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Differential Sandwich Theorems for Certain Subclasses of Analytic Functions Involving an Extended Multiplier Transformation

Mohamed Kamel AOUF and Robha Md. EL-ASHWAH

Mansoura University

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Abstract. In this paper we derive some subordination and superordination results for certain normalized analytic functions in the open unit disc, which are acted upon by a class of extended multiplier transformation. Relevant connection of the results, which are presented in this paper with various known results are also considered.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z \in C : |z| < 1\}$ and let $H[a, n]$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (a \in C)\,. \tag{1.1}$$

Also, let $A(n)$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k\,. \tag{1.2}$$

If $f, g \in H(U)$, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \ \ (z \in U) \quad \text{or} \quad f(z) \prec g(z) \ \ (z \in U)\,,$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalence (cf., e.g., [9]; see also [10, p. 4]):

$$f(z) \prec g(z) \ \ (z \in U) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U)\,.$$

Supposing that p, h are two analytic functions in U, let

$$\varphi(r, s, t; z) : C^3 \times U \to C\,.$$

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If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order subordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is called to be a solution of the differential superordination (1.3). (If f is subordinate to F, then F is superordinate to f). An analytic function q is called a subordinant of (1.3), if $q(z) \prec p(z)$ for all the functions $p(z)$ satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all the subordinants q of (1.3), is called the best subordinant (cf., e.g., [9], see also [10]).

Recently, Miller and Mocanu [11] obtained sufficient conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

Using the results of Miller and Mocanu [11], Bulboaca [4] considered certain classes of first-order differential superdordination as well as superordination-preserving integral operators [5]. Ali et al. [1] have used the results of Bulboaca [4] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \tag{1.5}$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = 1$. Shanmugam et al. [17] obtained sufficient conditions for normalized analytic function $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = 1$ and $q_2(0) = 1$, while Obradovic [12] introduced a class of function $f \in A = A(1)$, such that, for $0 < \alpha < 1$,

$$\mathrm{Re}\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\alpha}\right\} > 0, \quad z \in U.$$

He called this class of function as "non-Bazilevic" type. Using this non-Bazilevic class, Wang et al. [21] studied many subordination results for the class $N(\alpha, \lambda, A, B)$ defined by

$$N(\alpha, \lambda, A, B) = \left\{f \in A : (1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha} \prec \frac{1+Az}{1+Bz}\right\},$$

where $\lambda \in C$, $-1 \le B < A \le 1$, $A \ne B$, $0 < \alpha < 1$.

Many essentially equivalent definitions of muliplier transformation have been given in literature (see [7], [8] and [22]). In [6] Catas defined the operator $I^m(\lambda,\ell)$ as follows:

DEFINITION 1 [6]. Let the function $f(z) \in A(n)$. For $m \in N_0 = N \cup \{0\}$, where $N = \{1, 2, ...\}, \lambda \geq 0, \ell \geq 0$. The extended muliplier transformation $I^m(\lambda,\ell)$ on $A(n)$ is defined by the following infinite series

$$I^m(\lambda,\ell)f(z) = z + \sum_{k=n+1}^{\infty} \left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m a_k z^k . \tag{1.6}$$

It follows from (1.6) that

$$I^0(\lambda,\ell)f(z) = f(z) ,$$

$$(1+\ell)I^{m+1}(\lambda,\ell)f(z) = (1-\lambda+\ell)I^m(\lambda,\ell)f(z) + \lambda z(I^m(\lambda,\ell)f(z))' \quad (\lambda > 0) \tag{1.7}$$

and

$$I^{m_1}(\lambda,\ell)(I^{m_2}(\lambda,\ell))f(z) = I^{m_1+m_2}(\lambda,\ell)f(z) = I^{m2}(\lambda,\ell)(I^{m_1}(\lambda,\ell))f(z) . \tag{1.8}$$

We note that

1- $I^m(1,\ell)f(z) = I_\ell^m f(z)$(see [7] and [8]);

2- $I^m(\lambda,0)f(z) = D_\lambda^m f(z)$(see [2]);

3- $I^m(1,0)f(z) = D^m f(z)$(see [16]);

4- $I^m(1,1)f(z) = I_m f(z)$(see [22]).

