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Remarks on the Goursat Problems

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Introduction

Perhaps one of the most general form of the so-called Goursat Problem is that given by Hörmander [3]: Discuss the existence and the uniqueness of solutions of the partial differential equation:

$$D^{\beta}u=\sum_{|\alpha|\leqq|\beta|}a_{\alpha}D^{\alpha}u+f \tag{1}$$

such that

$$u(x)-\varphi(x)=O(x^{\beta})$$

as x approaches each coordinate planes in the space C^{n} of n complex variables where $\varphi(x)$ is a given function. We may roughly summarize Hörmander's results as follows. There exists a unique holomorphic solution of the equation (1) if the coefficient on the left-hand side of the equation (1) is sufficiently large compared with the coefficients a_{α}'s on the right-hand side of the equation.

On the other extremity of the formulation of the problem, a very special, and perhaps the simplest second order equation with constant coefficients;

$$\varepsilon u_{xy}=u_{xx}+u_{yy}+f(x, y) \tag{2}$$

was studied, by J. Leray, in [4] with the similar boundary condition. He proved that there exists a unique holomorphic solution of (2) for all complex value of the parameter ε except for the real interval $[-2, 2]$. Furthermore, he revealed an unexpected complication of the problem which can only be expressed by a continued fraction expansion of ε. He showed that both existence and uniqueness of holomorphic solutions of (2) depends on the transcendental algebraic behavior of the parameter

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ε on this particular interval.

If we want to describe explicitly the phrase "sufficiently large" in Hörmander's result for the equation (2), we have

$$|\varepsilon|>59.112\cdots .$$

In these two different types of approaches, the best result obtained up to now seems to be the work of L. Gårding, [1]. He studied higher order equations with variable coefficients and proved the existence and the uniqueness of holomorphic solutions outside the circle $|\varepsilon|>2$ in the complex ε-plane.

The object of the present study is to give a precise bound of the circle of convergence of the holomorphic solutions in terms of a non-euclidean distance between the parameter ε and the exceptional interval $[-2, 2]$ for a class of equations similar to (2) but with variable coefficients for which Leray's method is no-longer applicable. Thus our results will be stated for equations which are less general than that of Gårding's, but much more general than Leray's equation (2). Our results are not so deep as that of Leray's, but exceed Gårding's results by proving the existence and the uniqueness inside the circle $|\varepsilon|\leqq 2$ which he left untouched.

The method of the proof depends largely on the estimates of large sparse matrices with a parameter among their elements. We use Chebychef polynomials as solutions of certain difference equations. Our method of attack could be applicable to wider classes than the class of partial differential equations treated in this paper.

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§ 1. Notations and results.

Let x, y be complex variables and α be multi-index. We put,

$$\alpha=(\alpha_1, \alpha_2)\ , \quad |\alpha|=\alpha_1+\alpha_2\ , \quad D^\alpha=\left(\frac{\partial}{\partial x}\right)^{\alpha_1}\left(\frac{\partial}{\partial y}\right)^{\alpha_2} ,$$

where α_1 and α_2 are non-negative integers. We shall consider the following analytic differential equation.

$$\varepsilon\left(\frac{\partial}{\partial x}\right)^p\left(\frac{\partial}{\partial y}\right)^q u=\sum_{|\alpha|\leq p+q} a_\alpha(x, y)D^\alpha u+h(x, y)\ , \quad p, q\geqq 1\ , \tag{1.1}$$

where ε is a complex parameter and $a_\alpha(x, y)$, $h(x, y)$ are analytic in an open ball B;

$$B=\{(x, y);\ |x|<r_0,\ |y|<r_0\}\ .$$

We take, as the boundary conditions,

$$(1.2)\qquad \begin{aligned} &\left(\frac{\partial}{\partial x}\right)^m u\Big|_{x=0}=\phi_{1,m}(y)\ , \quad 0\leqq m\leqq p-1\ ,\\ &\left(\frac{\partial}{\partial y}\right)^n u\Big|_{y=0}=\phi_{2,n}(x)\ , \quad 0\leqq n\leqq q-1\ , \end{aligned}$$

where $\phi_{1,m}$ and $\phi_{2,n}$ are analytic in a neighborhood of the origin, and the following compatibility conditions are assumed:

$$(1.3)\qquad \left(\frac{\partial}{\partial x}\right)^m \phi_{2,n}\Big|_{x=0}=\left(\frac{\partial}{\partial y}\right)^n \phi_{1,m}\Big|_{y=0}\ ,$$

for all m, n satisfying

$$0\leqq m\leqq p-1\ ,\quad 0\leqq n\leqq q-1\ .$$

We assume the following conditions;

$$(1.4)\qquad \begin{aligned} &a_\alpha(0, 0)\neq 0\ , \quad \text{if} \quad \alpha=(p-1, q+1) \quad \text{or} \quad (p+1, q-1)\ ,\\ &a_\alpha(0, 0)=0\ , \quad \text{if} \quad \alpha\neq(p-1, q+1) \quad \text{or} \quad (p+1, q-1) \quad \text{and} \quad |\alpha|=p+q\ . \end{aligned}$$

REMARK 1. If, in the equation (1.1), (p, q) is not contained in the convex hull of the set of multi-indices $\{\alpha;\ a_\alpha(0, 0)\neq 0\}$, then the Goursat problem (1.1)-(1.2) has one and only one analytic solution, in case $\varepsilon\neq 0$, (see [3]). The equation (1.1) is not in general contained in this case.

