rangle + (\langle (v_2')^2 \rangle + 1) + \sum_{i \geq 3} \dim \mathcal{M}_H(v_i')^{ss} . \tag{2.33}$$

By induction on s, we get (2.30).

(Step 4) By Step 3 and Theorem 1.2, $\Phi_{I_\Delta^*[1]}^{-1}(\mathcal{F}^{HN}(v_1, v_2, \ldots, v_s)) \cap \mathcal{M}_H(v)^{ss}$ is a locally closed substack of $\mathcal{M}_H(v)^{ss}$ such that $\dim \Phi_{I_\Delta^*[1]}^{-1}(\mathcal{F}^{HN}(v_1, v_2, \ldots, v_s)) \cap \mathcal{M}_H(v)^{ss} < \dim \mathcal{M}_H(v)^{ss}$. Combining this with Theorem 1.2, we have $\Phi_{I_\Delta^*[1]}(\mathcal{M}_H(v)^{ss}) \cap \mathcal{M}_H(w)^s \neq \emptyset$. We set

$$\begin{aligned} M_H(v)^* &:= \{E \in M_H(v) | \Phi_{I_\Delta^*[1]}(E) \in M_H(w)\} , \\ M_H(w)^* &:= \{F \in M_H(w) | \Phi_{I_\Delta[1]}(F) \in M_H(v)\} . \end{aligned} \tag{2.34}$$

Then $M_H(v)^*$ and $M_H(w)^*$ are non-empty open subschemes of $M_H(v)$ and $M_H(w)$ respectively and $\Phi_{I_\Delta^*[1]}$ induces an isomorphism $M_H(v)^* \cong M_H(w)^*$. Since $M_H(v)^*$ and $M_H(w)^*$ are irreducible by Theorem 1.2, we get Theorem 0.1 (1) (b).

(II) We next assume that $r + a \geq 0$. Since $(-a) + (-r) \leq 0$ and $w := (-a, dH, -r)$ is $\Phi_{I_\Delta^*}^H(v)$, $\Phi_{I_\Delta^*[1]}$ induces a birational map $M_H(w) \cdots \to M_H(v)$. Since the inverse of $\Phi_{I_\Delta^*[1]}$ is $\Phi_{I_\Delta[1]}$, we get (1) (a). $\square$

REMARK 2.1. For $F \in M_H(r, dH, a)$ with $d > 0$ and $r + a \geq 0$, $\Phi_{I_\Delta[1]}(F)$ fits in the exact sequence

$$\begin{aligned} 0 \longrightarrow H^{-1}(\Phi_{I_\Delta[1]}(F)) &\longrightarrow H^0(X, F) \otimes \mathcal{O}_X \longrightarrow F \\ \longrightarrow H^0(\Phi_{I_\Delta[1]}(F)) &\longrightarrow H^1(X, F) \otimes \mathcal{O}_X \longrightarrow 0 . \end{aligned} \tag{2.35}$$

If $\Phi_{I_\Delta[1]}(F)$ is a semi-stable sheaf, then $H^1(X, F) = 0$ and $H^0(X, F) \otimes \mathcal{O}_X \to F$ is injective.

2.3. Proof of Theorem 0.1 (2). We note that $(\mathcal{D} \circ \Phi_{I_\Delta})^{-1} = \mathcal{D} \circ \Phi_{I_\Delta}$. Hence we shall prove that $\mathcal{D} \circ \Phi_{I_\Delta}$ induces a birational map

$$M_H(a, dH, r) \cdots \to M_H(r, dH, a) \tag{2.36}$$

for $a = 0, 1$.

PROPOSITION 2.3. *Let $v = (0, dH, r)$, $r \geq 0$, $d > 0$ be a Mukai vector. Then $\mathcal{D} \circ \Phi_{I_\Delta} = \Phi_{I_\Delta^*[2]} \circ \mathcal{D}$ induces a birational map $M_H(0, dH, r) \cdots \to M_H(r, dH, 0)$. Thus Theorem 0.1 (2) holds for $a = 0$.*

PROOF. We note that $\mathcal{D}$ induces an isomorphism $M_H(0, dH, a) \to M_H(0, dH, -a)$ by sending L to $\mathcal{E}xt^1_{\mathcal{O}_X}(L, \mathcal{O}_X)$. Hence the claim follows from Theorem 0.1 (1). □