Also if $f \in A(n)$, then we can write

$$I^m(\lambda,\ell)f(z) = (f * \varphi^m_{\lambda,\ell})(z) ,$$

where

$$\varphi^m_{\lambda,\ell}(z) = z + \sum_{k=n+1}^{\infty} \left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m z^k . \tag{1.9}$$

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and results.

DEFINITION 2 [11]. Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U}\backslash E(f)$, where

$$E(f) = \{\zeta : \zeta \in \partial \text{ and } \lim_{z\to\zeta} f(z) = \infty\} \tag{2.1}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

LEMMA 1 [10]. *Let the function $q(z)$ be univalent in the unit disc U and let $\theta$and ϕ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(i) *$Q(z)$ is starlike univalent in U,*

(ii) $\operatorname{Re}\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ *for $z \in U$.*

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then

$$p(z) \prec q(z)$$

and $q(z)$ is the best dominant.

LEMMA 2 [17]. *Let q be a convex univalent function in U and let $\psi \in C, \delta \in C^* = C\backslash\{0\}$ with*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, \operatorname{Re}\left(\frac{\psi}{\delta}\right)\right\}.$$

If $p(z)$ is analytic in U and

$$\psi p(z) + \delta zp'(z) \prec \psi q(z) + \delta zq'(z), \tag{2.3}$$

then

$$p(z) \prec q(z) \quad (z \in U)$$

and q is the best dominant.

LEMMA 3 [4]. *Let $q(z)$ be convex univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\operatorname{Re}\left\{\dfrac{\theta'(q(z))}{\varphi(q(z))}\right\} > 0$ *for $z \in U$;*

(ii) *$zq'(z)\varphi(q(z))$ is starlike univalent in U.*

If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.4}$$

then

$$q(z) \prec p(z) \quad (z \in U)$$

and $q(z)$ is the best subordinant.

LEMMA 4 [11]. *Let q be convex univalent in U and $\delta \in C$. Further assume that* $\mathrm{Re}(\overline{\delta}) > 0$. *If $p(z) \in H[q(0), 1] \cap Q$ and $p(z) + \delta z p'(z)$ is univalent in U, then*

$$q(z) + \delta z q'(z) \prec p(z) + \delta z p'(z)\,, \tag{2.5}$$

implies

$$q(z) \prec p(z) \quad (z \in U)$$

and q is the best subordinant.

This last lemma gives us a necessary and sufficient condition for the univalence of a special function which will be used in some particular cases.

LEMMA 5 [15]. *The function $q(z) = (1 - z)^{-2ab}$ is univalent in U if and only if* $|2ab - 1| \leq 1$ *or* $|2ab + 1| \leq 1$.

3. Subordination for analytic functions

THEOREM 1. *Let q be univalent in U, $\gamma \in C^*$, $\lambda > 0$ and $0 < \alpha < 1$. Suppose q satisfies*

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{\alpha}{\gamma}\right)\right\}. \tag{3.1}$$

If $f \in A(n)$, $I^m(\lambda, \ell)f(z) \neq 0$ ($z \in U^* = U \backslash \{0\}$) and satisfies the subordination

$$\Psi(f, \gamma, m, \lambda, \ell, \alpha) \prec q(z) + \frac{\gamma}{\alpha} z q'(z)\,, \tag{3.2}$$

where

$$\begin{aligned}\Psi(f, \gamma, m, \lambda, \ell, \alpha) = {} & \left[1 + \gamma\left(\frac{\ell + 1}{\lambda}\right)\right]\left(\frac{z}{I^m(\lambda, \ell)f(z)}\right)^{\alpha} \\ & - \gamma\left(\frac{\ell + 1}{\lambda}\right)\frac{I^{m+1}(\lambda, \ell)f(z)}{z}\left(\frac{z}{I^m(\lambda, \ell)f(z)}\right)^{\alpha+1},\end{aligned} \tag{3.3}$$

then

$$\left(\frac{z}{I^m(\lambda, \ell)f(z)}\right)^{\alpha} \prec q(z) \tag{3.4}$$

and q is the best dominant of (3.2).