By the assumptions (1.4), we may assume;

$$(1.5)\qquad a_\alpha(0, 0)=1\ , \quad \text{if} \quad \alpha=(p-1, q+1) \quad \text{or} \quad (p+1, q-1)\ ,$$

since this is always possible by making use of the change of variables; $x=c_1z_1$; $y=c_2z_2$, $(c_1=\sqrt{a_{(p+1,q-1)}(0, 0)}$; $c_2=\sqrt{a_{(p-1,q+1)}(0, 0)})$. For simplicity, we shall consider only the homogeneous cases.

$$(1.6)\qquad \begin{aligned} &\left(\frac{\partial}{\partial x}\right)^m u\Big|_{x=0}=0\ , \quad 0\leqq m\leqq p-1\ ,\\ &\left(\frac{\partial}{\partial y}\right)^n u\Big|_{y=0}=0\ , \quad 0\leqq n\leqq q-1\ . \end{aligned}$$

The general case then follows by using an appropriate transformation such as

$$v(x, y)=u(x, y)-\sum_{m=0}^{p-1}\frac{x^m}{(m!)}\phi_{1,m}(y)-\sum_{n=0}^{q-1}\frac{y^n}{(n!)}\phi_{2,n}(x)$$
$$+\sum_{m=0}^{p-1}\sum_{n=0}^{q-1}\frac{x^m}{(m!)}\frac{y^n}{(n!)}\left(\left(\frac{\partial}{\partial x}\right)^m\phi_{2,n}\right)(0)\ . \tag{1.7}$$

By the analyticity of $a_\alpha(x, y)$, $h(x, y)$ we have,

$$a_\alpha(x, y)=\sum a^\alpha_{ij}\frac{x^i}{(i!)}\frac{y^j}{(j!)}\ ,\quad h(x, y)=\sum h_{ij}\frac{x^i}{(i!)}\frac{y^j}{(j!)}\ , \tag{1.8}$$

where a^α_{ij} and h_{ij} have the following estimates,

$$|a^\alpha_{ij}|\leqq Mr^{i+j}((i+j)!)\ ,\quad |h_{ij}|\leqq Mr^{i+j}((i+j)!) \tag{1.9}$$

for some constants M and r which are independent of α, i, j. For each $c>1$ we set,

$$F_c=\left\{\varepsilon;\ \varepsilon=\varepsilon_1+\sqrt{-1}\varepsilon_2,\ \left(\frac{\varepsilon_1}{c+1/c}\right)^2+\left(\frac{\varepsilon_2}{c-1/c}\right)^2>1\right\}\ . \tag{1.10}$$

We denote the complement of the closed interval $[-2, 2]$ in the complex plane by $C\backslash[-2, 2]$. Then we can see that for each ε contained in $C\backslash[-2, 2]$, we can take a number c with the property,

$$c>1\ ,\quad \varepsilon\in F_c\ .$$

THEOREM. *If ε is contained in $C\backslash[-2, 2]$ and if we assume the condition* (1.4), (1.5), *the Goursat problem* (1.1)-(1.6) *has one and only one solution $u(x, y, \varepsilon)$ with is analytic with respect to x, y in the domain*

$$\{(x, y);\ |x|+|y|<1/(K_0\delta+e^{1/e})r,\ \delta=c^2/(c-1)^3\}\ ,$$

where K_0 is independent of ε and c is the number chosen as above. Moreover, $u(x, y, \varepsilon)$ is also analytic with respect to ε in $C\backslash[-2, 2]$.

Theorem will be proved in §§ 2-4.

§ 2. Construction of the formal solution.

Suppose there exists an analytic solution $u(x, y, \varepsilon)=\sum u_{ij}(x^iy^j/(i!)(j!))$. Then by (1.6) we get

$$u_{ij}=0\ ,\quad (0\leqq i\leqq p-1,\ \text{ or }\ 0\leqq j\leqq q-1)\ , \tag{2.1}$$

and by (1.1), (1.8) we get

$$\varepsilon \sum_{i,j} u_{i+p,j+q} \frac{x^i y^j}{(i!)(j!)} = \sum_{\alpha} \left(\sum_{i,j} u_{i+\alpha_1, j+\alpha_2} \frac{x^i y^j}{(i!)(j!)} \right) \times \left(\sum a^{\alpha}_{\nu,\mu} \frac{x^\nu}{(\nu!)} \frac{y^\mu}{(\mu!)} \right) + \sum_{i,j} h_{ij} \frac{x^i y^j}{(i!)(j!)} . \tag{2.2}$$