In order to treat the case where $a = 1$, we study the properties of $\mathcal{D} \circ \Phi_{I_\Delta}$. For a coherent sheaf E on X,

$$\mathcal{D} \circ \Phi_{I_\Delta}(E) = \Phi_{I_\Delta^*[2]} \circ \mathcal{D}(E) = \mathbf{R}\operatorname{Hom}_{p_2}(I_\Delta \otimes p_1^*(E), \mathcal{O}_{X \times X})[2] \tag{2.37}$$

and we have an exact triangle

$$\begin{aligned} \mathbf{R}\operatorname{Hom}_{p_2}(\mathcal{O}_\Delta \otimes p_1^*(E), \mathcal{O}_{X \times X}) &\xrightarrow{\phi} \mathbf{R}\operatorname{Hom}(E, \mathcal{O}_X) \otimes \mathcal{O}_X \to \\ \mathbf{R}\operatorname{Hom}_{p_2}(I_\Delta \otimes p_1^*(E), \mathcal{O}_{X \times X}) &\to \mathbf{R}\operatorname{Hom}_{p_2}(\mathcal{O}_\Delta \otimes p_1^*(E), \mathcal{O}_{X \times X})[1]\,. \end{aligned} \tag{2.38}$$

Since $\mathbf{R}\operatorname{Hom}_{p_X}(E \otimes \mathcal{O}_\Delta, \mathcal{O}_{X \times X}) = \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$, we have an exact sequence
(2.39)

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}(E, \mathcal{O}_X) \otimes \mathcal{O}_X & \longrightarrow & H^0(\Phi_{I_\Delta^*} \circ \mathcal{D}(E)) \\ & \longrightarrow & 0 & \longrightarrow & \operatorname{Ext}^1(E, \mathcal{O}_X) \otimes \mathcal{O}_X & \longrightarrow & H^1(\Phi_{I_\Delta^*} \circ \mathcal{D}(E)) \\ & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X) & \xrightarrow{H^2(\phi)} & \operatorname{Ext}^2(E, \mathcal{O}_X) \otimes \mathcal{O}_X & \longrightarrow & H^2(\Phi_{I_\Delta^*} \circ \mathcal{D}(E)) \\ & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}_X}(E, \mathcal{O}_X) & \longrightarrow & 0. & & \end{array}$$

Assume that E is a stable sheaf with $(c_1(E), H) > 0$. Then $\operatorname{Hom}(E, \mathcal{O}_X) = 0$, which implies that $H^0(\Phi_{I_\Delta^*} \circ \mathcal{D}(E)) = 0$.

LEMMA 2.4. (1) *If $H^0(X, E) \otimes \mathcal{O}_X \to E$ is generically surjective, then $H^1(\Phi_{I_\Delta^*} \circ \mathcal{D}(E)) \cong \operatorname{Ext}^1(E, \mathcal{O}_X) \otimes \mathcal{O}_X$.*

(2) *If E is a stable purely 1-dimensional sheaf on X, then $H^1(\Phi_{I_\Delta^*} \circ \mathcal{D}(E)) \cong \operatorname{Ext}^1(E, \mathcal{O}_X) \otimes \mathcal{O}_X$ and $H^2(\Phi_{I_\Delta^*} \circ \mathcal{D}(E))$ is the universal extension of $\mathcal{E}xt^1_{\mathcal{O}_X}(E, \mathcal{O}_X)$ by $\mathcal{O}_X$.*

PROOF. (1) By the Serre duality, the dual of ϕ is the evaluation map ev : $\mathbf{R}$ Hom $(\mathcal{O}_X, E) \otimes \mathcal{O}_X \to E$. Since $H^0(\mathrm{ev})$ is generically surjective, $H^2(\phi)$ is generically injective. Since $\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ is locally free, $H^2(\phi)$ is injective. Therefore (1) holds.