PROOF. Define the function $p(z)$ by

$$p(z) = \left(\frac{z}{I^m(\lambda, \ell)f(z)}\right)^{\alpha} \quad (z \in U)\,. \tag{3.5}$$

Then the function p is analytic in U and $p(0) = 1$. Therefore, differentiating (3.5) logarithmically with respect to z and using the identity (1.7) in the resulting equation, we have

$$\left[1+\gamma\left(\frac{\ell+1}{\lambda}\right)\right]\left(\frac{z}{I^m(\lambda,\ell)f(z)}\right)^{\alpha} - \gamma\left(\frac{\ell+1}{\lambda}\right)\frac{I^{m+1}(\lambda,\ell)f(z)}{z}\left(\frac{z}{I^m(\lambda,\ell)f(z)}\right)^{\alpha+1}$$
$$= p(z) + \frac{\gamma}{\alpha} z p'(z)\,. \tag{3.6}$$

Using (3.6) and (3.2), we have

$$p(z) + \frac{\gamma}{\alpha} z p'(z) \prec q(z) + \frac{\gamma}{\alpha} z q'(z)\,. \tag{3.7}$$

The assertion (3.4) of Theorem 1 now follows by an application of Lemma 2 with $\delta = \frac{\gamma}{\alpha}$, $0 < \alpha < 1$, and $\psi = 1$.

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 1, we obtain the following corollary.

COROLLARY 1. *Let* $-1 \le B < A \le 1, \gamma \in C^*, \lambda > 0, 0 < \alpha < 1$ *and* $\operatorname{Re}\left\{\dfrac{1-Bz}{1+Bz}\right\} > \max\left\{0, -\operatorname{Re}\left(\dfrac{\alpha}{\gamma}\right)\right\}$. *If* $f(z) \in A(n)$, $I^m(\lambda,\ell)f(z) \ne 0$ $(z \in U^*)$ *and*

$$\Psi(f,\gamma,m,\lambda,\ell,\alpha) \prec \frac{1+Az}{1+Bz} + \frac{\gamma(A-B)z}{\alpha(1+Bz)^2}\,, \tag{3.8}$$

where $\Psi(f,\gamma,m,\lambda,\ell,\alpha)$ *is given by* (3.3), *then*

$$\left(\frac{z}{I^m(\lambda,\ell)f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz} \tag{3.9}$$

and $\dfrac{1+Az}{1+Bz}$ *is the best dominant of* (3.8).

Taking $A = 1$ and $B = -1$ in Corollary 1, we obtain the following corollary.

COROLLARY 2. *Let* $\gamma \in C^*, \lambda > 0, 0 < \alpha < 1$ *and* $\operatorname{Re}\left\{\dfrac{1+z}{1-z}\right\} > \max\left\{0, -\operatorname{Re}\left(\dfrac{\alpha}{\gamma}\right)\right\}$. *if* $f(z) \in A(n)$, $I^m(\lambda,\ell)f(z) \ne 0 (z \in U^*)$ *and*

$$\Psi(f,\gamma,m,\lambda,\ell,\alpha) \prec \frac{1+z}{1-z} + \frac{2\gamma z}{\alpha(1-z)^2}\,, \tag{3.10}$$

where $\Psi(f,\gamma,m,\lambda,\ell,\alpha)$ *is given by* (3.3), *then*

$$\left(\frac{z}{I^m(\lambda,\ell)f(z)}\right)^{\alpha} \prec \frac{1+z}{1-z} \tag{3.11}$$

and $\dfrac{1+z}{1-z}$ *is the best dominant of* (3.10).

THEOREM 2. *Let q be univalent in U, $\gamma, \mu \in C^*, \lambda > 0$, and $0 \leq \beta \leq 1$. Let $f(z) \in A(n)$. Suppose q satisfies*

$$\operatorname{Re}\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0\,. \tag{3.12}$$

If

$$1+\gamma\mu[\Phi(f,\beta,m,\lambda,\ell)-1] \prec 1+\gamma\frac{zq'(z)}{q(z)}\,, \tag{3.13}$$

where

$$\Phi(f,\beta,m,\lambda,\ell)$$
$$=\frac{\beta\left(\frac{\ell+1}{\lambda}\right)I^{m+2}(\lambda,\ell)f(z)+\left[(1-2\beta)\left(\frac{\ell+1}{\lambda}\right)+\beta\right]I^{m+1}(\lambda,\ell)f(z)-(1-\beta)\left(\frac{\ell+1}{\lambda}-1\right)I^{m}(\lambda,\ell)f(z)}{(1-\beta)I^{m}(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}\,, \tag{3.14}$$

then

$$\left\{\frac{(1-\beta)I^{m}(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}{z}\right\}^{\mu} \prec q(z) \tag{3.15}$$

and q is the best dominant of (3.13).