By comparing the coefficients of $x^i y^j$ in (2.2), we obtain,

$$\begin{aligned} & -u_{i+p+1,j+q-1} + \varepsilon u_{i+p,j+q} - u_{i+p-1,j+q+1} \\ & = \sum_{\substack{0 \leq \nu \leq i \\ 0 \leq \mu \leq j \\ (\nu,\mu) \neq (i,j)}} \binom{i}{\nu} \binom{j}{\mu} \{ u_{\nu+p+1,\mu+q-1} a^{(p+1,q-1)}_{i-\nu,j-\mu} + u_{\nu+p-1,\mu+q+1} a^{(p-1,q+1)}_{i-\nu,j-\mu} \} \\ & \quad + \sum_{\alpha \neq (p-1,q+1),(p+1,q-1)} \sum_{\nu=0}^{i} \sum_{\mu=0}^{j} \left\{ \binom{i}{\nu} \binom{j}{\mu} u_{\nu+\alpha_1,\mu+\alpha_2} a^{\alpha}_{i-\nu,j-\mu} \right\} + h_{ij} , \end{aligned} \tag{2.3}$$

where

$$\binom{i}{\nu} = \frac{i!}{\nu!(i-\nu)!} , \quad \binom{j}{\mu} = \frac{j!}{\mu!(j-\mu)!} , \quad \text{for all} \quad i, j \geqq 0 .$$

Specifically, if we put $i=j=0$, $i=1$, $j=0$ or $i=0$, $j=1$, then we have, respectively

$$\varepsilon u_{pq} = h_{00} , \tag{2.4}$$

$$\begin{aligned} \varepsilon u_{p+1,q} &= u_{p,q+1} + a^{(p,q)}_{1,0} u_{p,q} + h_{1,0} , \\ \varepsilon u_{p,q+1} &= u_{p+1,q} + a^{(p,q)}_{0,1} u_{p,q} + h_{0,1} . \end{aligned} \tag{2.5}$$

We denote by $A_k (k=1, 2, \cdots)$ the following k by k matrix.

$$A_k = \begin{bmatrix} \varepsilon & -1 & 0 & \cdot & \cdot & \cdot & & & \\ -1 & \varepsilon & -1 & 0 & \cdot & \cdot & \cdot & & \\ 0 & -1 & \varepsilon & -1 & 0 & \cdot & \cdot & \cdot & \\ & 0 & -1 & \varepsilon & & \cdot & & & \\ & & 0 & & & & \cdot & & \\ & & & \ddots & & & & \ddots & \\ & & & & & & 0 & & \\ & & & & & & \varepsilon & -1 & 0 \\ & & & & & 0 & -1 & \varepsilon & -1 \\ & & & & & & 0 & -1 & \varepsilon \end{bmatrix} \tag{2.6}$$

(Large 0 in upper right and lower left of the matrix.)

Let U_k and $H_k(k=1, 2, \cdots)$ be the vectors defined by

$$U_k={}^t(u_{k+p-1,q}, u_{k+p-2,q+1}, \cdots, u_{p+1,k+q-2}, u_{p,k+q-1}) , \tag{2.7}$$

$$H_k={}^t(h_{k-1,0}, h_{k-2,1}, \cdots, h_{1,k-2}, h_{0,k-1}) . \tag{2.8}$$

Then (2, 4), (2.5) are written as follows;

$$\begin{bmatrix} \varepsilon & 0 & 0 \\ -a_{1,0}^{(p,q)} & \varepsilon & -1 \\ -a_{0,1}^{(p,q)} & -1 & \varepsilon \end{bmatrix} \begin{bmatrix} u_{pq} \\ u_{p+1,q} \\ u_{p,q+1} \end{bmatrix} = \begin{bmatrix} h_{00} \\ h_{1,0} \\ h_{0,1} \end{bmatrix} . \tag{2.9}$$

Generally, (2.3) is a system of linear equations with unknown variables $u_{i+p,j+q}(i+j\leqq k-1,\ i,\ j\geqq 0)$; recalling (2.1), (1.5), it can be written as follows.

$$\begin{pmatrix} A_1 & 0 & \cdot & \cdot & \cdot & & & & \\ B_{21} & A_2 & 0 & \cdot & \cdot & & & & \\ B_{31} & B_{32} & A_3 & 0 & \cdot & \cdot & \cdot & & \\ B_{41} & B_{42} & B_{43} & A_4 & 0 & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & & \cdot & \cdot & & & \\ & & & & & \cdot & & & \\ B_{k1} & B_{k2} & B_{k3} & B_{k4} & \cdot & \cdot & \cdot & B_{k,k-1} & A_k \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \cdot \\ \cdot \\ \cdot \\ \\ U_k \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ \cdot \\ \cdot \\ \cdot \\ \\ H_k \end{pmatrix} , \tag{2.10}$$

$$k=1, 2, \cdots ,$$

where $B_{m,n}$ is a (m, n) matrix.