(2) Since E is purely 1-dimensional, we can apply (1) to prove the first claim. For the second claim, we use Lemma 3.1. Since

$$\begin{aligned}\mathrm{Hom}(H^2(\Phi_{I_\Delta^*} \circ \mathcal{D}(E)), \mathcal{O}_X) &= \mathrm{Hom}(\Phi_{I_\Delta^*} \circ \mathcal{D}(E)[2], \mathcal{O}_X)\\ &= \mathrm{Hom}(\Phi_{I_\Delta}(\Phi_{I_\Delta^*} \circ \mathcal{D}(E)[2]), \Phi_{I_\Delta}(\mathcal{O}_X))\\ &= \mathrm{Hom}(\mathcal{D}(E), \mathcal{O}_X[-2])\\ &= \mathrm{Hom}(\mathcal{O}_X, E[-2]) = 0\,,\end{aligned} \tag{2.40}$$

we get our claim. □

PROOF OF THEOREM 0.1 (2). We take an irreducible and reduced curve $C \in |dH|$. Assume that there are distinct n points $p_1, p_2, \dots, p_n$ of C such that $Z_n := \{p_1, p_2, \dots, p_n\}$ satisfies $H^1(X, I_{Z_n}(dH)) = 0$. This condition is equivalent to the surjectivity of the restriction map $\xi_n : H^0(X, \mathcal{O}_X(dH)) \to H^0(Z_n, \mathcal{O}_{Z_n}(dH))$. If $\dim H^0(X, I_{Z_n}(dH)) \geq 2$, then there is a section of $H^0(X, I_{Z_n}(dH))$ whose support D is not C. Then for $Z_{n+1} := Z_n \cup \{p_{n+1}\}$ with $p_{n+1} \in C \setminus D$, $H^1(X, I_{Z_{n+1}}(dH)) = 0$. In this way, we can construct $I_Z(dH) \in \mathcal{M}_H(1, dH, r)^{ss}$ with a section $\phi : \mathcal{O}_X \to I_Z(dH)$ such that $\operatorname{coker}\phi$ is a torsion free sheaf on an irreducible and reduced curve C and $H^1(X, I_Z(dH)) = 0$. We shall study the relation of $\Phi_{I_\Delta^*} \circ \mathcal{D}(I_Z(dH))$ and $\Phi_{I_\Delta^*} \circ \mathcal{D}(\operatorname{coker}\phi)$. Since $\Phi_{I_\Delta^*} \circ \mathcal{D}(\mathcal{O}_X) = \mathcal{O}_X$, we have an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\Phi_{I_\Delta^*} \circ \mathcal{D}(\operatorname{coker}\phi)) & \longrightarrow & H^0(\Phi_{I_\Delta^*} \circ \mathcal{D}(I_Z(dH))) & \longrightarrow & \mathcal{O}_X \\ & \longrightarrow & H^1(\Phi_{I_\Delta^*} \circ \mathcal{D}(\operatorname{coker}\phi)) & \longrightarrow & H^1(\Phi_{I_\Delta^*} \circ \mathcal{D}(I_Z(dH))) & \longrightarrow & 0 \\ & \longrightarrow & H^2(\Phi_{I_\Delta^*} \circ \mathcal{D}(\operatorname{coker}\phi)) & \longrightarrow & H^2(\Phi_{I_\Delta^*} \circ \mathcal{D}(I_Z(dH))) & \longrightarrow & 0\,. \end{array} \tag{2.41}$$

By Lemma 2.4, $F := \Phi_{I_\Delta^*[2]} \circ \mathcal{D}(I_Z(dH)) \in \mathrm{Coh}(X)$ and is the universal extension of $L := \mathcal{E}xt^1_{\mathcal{O}_X}(\operatorname{coker}\phi, \mathcal{O}_X)$ by $\mathcal{O}_X$

$$0 \to \mathrm{Ext}^2(I_Z(dH), \mathcal{O}_X) \otimes \mathcal{O}_X \to H^2(\Phi_{I_\Delta^*} \circ \mathcal{D}(I_Z(dH))) \to L \to 0\,. \tag{2.42}$$

We shall prove that F is a semi-stable sheaf for a general L.

(Step 1) Assume that F is not semi-stable. For the Harder-Narasimhan filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F \tag{2.43}$$

of F, we set

$$\begin{aligned} E_i &:= F_i/F_{i-1}\,,\\ v_i &:= v(E_i) = (r_i, d_i H, a_i)\,. \end{aligned} \tag{2.44}$$

Then we see that

$$\frac{d_1}{r_1} \geq \frac{d_2}{r_2} \geq \cdots \geq \frac{d_s}{r_s} > 0 \tag{2.45}$$

and

$$\frac{a_i}{r_i} > \frac{a_{i+1}}{r_{i+1}}, \text{ if } \frac{d_i}{r_i} = \frac{d_{i+1}}{r_{i+1}}. \tag{2.46}$$

by a similar way as in the proof of (2.25).