PROOF. Define the function $p(z)$ by

$$p(z)=\left\{\frac{(1-\beta)I^{m}(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}{z}\right\}^{\mu}. \tag{3.16}$$

Then a computation shows that

$$\frac{zp'(z)}{p(z)}=\mu[\Phi(f,\beta,m,\lambda,\ell)-1]\,, \tag{3.17}$$

where $\Phi(f,\beta,m,\lambda,\ell)$ is given by (3.14). By setting

$$\theta(w)=1 \quad\text{and}\quad \varphi(w)=\frac{\gamma}{w}\,, \tag{3.18}$$

it can be easily observed that $\theta(w)$ is analytic in C, $\varphi(w)$ is analytic in C^*, and that $\varphi(w)\neq 0$ $(w\in C^*)$. Also, we let

$$Q(z)=zq'(z)\varphi(q(z))=\gamma\frac{zq'(z)}{q(z)}\,, \tag{3.19}$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \gamma \frac{zq'(z)}{q(z)}\,. \tag{3.20}$$

From (3.12), we find that $Q(z)$ is starlike univalent in U and that

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \tag{3.21}$$

by the assertion (3.12) of Theorem 2. Thus, by applying Lemma 1, our proof of Theorem 2 is completed.

Putting $n = 1, \beta = m = \ell = 0, \lambda = 1, \gamma = \dfrac{1}{ab}$ $(a, b \in C^*)$, $\mu = a$ and $q(z) = (1 - z)^{-2ab}$ in Theorem 2, then combining this together with Lemma 5, we obtain the next result due to Obradovic et al. [13 , Theorem 1]:

COROLLARY 3 [13]. *Let $a, b \in C^*$ such that $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$. Let $f(z) \in A$ and suppose that $\dfrac{f(z)}{z} \neq 0$ for all $z \in U$. If*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z} \tag{3.22}$$

then

$$\left(\frac{f(z)}{z}\right)^a \prec (1 - z)^{-2ab}\,, \tag{3.23}$$

and $(1 - z)^{-2ab}$ is the best dominant of (3.22). (The power is the principal one).

REMARK 1. For $a = 1$, Corollary 3 reduces to the recent result of Srivastava and Lashin [20,Theorem 3].

Putting $n = 1, \beta = m = \ell = 0, \lambda = 1, \gamma = \dfrac{e^{i\lambda}}{ab\cos\lambda}\left(a, b \in C^*;\ |\lambda| < \dfrac{\pi}{2}\right), \mu = a$ and $q(z) = (1 - z)^{-2ab\cos\lambda e^{-i\lambda}}$ in Theorem 2, we obtain the result due to Aouf et al. [3, Theorem 1]:

COROLLARY 4 [3]. *Let $a, b \in C^*$ and $|\lambda| < \dfrac{\pi}{2}$, and suppose that $\left|2ab\cos\lambda e^{-i\lambda} - 1\right| \leq 1$ or $\left|2ab\cos\lambda e^{-i\lambda} + 1\right| \leq 1$. Let $f(z) \in A$ such that $\dfrac{f(z)}{z} \neq 0$ for all $z \in U$. If*

$$1 + \frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z}\,, \tag{3.24}$$

then

$$\left(\frac{f(z)}{z}\right)^{a} \prec (1-z)^{-2ab\cos\lambda e^{i\lambda}}, \tag{3.25}$$

and $(1-z)^{-2ab\cos\lambda e^{i\lambda}}$ *is the best dominant of* (3.24). (The power is the principal one).