In order to consider the structure of the matrices $B_{m,n}(2\leqq m\leqq k,\ 1\leqq n\leqq m-1)$ more precisely, we first note that the term which appears in (2.3) is the term h_{ij} or $a_{i-\nu,j-\mu}^{\alpha}\binom{i}{\nu}\binom{j}{\mu}u_{\alpha_1+\nu,\alpha_2+\mu}$ for some α, i, j, ν, μ. Therefore, if we write (2.3) in the matrix notation (2.10), then each component of $B_{m,n}$ must be the finite sum of the coefficients of $u_{\nu'+p,\mu'+q}(\nu'+\mu'=m-1)$ in (2.3). More precisely, we can prove the following lemma.

LEMMA 1. *Let N be the number of terms which appear in the right-hand side of* (1.1). *Then the number of the terms*

$$a_{i-\nu,j-\mu}^{\alpha}\binom{i}{\nu}\binom{j}{\mu} \quad \textit{for some} \quad \alpha, i, j, \nu, \mu ,$$

which appear in each fixed row or column of $B_{m,n}(2\leqq m\leqq k,\ 1\leqq n<m;\ k=1,2,\cdots)$, *does not exceed* $N(m-n+1)$. *And, if we choose a positive constant* M_0 *sufficiently large, then the absolute values of these terms can be estimated by*

$$M_0 r^{m-n}((m-1)!)/((n-1)!)\ .$$

PROOF. Since each row of $B_{m,n}$ is assigned by (2.3) with some (i, j) such that $i+j=m-1$, the number of the terms in one row does not exceed the number of the pairs, (ν, μ) which satisfy the following set of inequalities:

$$\nu+\mu=p+q-|\alpha|+n-1,\ \nu+\alpha_1\geqq p,\ 0\leqq\nu\leqq i,\ 0\leqq\mu\leqq j,\ \mu+\alpha_2\geqq q\ , \tag{2.11}$$

where $\alpha=(\alpha_1, \alpha_2)$. By (2.11) we get,

$$p+q-|\alpha|-\nu+n-1\leqq j=m-i-1\ ,$$

and hence,

$$p+q-|\alpha|+i-m+n\leqq\nu\ .$$

By using (2.11) we have

$$p+q-|\alpha|+i-m+n\leqq\nu\leqq i\ .$$

Therefore ν can take at most $i-(p+q-|\alpha|+i-m+n)+1=m-n+|\alpha|-p-q+1$ different values. Since $|\alpha|\leqq p+q$ (the order of the equation), we obtain the desired fact.

Similarly, we can prove the same property for a column. The remaining part of Lemma 1 is proved as follows. First note that all the terms which appear in $B_{m,n}$ are of the form $a^{\alpha}_{i-\nu,j-\mu}\binom{i}{\nu}\binom{j}{\mu}$ for some i, j, ν, μ, α such that

$$i+j=m-1,\ \nu+\mu=n-1+p+q-|\alpha|,\ \nu+\alpha_1\geqq p,\ \mu+\alpha_2\geqq q\ .$$

By using (1.9) we have

$$\left|a^{\alpha}_{i-\nu,j-\mu}\binom{i}{\nu}\binom{j}{\mu}\right|\leqq Mr^{m-n+|\alpha|-p-q}((m-1)!)/((n-1+p+q-|\alpha|)!)\ .$$

On the other hand we have

$$r^{p+q-|\alpha|}(n-1+p+q-|\alpha|)(n-2+p+q-|\alpha|)\cdots n\geqq 1\ ,$$

if $p+q-|\alpha|\geqq 1$, and n is sufficiently large. From this inequality, choosing

$M_0(\geqq M)$ sufficiently large, we get

$$Mr^{m-n+|\alpha|-p-q}((m-1)!)/((n-1+p+q-|\alpha|)!)\leqq M_0 r^{m-n}((m-1)!)/((n-1)!)\ .$$

This completes the proof of Lemma 1.

To calculate $u_{i+p,j+q}(i+j=k-1,\ i,\ j\geqq 0)$ by using (2.10) we need:

LEMMA 2. *Let $\mathfrak{A}_k$ be the matrix in the left-hand side of* (2.10). *Suppose that A_k^{-1} exists for all $k=1, 2, \cdots$. Then $\mathfrak{A}_k^{-1}(k=2, 3, \cdots)$ exist and are given by;*

$$\text{(2.12)}\qquad \mathfrak{A}_k^{-1}=\begin{bmatrix} A_1^{-1} & 0 & \cdot & \cdot & \cdot & & \\ C_{21} & A_2^{-1} & 0 & \cdot & \cdot & \cdot & \\ C_{31} & C_{32} & A_3^{-1} & 0 & \cdot & \cdot & \cdot \quad 0 \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & \cdot & & & \cdot & \\ & & & & & & 0 \\ C_{k1} & C_{k2} & C_{k3} & \cdot\ \cdot\ \cdot & & C_{k,k-1} & A_k^{-1} \end{bmatrix},\quad k=2, 3, \cdots ,$$

where $C_{m,n}(2\leqq m\leqq k,\ 1\leqq n<m)$ is a (m, n) matrix and is calculated as follows;

$$\text{(2.13)}\qquad C_{m,n}=\sum_{s=1}^{m-n}\sum_{n=n(1)<n(2)<\cdots<n(s)<m}(-1)^s A_m^{-1}B_{m,n(s)}A_{n(s)}^{-1}B_{n(s),n(s-1)}A_{n(s-1)}^{-1}\cdots$$
$$\cdots A_{n(3)}^{-1}B_{n(3),n(2)}A_{n(2)}^{-1}B_{n(2),n}A_n^{-1}\ ,$$

where the summation $\sum_{n=n(1)<n(2)<\cdots<n(s)<m}$ is taken over all the combinations.