(Step 2) We shall next prove that

$$\begin{aligned} a_1 &\leq 0, \\ a_1 + a_2 &\leq 0, \\ &\vdots \\ a_1 + a_2 + \cdots + a_{s-1} &\leq 0. \end{aligned} \tag{2.47}$$

Proof of (2.47): We shall consider an exact sequence

$$0 \to \mathcal{O}_X^{\oplus r} \cap F_i \to F_i \to F_i/(\mathcal{O}_X^{\oplus r} \cap F_i) \to 0. \tag{2.48}$$

We shall prove that $\dim H^0(X, F_i) \leq \mathrm{rk}(F_i)$ for $i \leq s-1$. We note that $F_i/(\mathcal{O}_X^{\oplus r} \cap F_i)$ is regarded as a subsheaf of L. Since $\dim H^0(X, L) = 1$, it is sufficient to prove that $\dim H^0(X, \mathcal{O}_X^{\oplus r} \cap F_i) < \mathrm{rk}(F_i)$. If $\dim H^0(X, \mathcal{O}_X^{\oplus r} \cap F_i) = \mathrm{rk}(F_i)$, then since $H^0(X, \mathcal{O}_X^{\oplus r} \cap F_i) \otimes \mathcal{O}_X$ is a subsheaf of $\mathcal{O}_X^{\oplus r} \cap F_i$, we get $\mathcal{O}_X^{\oplus r} \cap F_i = \mathcal{O}_X^{\oplus \mathrm{rk}(F_i)}$. Since F_i is a filter of the Harder-Narasimhan filtration of F, $F_i/(\mathcal{O}_X^{\oplus r} \cap F_i) \neq 0$. We note that L is a torsion free sheaf on an irreducible and reduced curve C. Hence $c_1(F_i/(\mathcal{O}_X^{\oplus r} \cap F_i)) = dH$. Then F/F_i is a torsion free sheaf with $c_1(F/F_i) = 0$. Since $d_s/r_s > 0$, this is impossible. Therefore $\dim H^0(X, \mathcal{O}_X^{\oplus r} \cap F_i) < \mathrm{rk}(F_i)$.

(Step 3) We shall prove that

$$\frac{d_s}{r_s} < \frac{\sum_{i=1}^{s-1} d_i}{\sum_{i=1}^{s-1} r_i}. \tag{2.49}$$

Proof of (2.49): By (2.45), $d_s/r_s \leq (\sum_{i=1}^{s-1} d_i)/(\sum_{i=1}^{s-1} r_i)$. If the equality holds, then (2.45) and (2.46) imply that $d_i/r_i = d_s/r_s$ for all i and $a_s/r_s < (\sum_{i=1}^{s-1} a_i)/(\sum_{i=1}^{s-1} r_i)$. By (2.47), we have $a_s \leq 0$. On the other hand, $\sum_{i=1}^{s} a_i = 1$. Therefore (2.49) holds.

(Step 4) We shall prove that

$$\dim \mathcal{F}^{HN}(v_1, v_2, \ldots, v_s) < \dim \mathcal{M}_H(v)^{ss} \tag{2.50}$$

unless $(H^2) = 2$, $v = (2d-1, dH, 1)$, $d \geq 2$, $s = 2$, $v_1 = (2d-3, (d-1)H, 0)$ and $v_2 = (2, H, 1)$.

Proof of (2.50): We set $v' := \sum_{i=1}^{s-1} v_i$. By (2.47), we can apply Lemma 2.1 successively to prove

$$\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) \le \langle v_s, v' \rangle + (\langle v', v' \rangle + 1) + \dim \mathcal{M}_H(v_s)^{ss} \tag{2.51}$$

as in the proof of Theorem 0.1 (1). Moreover if the equality holds, then we have $s = 2$. Applying Lemma 2.2 to the pair v' and v_s, we get

$$\langle v_s, v' \rangle + (\langle v', v' \rangle + 1) + \dim \mathcal{M}_H(v_s)^{ss} \le \langle v^2 \rangle + 1 = \dim \mathcal{M}_H(v)^{ss}\,. \tag{2.52}$$