Putting $m=\ell=0, \lambda=\beta=1, \gamma=\dfrac{1}{ab}(a,b\in C^{*}), \mu=a$ and $q(z)=(1-z)^{-2ab}$ in Theorem 2, then combining this together with Lemma 5, we obtain the next result.

COROLLARY 5. *Let $a,b\in C^{*}$ such that $|2ab-1|\le 1$ or $|2ab+1|\le 1$.Let $f(z)\in A(n)$ and suppose that $f'(z)\neq 0$ for all $z\in U$. If*

$$1+\frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}, \tag{3.26}$$

then

$$(f'(z))^{a} \prec (1-z)^{-2ab} \tag{3.27}$$

and $(1-z)^{-2ab}$ *is the best dominant of* (3.26) . (The power is the principal one).

REMARK 2. For $a=n=1$, Corollary 5 reduces to the recent result of Srivastava and Lashin [20, Corollary 1].

Taking $n=1, m=\ell=\beta=0, \lambda=1, \gamma=\dfrac{1}{\mu}$ $(\mu\in C^{*})$ and $q(z)=(1+Bz)^{\mu\left(\frac{A-B}{B}\right)}$ $(-1\le B<A\le 1, B\neq 0)$ in Theorem 2, we get the following known result obtained by Obradovic and Owa [14].

COROLLARY 6 [14]. *Let $-1\le B<A\le 1, B\neq 0, \mu\in C^{*}$ such that $\left|\mu\left(\frac{A-B}{B}\right)-1\right|\le 1$ or $\left|\mu\left(\frac{A-B}{B}\right)+1\right|\le 1$. Let $f(z)\in A$ and suppose that $\dfrac{f(z)}{z}\neq 0$ for all $z\in U$. If*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \ (z\in U), \tag{3.28}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec (1+Bz)^{\mu\left(\frac{A-B}{B}\right)} \ (\mu\in C^{*}; B\neq 0) \tag{3.29}$$

and $(1+Bz)^{\mu\left(\frac{A-B}{B}\right)}$ *is the best dominant of* (3.28).

Taking $n=1,\ m=\ell=\beta=0, \lambda=1, \gamma=\dfrac{1}{\mu}$ and $q(z)=e^{\mu Az}$ $(-1<A\le 1)$ in Theorem 2, we get the following known result obtained by Obradovic and Owa [14].

COROLLARY 7 [14]. *Let* $-1 < A \leq 1, \mu \in C^*$ *such that* $|\mu A| \leq \pi$. *Let* $f(z) \in A$ *and suppose that* $\frac{f(z)}{z} \neq 0$ *for all* $z \in U$. *If*

$$\frac{zf'(z)}{f(z)} \prec 1 + Az \quad (z \in U)\,, \tag{3.30}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec e^{\mu Az} \quad (\mu \in C^*) \tag{3.31}$$

and $e^{\mu Az}$ *is the best dominant of* (3.30).

THEOREM 3. *Let* $q(z)$ *be univalent in* U, $\gamma \neq 0, \delta, \alpha \in C$, *and let* $0 \leq \beta \leq 1$. *Let* $f(z) \in A(n)$. *Suppose* q *satisfies*

$$\operatorname{Re}\left\{\frac{\alpha}{\gamma} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0\,, \tag{3.32}$$

and also $\operatorname{Re}(\frac{\alpha}{\gamma}) > 0$. *Let*

$$\Psi(z) = \left[\frac{(1-\beta)I^m(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu}\{\alpha + \gamma\mu[\Phi(f,\beta,m,\lambda,\ell)-1]\} + \delta\,, \tag{3.33}$$

where $\Phi(f, \beta, m, \lambda, \ell)$ *is given by* (3.14). *If*

$$\Psi(z) \prec \alpha q(z) + \delta + \gamma z q'(z)\,, \tag{3.34}$$

then

$$\left[\frac{(1-\beta)I^m(\lambda,\ell)f(z) + \beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu} \prec q(z) \tag{3.35}$$

and $q(z)$ *is the best dominant of* (3.34).