The proof of Lemma 2 is based on a straight forward computation. The proof is nearly trivial for $k=2$. So it should be done for $k=3$ and so on.

In view of Lemma 2, the estimates of the coefficients depend heavily on A_k^{-1}. Let us write

$$\text{(2.14)}\qquad I_k=\det A_k\quad (k=1, 2, \cdots)\ ,$$

then by (2.6) and simple calculations we get

$$\text{(2.15)}\qquad I_1=\varepsilon\ ,\quad I_2=\varepsilon^2-1\ .$$

By expanding det A_k with respect to the first row, we obtain the following difference equation.

$$I_{k+1}=\varepsilon I_k-I_{k-1} \quad (k=2, 3, \cdots) . \tag{2.16}$$

By the elementary theory of difference equations, the solution of (2.16) with initial conditions (2.15) is given by;

$$I_k=\frac{1}{\sqrt{\varepsilon^2-4}}\left\{\left(\frac{\varepsilon+\sqrt{\varepsilon^2-4}}{2}\right)^{k+1}-\left(\frac{\varepsilon-\sqrt{\varepsilon^2-4}}{2}\right)^{k+1}\right\} , \tag{2.17}$$

when $\varepsilon\neq\pm 2$, and

$$I_k=(\pm 1)^k(k+1) , \tag{2.18}$$

when $\varepsilon=\pm 2$.

By (2.17) and in view of the Remark 4 which follows, we can calculate the zeros of I_k as follows;

$$\varepsilon=2\cos\frac{l\pi}{k+1} , \quad l=1, 2, \cdots, k .$$

Put

$$E=\bigcup_{k=1}^{\infty}\bigcup_{l=1}^{\infty}\left\{\varepsilon;\ \varepsilon=2\cos\frac{l\pi}{k+1}\right\}$$

and let $\bar{E}$ be the closure (in an ordinary sense) of E, that is,

$$\bar{E}=[-2, 2] .$$

DEFINITION. We call $\bar{E}$ "the exceptional set" of the Goursat problem (1.1)-(1.6).

Therefore, if ε is not contained in E, then $A_k^{-1}(k=1, 2, \cdots)$ exists. Consequently, in view of Lemma 2 and (2.10), the formal solution of (1.1)-(1.6) exists and it is unique.

REMARK 3. If the order of the equation (1.1) is 2, (1.1)-(1.6) are reduced to the following ones,

$$\varepsilon\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}\right)u=\sum_{|\alpha|\leq 2}a_\alpha(x, y)D^\alpha u+h(x, y) , \tag{2.19}$$

$$u(x, 0)\equiv u(0, y)\equiv 0 , \quad (x, y)\in C^2 . \tag{2.20}$$

Now, if we restrict the variables (x, y) in R^2, then the exceptional set coincides with the set where the equation is elliptic or parabolic, that is, the characteristic form is nonnegative.

REMARK 4. (The representation of I_k). If we substitute ε in (2.17) with $2z$, we have,

$$I_k(2z)=\frac{1}{2\sqrt{z^2-1}}\{(z+\sqrt{z^2-1})^{k+1}-(z-\sqrt{z^2-1})^{k+1}\}.$$

On the other hand, recalling two formulas concerning the Chebychef's function $U_\nu(\zeta)$ and hypergeometric series $F(\alpha, \beta, \gamma; \zeta)$, we have

$$\begin{aligned}U_\nu(\zeta)&=\nu(1-\zeta^2)^{1/2}F\Big(\frac{1+\nu}{2}, \frac{1-\nu}{2}, \frac{3}{2}; 1-\zeta^2\Big)\\&=\frac{1}{2\sqrt{-1}}\{(\zeta+\sqrt{\zeta^2-1})^\nu-(\zeta-\sqrt{\zeta^2-1})^\nu\},\end{aligned}$$

$$F(\alpha, \beta, \gamma; \zeta)=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(\gamma)\zeta^n}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+n)(n!)},$$

where ν is an arbitrary complex number. Then we have the representations

$$I_k(2z)=(1-z^2)^{-1/2}U_{k+1}(z)=(k+1)F\Big(\frac{k+2}{2}, -\frac{k}{2}, \frac{3}{2}; 1-z^2\Big).$$

§ 3. Some lemmas.

The proof of Theorem essentially depends on the following lemma.

LEMMA 3. *Let b_{ij}^k be the (i, j) component of A_k^{-1}, then we have*

$$\sum_{j=1}^{k}|b_{ij}^k|\leqq c^2/(c-1)^3,\ i=1, \cdots, k;\ k=1, 2, \cdots,\ \forall\varepsilon\in F_c, \tag{3.1}$$

here F_c and A_k are given in (1.10), (2.6) *respectively.*

REMARK 5. Note that the estimate (3.1) is uniform in k. Since A_k is a symmetric matrix, we can easily prove

$$b_{ij}^k=b_{ji}^k,$$

therefore we can interchange the parts of i, j in (3.1).