Moreover if the equality holds, then $(H^2) = 2$, $v' = (2d_1 - 1, d_1 H, 0)$ and $v_s = (2, H, 1)$. Therefore

$$\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) < \dim \mathcal{M}_H(v)^{ss} \tag{2.53}$$

unless $(H^2) = 2$, $v = (2d - 1, dH, 1)$, $d \ge 2$, $s = 2$, $v_1 = (2d - 3, (d - 1)H, 0)$ and $v_2 = (2, H, 1)$. Thus Theorem 0.1 (2) holds. □

2.4. Proof of Theorem 0.1 (3). Assume that $(H^2) = 2$. We set $v := (1, dH, 2d-1)$ and assume that $d \ge 2$. For a simple and rigid vector bundle G on X, we set

$$\mathcal{E}_G := \ker(G^\vee \boxtimes G \to \mathcal{O}_\Delta)\,. \tag{2.54}$$

$\Phi_{\mathcal{E}_G}$ is a generalization of Φ_{I_Δ} and has similar properties. For example, if $\mathrm{Hom}(G, E) = \mathrm{Ext}^2(G, E) = 0$, $E \in \mathrm{Coh}(X)$, then $\Phi_{\mathcal{E}_G^*[1]}(E)$ is the universal extension of E by G.

We shall show that $\Phi_{\mathcal{E}_{\mathcal{O}_X(H)}[1]}$ induces a birational map

$$M_H(1, dH, 2d - 1) \cdots \to M_H(0, dH, 2d - 3)\,. \tag{2.55}$$

In particular, a general member $I_Z(dH) \in M_H(1, dH, 2d - 1)$ fits in the following exact sequence

$$0 \to \mathcal{O}_X(H) \to I_Z(dH) \to L \to 0 \tag{2.56}$$

where $L \in M_H(0, (d - 1)H, 2d - 3)$ and $\mathrm{Ext}^1(L, \mathcal{O}_X(H)) \cong \mathbb{C}$.

Proof of the claim: We have isomorphisms $M_H(1, dH, 2d - 1) \cong M_H(1, (d - 1)H, 0)$ and $M_H(0, (d-1)H, 2d-3) \cong M_H(0, (d-1)H, -1)$ by the operation $E \mapsto E(-H)$. Since $(\Phi_{\mathcal{E}_{\mathcal{O}_X(H)}[1]}(E))(-H) = \Phi_{I_\Delta[1]}(E(-H))$ for $E \in \mathrm{Coh}(X)$, the claim follows from Theorem 0.1 (1).

Applying Theorem 0.1 (2) to $\mathcal{O}_X(H)$ and a general $L \in M_H(0, (d - 1)H, 2d - 3)$, we get stable sheaves $E_1 := \mathcal{D} \circ \Phi_{I_\Delta}(\mathcal{O}_X(H)) \in M_H(2, H, 1)$ and $F := \mathcal{D} \circ \Phi_{I_\Delta}(L) \in M_H(2d - 3, (d - 1)H, 0)$. Hence $\mathcal{D} \circ \Phi_{I_\Delta}(I_Z(dH))$ fits in an exact sequence

$$0 \to F \to \mathcal{D} \circ \Phi_{I_\Delta}(I_Z(dH)) \to E_1 \to 0\,. \tag{2.57}$$

Hence $\mathcal{D} \circ \Phi_{I_\Delta}(I_Z(dH))$ is not stable.

By the stability of E_1 and F, $\mathrm{Ext}^2(E_1, F) = 0$. Since $\mathrm{Hom}(E_1, F) = \mathrm{Hom}(L, \mathcal{O}_X(H)) = 0$, $\mathrm{Ext}^1(E_1, F) \cong \mathbb{C}$ and $\Phi_{\mathcal{E}_{E_1}[1]}(F)$ fits in an exact sequence

$$0 \to F \to \Phi_{\mathcal{E}_{E_1}[1]}(F) \to \mathrm{Ext}^1(E_1, F) \otimes E_1 \to 0\,. \tag{2.58}$$