PROOF. Define the function $p(z)$ by

$$p(z) = \left[\frac{(1-\beta)I^m(\lambda,\ell)f(z) + \beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu}. \tag{3.36}$$

Differentiating (3.36) logarithmically with respect to z and using the identity (1.7) in the resulting equation, we have

$$\frac{zp'(z)}{p(z)} = \mu[\Phi(f,\beta,m,\lambda,\ell) - 1]\,, \tag{3.37}$$

where $\Phi(f, \beta, m, \lambda, \ell)$ is defined by (3.14). From (3.37), we have

$$zp'(z) = \mu p(z)[\Phi(f,\beta,m,\lambda,\ell) - 1]\,. \tag{3.38}$$

By setting

$$\theta(w) = \alpha w + \delta \quad \varphi(w) = \gamma\,, \tag{3.39}$$

it can be easily observed that $\theta(w)$ and $\varphi(w)$ are analytic in C. Also, we let

$$Q(z) = zq'(z)\varphi(q(z)) = \gamma zq'(z) \tag{3.40}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \delta + \gamma zq'(z)\,. \tag{3.41}$$

From (3.40), we find that $Q(z)$ is starlike univalent in U, and that

$$\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \mathrm{Re}\left\{\frac{\alpha}{\gamma} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \tag{3.42}$$

by the hypothesis (3.32) of Theorem 3. Thus, by applying Lemma 3, our proof of Theorem 3 is completed.

Taking $m = \ell = 0, \lambda = \beta = 1$, $\delta = -\alpha$ and $\gamma = 1$ in Theorem 3, we obtain the following result obtained by Shanmugam et al. [18, Corollary 3.10].

COROLLARY 8 [18]. *Let q be univalent in U. Also let $f \in A(n)$ and* $1 + \alpha > 0$. *Suppose q satisfies*

$$\mathrm{Re}\left\{\alpha + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0\,, \tag{3.43}$$

then

$$\alpha\{(f'(z))^{\mu} - 1\} + \mu\left\{\frac{zf''(z)}{f'(z)}(f'(z))^{\mu}\right\} \prec \alpha q(z) - \alpha + zq'(z) \tag{3.44}$$

then

$$(f'(z))^{\mu} \prec q(z)$$

and q is the best dominant of (3.44).

REMARK 3. Taking $q(z) = 1 + \frac{\lambda}{(1+\alpha)}z$, $\alpha \geq 0$ and $0 < \lambda \leq 1 + \alpha$,in Corollary 8, we obtain a recent result of Singh [19, Theorem 1(ii)].

4. Superordination for analytic function

THEOREM 4. *Let q be convex univalent in U, $\gamma \in C$, $\lambda > 0$ and $0 < \alpha < 1$. Suppose*

$$\mathrm{Re}\{\gamma\} > 0\,. \tag{4.1}$$

Let $f(z) \in A(n)$, $I^m(\lambda, \ell)f(z) \neq 0\,(z \in U^*)$ *and* $\left(\frac{z}{I^m(\lambda, \ell)f(z)}\right)^{\alpha} \in H[q(0), 1] \cap Q$. *Let* $\Psi(f, \gamma, m, \lambda, \ell, \alpha)$ *is univalent in* U, *where* $\Psi(f, \gamma, m, \lambda, \ell, \alpha)$ *is defined by* (3.3). *If*

$$q(z) + \frac{\gamma}{\alpha} zq'(z) \prec \Psi(f, \gamma, m, \lambda, \ell, \alpha)\,, \tag{4.2}$$

then

$$q(z) \prec \left(\frac{z}{I^m(\lambda, \ell)f(z)}\right)^{\alpha} \tag{4.3}$$

and q *is the best subordinant of* (4.1).

PROOF. Define the function $p(z)$ by

$$p(z) = \left(\frac{z}{I^m(\lambda, \ell)f(z)}\right)^{\alpha} \quad (z \in U)\,. \tag{4.4}$$

Differentiating (4.4) logarithmically with respect to z and using the identity (1.7) in the resulting equation, we have

$$p(z) + \frac{\gamma}{\alpha} zp'(z) \prec \Psi(f, \gamma, m, \lambda, \ell, \alpha)\,. \tag{4.5}$$

Theorem 4 follows as an applying of Lemma 4.

Taking $q(z) = \frac{1 + Az}{1 + Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 4, we obtain the following corollary.