To prove Lemma 3 we prepare the following Lemmas 4, 5.

LEMMA 4.

$$b_{ij}^k=\begin{cases}I_{j-1}I_{k-i}/I_k & (i>j)\\ I_{i-1}I_{k-j}/I_k & (i\leqq j),\end{cases} \tag{3.2}$$

where we suppose $I_0=1$.

PROOF. By the formula of an inverse matrix, we get

$$b_{ij}^k=\Delta_{ji}^k/I_k ,$$

where Δ_{ji}^k is a (j, i) cofactor of A_k. A simple computation gives

$$\Delta_{ji}^k=\begin{cases} I_{j-1}I_{k-i} & (i>j) , \\ I_{i-1}I_{k-j} & (i\leqq j) . \end{cases}$$

Hence, Lemma 4 is proved.

LEMMA 5. *We put,*

$$\theta=\frac{1}{\sqrt{-1}}\log\left(\frac{\varepsilon+\sqrt{\varepsilon^2-4}}{2}\right)=\theta'+\sqrt{-1}\theta'' , \tag{3.3}$$

where θ', θ'' *are the real and imaginary part of* θ *respectively, and we take the branch of* $\sqrt{\varepsilon^2-4}$ *such that*

$$\varepsilon^2-4>0 ,$$

which implies,

$$\sqrt{\varepsilon^2-4}>0$$

and we fix a branch of

$$\log\left(\frac{\varepsilon+\sqrt{\varepsilon^2-4}}{2}\right)$$

appropriately, then we have

$$\theta''\leqq-\log c , \quad \forall\varepsilon\in F_c . \tag{3.4}$$

PROOF. Put

$$w=\frac{\varepsilon+\sqrt{\varepsilon^2-4}}{2} ,$$

then the mapping $\varepsilon\to w$ maps F_c into the exterior of the circle $\{w; |w|=c\}$. Therefore we have

$$|w|>c , \quad (\text{if} \quad \varepsilon\in F_c) . \tag{3.5}$$

On the other hand, we have

$$\sqrt{-1}\theta=\log w=\log|w|+\sqrt{-1}\arg w\ ,$$

that is,

$$\theta=-\sqrt{-1}\log|w|+\arg w\ .$$

So we get

$$\theta''=-\log|w|\ .$$

Using (3.5), we get

$$\theta''\leqq-\log c\ .$$

PROOF OF LEMMA 2. By (2.17), (3.3) we have

$$I_k(\varepsilon)=\frac{\sin(k+1)\theta}{\sin\theta}\ . \tag{3.6}$$

By Lemma 4 and (3.6), we have, when $i\geqq j$,

$$b^k_{ij}=\frac{\sin j\theta\sin(k-i+1)\theta}{\sin\theta\sin(k+1)\theta}=\frac{1}{2\sqrt{-1}\sin\theta} \times\frac{\left\{\begin{array}{l}\exp(\sqrt{-1}(k+1-i+j)\theta)+\exp(-\sqrt{-1}(k+1-i+j)\theta)\\ -\exp(\sqrt{-1}(k+1-i-j)\theta)-\exp(-\sqrt{-1}(k+1-i-j)\theta)\end{array}\right\}}{\exp(\sqrt{-1}(k+1)\theta)-\exp(-\sqrt{-1}(k+1)\theta)}\ . \tag{3.7}$$

And when $i<j$,

$$b^k_{ij}=\frac{\sin i\theta\sin(k-j+1)\theta}{\sin\theta\sin(k+1)\theta}=\frac{1}{2\sqrt{-1}\sin\theta} \times\frac{\left\{\begin{array}{l}\exp(\sqrt{-1}(k+1+i-j)\theta)+\exp(-\sqrt{-1}(k+1+i-j)\theta)\\ -\exp(\sqrt{-1}(k+1-i-j)\theta)-\exp(-\sqrt{-1}(k+1-i-j)\theta)\end{array}\right\}}{\exp(\sqrt{-1}(k+1)\theta)-\exp(-\sqrt{-1}(k+1)\theta)}\ . \tag{3.8}$$

By Lemma 5, we get

$$\theta''\leqq-\log c\ ,$$

so, we can estimate the dominators of (3.7), (3.8) as follows.