Therefore $\Phi_{\mathcal{E}_{E_1}[1]}(F) = \mathcal{D} \circ \Phi_{I_\Delta}(I_Z(dH))$. On the other hand, since

$$(\mathrm{rk}\, E_1)c_1(F) - (\mathrm{rk}\, F)c_1(E_1) = H\,, \tag{2.59}$$

[11, Thm. 2.5] implies that $\Phi_{\mathcal{E}^*_{E_1}[1]}$ induces a birational map

$$M_H(2d-3, (d-1)H, 0) \cdots \to M_H(2d-1, dH, 1)\,. \tag{2.60}$$

We define $\Phi : \mathbf{D}(X) \to \mathbf{D}(X)$ by $\Phi := \mathcal{D} \circ \Phi_{\mathcal{E}^*_{E_1}[1]} \circ \Phi_{\mathcal{E}^*_{E_1}[1]} \circ \mathcal{D} \circ \Phi_{I_\Delta} = \Phi_{\mathcal{E}_{E_1}[1]} \circ \Phi_{\mathcal{E}_{E_1}[1]} \circ \Phi_{I_\Delta}$. Then $(\mathcal{D} \circ \Phi)^{-1} = \mathcal{D} \circ \Phi$ gives a desired birational map $M_H(2d-1, dH, 1) \cdots \to M_H(1, dH, 2d-1)$. Thus Theorem 0.1 (3) holds. □

3. Appendix

LEMMA 3.1. *Let E, G be coherent sheaves on X and V a finite dimensional vector space. For an extension*

$$0 \to V \otimes G \to F \to E \to 0 \tag{3.1}$$

of E by $V \otimes G$, we assume that $\mathrm{Hom}(F, G) = 0$. *Then the extension class $e \in \mathrm{Ext}^1(E, G) \otimes V$ induces an injective homomorphism $V^\vee \to \mathrm{Ext}^1(E, G)$. In particular, if* $\mathrm{Hom}(F, G) = 0$ *and* $\dim V = \dim \mathrm{Ext}^1(E, G)$, *then Then* (3.1) *is the universal extension of E by G, that is, $e \in \mathrm{Ext}^1(E, G) \otimes V$ induces an isomorphism $V^\vee \to \mathrm{Ext}^1(E, G)$.*

PROOF. Assume that the induced homomorphism $\varepsilon : V^\vee \to \mathrm{Ext}^1(E, G)$ is not injective. Then there is a non-zero homomorphism $\phi : V \to \mathbb{C}$ belonging to $\ker \varepsilon$. For $V \otimes G \xrightarrow{\phi} \mathbb{C} \otimes G$, we take the induced extension

$$\begin{tikzcd} 0 \arrow[r] & V \otimes G \arrow[r] \arrow[d, "\phi"'] & F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C} \otimes G \arrow[r] & F' \arrow[r] & E \arrow[r] & 0\,. \end{tikzcd} \tag{3.2}$$

Since $\phi \in \ker \varepsilon$, the induced extension is trivial, that is, $F' = \mathbb{C} \otimes G \oplus E$. Then we get $\mathrm{Hom}(F, G) \neq 0$. Therefore ε is injective. □

LEMMA 3.2. *Let E be a μ-semi-stable sheaf with $(c_1(E), H) = 0$. If there is a surjective homomorphism $\psi : \mathcal{O}_X^{\oplus n} \to E$, then $H^0(X, E) \otimes \mathcal{O}_X \to E$ is an isomorphism.*

PROOF. We have a commutative diagram

$$(3.3)\qquad \begin{array}{ccc} H^0(X, \mathcal{O}_X^{\oplus n}) \otimes \mathcal{O}_X & \xrightarrow{\phi_1} & \mathcal{O}_X^{\oplus n} \\ \downarrow & & \downarrow \psi \\ H^0(X, E) \otimes \mathcal{O}_X & \xrightarrow{\phi_2} & E \end{array}$$

where ϕ_1 and ϕ_2 are evaluation maps. Since ϕ_1 is an isomorphism, the surjectivity of ψ implies that ϕ_2 is also surjective. We shall prove that ϕ_2 is injective. Assume that $\ker \phi_2 \neq 0$. Then $\ker \phi_2$ is a μ-semi-stable locally free sheaf with $(c_1(\ker \phi_2), H) = 0$. We take a μ-stable subsheaf F of $\ker \phi_2$ with $(c_1(\ker \phi_2), H) = 0$. Then there is a non-zero homomorphism $F \to \mathcal{O}_X$, which is an isomorphism. Then $\ker \phi_2$ contains $\mathcal{O}_X$, which is a contradiction. Therefore ϕ_2 is injective and we get our claim. □

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