COROLLARY 9. *Let* $-1 \leq B < A \leq 1, \gamma \in C, \operatorname{Re}(\gamma) > 0, \lambda > 0$ *and* $0 < \alpha < 1$. *Also let* q *be convex univalent in* U. *Suppose* $I^m(\lambda, \ell)f(z) \neq 0$ $(z \in U^*)$ *and* $\left(\frac{z}{I^m(\lambda, \ell)f(z)}\right)^{\alpha} \in H[q(0), 1] \cap Q$. *Let* $\Psi(f, \gamma, m, \lambda, \ell, \alpha)$ *is univalent in* U, *where* $\Psi(f, \gamma, m, \lambda, \ell, \alpha)$ *is given by* (3.3). *If*

$$\frac{\gamma(A - B)z}{\alpha(1 + Bz)^2} + \frac{1 + Az}{1 + Bz} \prec \Psi(f, \gamma, m, \lambda, \ell, \alpha)\,, \tag{4.6}$$

then

$$\frac{1 + Az}{1 + Bz} \prec \left(\frac{z}{I^m(\lambda, \ell)f(z)}\right)^{\alpha} \tag{4.7}$$

and $\frac{1 + Az}{1 + Bz}$ *is the best subordinant of* (4.6).

The proof of the following theorem is similar to the proof of Theorem 4, so we state the theorem without proof.

THEOREM 5. *Let q be convex univalent in U, $\gamma \in C$, $0 \leq \beta \leq 1$, and $f \in A(n)$. Suppose*

$$0 \neq \left[\frac{(1-\beta)I^m(\lambda,\ell)f(z) + \beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu} \in H[q(0),1] \cap Q\,,$$

and $1+\gamma\mu[\Phi(f,\beta,m,\lambda,\ell)-1]$ is univalent in U, where $\Phi(f,\beta,m,\lambda,\ell)$ is given by (3.14). *If*

$$1 + \gamma\frac{zq'(z)}{q(z)} \prec 1 + \gamma\mu[\Phi(f,\beta,m,\lambda,\ell) - 1]\,, \tag{4.8}$$

then

$$q(z) \prec \left[\frac{(1-\beta)I^m(\lambda,\ell)f(z) + \beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu} \tag{4.9}$$

and q is the best subordinant of (4.8).

THEOREM 6. *Let q be convex univalent in U, $\gamma \in C^*$, $\delta, \alpha \in C$ and let $0 \leq \beta \leq 1$. Let $f \in A(n)$ and $0 \neq \left[\frac{(1-\beta)I^m(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu} \in H[q(0),1] \cap Q$. Suppose q satisfies*

$$\mathrm{Re}\left\{\frac{\alpha}{\gamma}q'(z)\right\} > 0\,. \tag{4.10}$$

If

$$\alpha q(z) + \delta + \gamma z q'(z) \prec$$

$$\left[\frac{(1-\beta)I^m(\lambda,\ell)f(z) + \beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu}\{\alpha + \gamma\mu[\Phi(f,\beta,m,\lambda,\ell) - 1]\} + \delta\,, \tag{4.11}$$

where $\Phi(f,\beta,m,\lambda,\ell))$ is given by (3.14). *Then*

$$q(z) \prec \left[\frac{(1-\beta)I^m(\lambda,\ell)f(z) + \beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu} \tag{4.12}$$

and q is the best subordinant of (4.11).

PROOF. Define the function $p(z)$ by

$$p(z) = \left[\frac{(1-\beta)I^m(\lambda,\ell)f(z) + \beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu}. \tag{4.13}$$

Then a computation shows that

$$\frac{zp'(z)}{p(z)} = \mu[\Phi(f,\beta,m,\lambda,\ell) - 1]\,, \tag{4.14}$$

where $\Phi(f,\beta,m,\lambda,\ell)$ is given by (3.14). Therefore, we have

$$zp'(z)=\mu p(z)[\Phi(f,\beta,m,\lambda,\ell)-1]\,. \tag{4.15}$$

By setting

$$\theta(w)=\alpha w+\delta\,,\quad \varphi(w)=\gamma\,, \tag{4.16}$$

it can be easily observed that both $\theta(w)$ and $\varphi(w)$ are analytic in C. Now,

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\}=\operatorname{Re}\left\{\frac{\alpha q'(z)}{\gamma}\right\}>0\,, \tag{4.17}$$

by the hypothesis (4.10) of Theorem 6. Thus, by applying Lemma 3, our proof of Theorem 6 is completed.