$$|\exp(\sqrt{-1}(k+1)\theta)-\exp(-\sqrt{-1}(k+1)\theta)|\geqq(1-1/c^4)\exp(-(k+1)\theta'')\ . \tag{3.9}$$

$$|\sin\theta|\geqq\frac{1}{2}(1-1/c^2)\exp(-\theta'')\ . \tag{3.10}$$

Hence, by using (3.7), (3.9), (3.10) we have, for $i\geqq j$,

$$(3.11)\quad |b_{ij}^k| \leqq \frac{1}{2|\sin\theta|(|\exp(\sqrt{-1}(k+1)\theta)-\exp(-\sqrt{-1}(k+1)\theta)|)}$$
$$\times|\{\exp(\sqrt{-1}(k+1-i+j)\theta)+\exp(-\sqrt{-1}(k+1-i+j)\theta)$$
$$-\exp(\sqrt{-1}(k+1-i-j)\theta)-\exp(-\sqrt{-1}(k+1-i-j)\theta)\}|$$
$$=\frac{c^6\exp((k+2)\theta'')}{(c^2-1)(c^4-1)}\{\exp(-\theta''(k+1-i+j))+\exp(\theta''(k+1-i+j))$$
$$+\exp(-\theta''(k+1-i-j))+\exp(\theta''(k+1-i-j))\}\,.$$

Similarly, by using (3.8), (3.9), (3.10), we get, when $i<j$,

$$(3.12)\quad |b_{ij}^k| \leqq \frac{c^6\exp(\theta''(k+2))}{(c^2-1)(c^4-1)}\{\exp(-\theta''(k+1+i-j))+\exp(\theta''(k+1+i-j))$$
$$+\exp(-\theta''(k+1-i-j))+\exp(\theta''(k+1-i-j))\}\,.$$

For each positive integer k and i such that $1\leqq i\leqq k$, we have

$$\sum_{j=1}^{k}|b_{ij}^k|=\sum_{j=1}^{i}|b_{ij}^k|+\sum_{j=i+1}^{k}|b_{ij}^k|=\frac{c^6\exp(\theta''(k+2))}{(c^2-1)(c^4-1)}$$
$$\times\Big\{\sum_{j=1}^{i}[\exp(-\theta''(k+1-i+j))+\exp(\theta''(k+1-i+j))]$$
$$+\sum_{j=i+1}^{k}[\exp(-\theta''(k+1+i-j))+\exp(\theta''(k+1+i-j))]$$
$$+\sum_{j=1}^{k}[\exp(-\theta''(k+1-i-j))+\exp(\theta''(k+1-i-j))]\Big\}$$
$$=\frac{c^6\exp(\theta''(k+2))}{(c^2-1)(c^4-1)(1-e^{\theta''})}\{-\exp(-\theta''(k+1-i))+\exp(-\theta''(k+1))$$
$$+\exp(\theta''(k+2-i))+\exp(-\theta''k)-\exp(\theta''(k+2))-\exp(-\theta''i)$$
$$-\exp(\theta''(k+1))+\exp(\theta''(i+1))-\exp(\theta''i)+\exp(-\theta''(k-i))$$
$$+\exp(-\theta''(i-1))-\exp(\theta''(k-i+1))\}\,.$$

On the other hand, we can prove

$$e^{\theta''(i+1)}-e^{i\theta''}<0,\ e^{-\theta''(i-1)}-e^{-\theta''i}<0,\ 1-e^{\theta''}\geqq 1-1/c\,,$$

by using the estimate,

$$\theta''\leqq-\log c<0\,,$$

for $c>1$ by definition. Hence

$$\leqq\frac{c^7}{(c-1)^3(c+1)^2(c^2+1)}\{e^{-\log c}+e^{-5\log c}+e^{-2\log c}+e^{-3\log c}\}$$

$$\leqq \frac{c^2(c^4+c^3+c^2+1)}{(c-1)^3(c^4+2c^3+2c^2+2c+1)} \leqq \frac{c^2}{(c-1)^3} .$$

This completes the proof of Lemma 3.

§4. Completion of the proof of Theorem.

First, we shall prepare two lemmas.

LEMMA 6. *Let λ be a positive integer and μ be a positive number and let $x_1, x_2, \cdots, x_\lambda$ be real variables. Then, the maximum of the product $x_1x_2\cdots x_\lambda$ uuder the conditions,*

$$x_1+x_2+\cdots+x_\lambda=\mu,\ x_1,\ x_2,\ \cdots,\ x_\lambda \geqq 0 ,$$

does not exceed $e^{\mu/e}$.

PROOF. By using the Lagrange's method of indeterminate coefficients, we can see that the maximum of $x_1x_2\cdots x_\lambda$ under the above conditions is $(\mu/\lambda)^\lambda$. Then, considering the maximum of the function $g(x)=(\mu/x)^x$ in the interval $[1, \infty)$, we can prove Lemma 6.

LEMMA 7. (*The estimate of $C_{m,n}$*). *The absolute value of each component of $C_{m,n}$ can be estimated by*

$$\frac{\delta((m-1)!)}{N((n-1)!)} r^{m-n}(M_0\delta N e^{2/e}+e^{1/e})^{m-n} ,$$

where $\delta=c^2/(c-1)^3$ and M_0, N, r are defined in Lemma 1.

PROOF. By using (2.13), Lemmas 1,3 we can see that the absolute value of each component of $(-1)^s A_m^{-1} B_{m,n(s)} A_{n(s)}^{-1} B_{n(s),n(s-1)} A_{n(s-1)}^{-1} \cdots A_{n(1)}^{-1}$ does not exceed the following value;

$$M_0^s \delta^{s+1} r^{m-n} N^{s-1} \frac{((m-1)!)}{((n-1)!)} (m-n(s)+1)(n(s)-n(s-1)+1)\cdots(n(3)-n(2)+1) .$$

On the other hand, using Lemma 6 with $\lambda=s$, $\mu=m-n+s$, we have

$$\begin{aligned}&(m-n(s)+1)(n(s)-n(s-1)+1)\cdots(n(3)-n(2)+1)\\ &\quad\leqq(m-n(s)+1)(n(s)-n(s-1)+1)\cdots(n(3)-n(2)+1)(n(2)-n(1)+1)\\ &\quad\leqq e^{(m-n+s)/e}.\end{aligned}$$

Hence the absolute value of each component of $C_{m,n}$ has the following estimate.

$$\sum_{s=1}^{m-n} \sum_{n=n(1)<\cdots<n(s)<m} M_0^s \delta^{s+1} r^{m-n} N^{s-1} e^{(m-n)/e} e^{s/e} \frac{((m-1)!)}{((n-1)!)}$$

$$= \sum_{s=1}^{m-n} \binom{m-n}{s} M_0^s \delta^{s+1} r^{m-n} N^{s-1} e^{(m-n)/e} e^{s/e} \frac{((m-1)!)}{((n-1)!)}$$

$$\leqq \frac{\delta((m-1)!)}{N((n-1)!)} r^{m-n} e^{(m-n)/e} (M_0 \delta N e^{1/e} + 1)^{m-n}$$

$$= \frac{\delta((m-1)!)}{N((n-1)!)} r^{m-n} (M_0 \delta N e^{2/e} + e^{1/e})^{m-n} .$$

Consequently, we have proved Lemma 7.

PROOF OF THEOREM. To prove Theorem, we have only to prove the convergence of the formal solution obtained in § 2. By Lemma 7 and (2.10), we have,

$$|u_{k+p-1,q}|, \cdots, |u_{p,k+q-1}|$$
$$\leqq \sum_{l=1}^{k-1} \left\{ \frac{\delta((k-1)!)}{N((l-1)!)} r^{k-l} l (M_0 \delta N e^{2/e} + e^{1/e})^{k-l} M_0 r^{l-1} ((l-1)!) \right\}$$
$$+ M_0 r^{k-1} ((k-1)!) \delta \leqq \frac{\delta M_0 (k!) r^{k-1}}{N} \left\{ \frac{(M_0 N \delta e^{2/e} + e^{1/e})^k - (M_0 \delta N e^{2/e} + e^{1/e})}{M_0 \delta N e^{2/e} + e^{1/e} - 1} \right\}$$
$$+ M_0 \delta ((k-1)!) r^{k-1} \leqq \frac{(k!) r^{k-1}}{N^2 e^{2/e}} (M_0 N \delta e^{2/e} + e^{1/e})^k + M_0 \delta r^{k-1} ((k-1)!)$$
$$\leqq (k!) r^{k-1} (M_0 \delta N e^{2/e} + e^{1/e})^{k-1} \left(M_0 \delta + \frac{\delta M_0 N e^{1/e} + 1}{N^2 e^{1/e}} \right) .$$

Put,

$$r_1 = r(M_0 N \delta e^{2/e} + e^{1/e}) , \quad K = M_0 \delta + \frac{M_0 N \delta e^{1/e} + 1}{N^2 e^{1/e}} ,$$

then we have,

$$|u_{p+k-1,q}|, \cdots, |u_{p,q+k-1}| \leqq K r_1^{k-1} (k!) .$$

Therefore we have,

$$\left| \sum_{i,j\geqq 0} u_{ij} \frac{x^i}{(i!)} \frac{y^j}{(j!)} \right| = \left| \sum_{k=1}^{\infty} \sum_{i+j=k-1, i,j\geqq 0} \left(u_{i+p,j+q} \frac{x^{i+p} y^{j+q}}{((i+p)!)((j+q)!)} \right) \right|$$
$$\leqq \sum_{k=1}^{\infty} \sum_{i+j=k-1} K r_1^{k-1} \frac{(k!)((k+p+q-1)!)}{((k+p+q-1)!)((i+p)!)((j+q)!)} |x|^{i+p} |y|^{j+q}$$

$$\leq \sum_{k=1}^{\infty} Kr_1^{k-1} \sum_{i+j=k-1} \binom{k+p+q-1}{i+p} |x|^{i+p} |y|^{k+p+q-1-i-p}$$
$$= \sum_{k=1}^{\infty} Kr_1^{k-1} (|x|+|y|)^{k+p+q-1} .$$

Hence the formal solution obtained in §2 actually converges, if

$$|x|+|y| < \frac{1}{r_1} = \frac{1}{r(M_0 N \delta e^{2/\varepsilon} + e^{1/\varepsilon})} .$$

Note that this convergence is uniform in ε on any compact subset of $C\backslash[-2, 2]$. Therefore this solution is also analytic with respect to ε in the domain $C\backslash[-2, 2]$. Consequently we have proved Theorem.

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