5. Sandwich results

Combining the results of differential subordination and supordination, we state the following "sandwich results".

THEOREM 7. *Let q_1 be convex univalent and let q_2 be univalent in U, $\gamma\in C^*$, and $0<\alpha<1$. Suppose q_1 satisfies* (4.1) *and q_2 satisfies* (3.1). *If* $0\neq\left(\frac{z}{I^m(\lambda,\ell)f(z)}\right)\in H[q(0),1]\cap Q$, *$\Psi(f,\gamma,m,\lambda,\ell,\alpha)$ is univalent in U, where $\Psi(f,\gamma,m,\lambda,\ell,\alpha)$ is given by* (3.3), *and*

$$q_1(z)+\frac{\gamma}{\alpha}zq_1'(z)\prec\Psi(f,\gamma,m,\lambda,\ell,\alpha)\prec q_2(z)+\frac{\gamma}{\alpha}zq_2'(z)\,, \tag{5.1}$$

then

$$q_1(z)\prec\left(\frac{z}{I^m(\lambda,\ell)f(z)}\right)^{\alpha}\prec q_2(z) \tag{5.2}$$

and q_1 and q_2 are, respectively, the best subordinant and best dominant.

THEOREM 8. *Let q_1 be convex univalent and let q_2 be univalent in U, $\gamma,\mu\in C^*,\lambda>0$, and $0\le\beta\le1$. Let $f(z)\in A(n)$. Suppose q_2 satisfies* (3.12), *and* $0\neq\left[\frac{(1-\beta)I^m(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu}\in H[q(0),1]\cap Q, 1+\gamma\mu[\Phi(f,\beta,m,\lambda,\ell)-1]$ *is univalent in U, where $\Phi(f,\beta,m,\lambda,\ell)$ is given by* (3.14). *If*

$$1+\gamma\frac{zq_1'(z)}{q_1(z)}\prec1+\gamma\mu[\Phi(f,\beta,m,\lambda,\ell)-1]\prec1+\gamma\frac{zq_2'(z)}{q_2(z)}\,, \tag{5.3}$$

then

$$q_1(z)\prec\left[\frac{(1-\beta)I^m(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^{\mu}\prec q_2(z) \tag{5.4}$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

THEOREM 9. *Let q_1 be convex univalent and let q_2 be univalent in U, $\gamma,\mu \in C^*, \lambda > 0$ and $0 \le \beta \le 1$. Suppose q_1 satisfies* (4.10), *q_2 satisfies* (3.32), *and* $0 \neq \left[\dfrac{(1-\beta)I^m(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^\mu \in H[q(0),1]\cap Q$. *Let*

$$\left[\frac{(1-\beta)I^m(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^\mu\{\alpha+\gamma\mu[\Phi(f,\beta,m,\lambda,\ell)-1]\}+\delta \tag{5.5}$$

is univalent in U. If

$$\alpha q_1(z)+\delta+\gamma z q_1'(z) \prec$$

$$\left[\frac{(1-\beta)I^m(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^\mu\{\alpha+\gamma\mu[\Phi(f,\beta,m,\lambda,\ell)-1]\}+\delta$$

$$\prec \alpha q_2(z)+\delta+\gamma z q_2'(z)\,, \tag{5.6}$$

then

$$q_1(z) \prec \left[\frac{(1-\beta)I^m(\lambda,\ell)f(z)+\beta I^{m+1}(\lambda,\ell)f(z)}{z}\right]^\mu \prec q_2(z) \tag{5.7}$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

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Present Addresses:
MOHAMED KAMEL AOUF
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,
MANSOURA UNIVERSITY,
MANSOURA 35516, EGYPT.
e-mail: mkaouf127@yahoo.com

ROBHA MD. EL-ASHWAH
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,
MANSOURA UNIVERSITY,
MANSOURA 35516, EGYPT.
e-mail: r_elashwah@yahoo.com